On directional regularity of mappings

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Outline of the talk

- Notation, preliminaries, and motivation

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- A generalized directional Ekeland Variational Principle
- Ioffe-type criteria for directional regularity
- Stability of the directional regularity
- Necessary and sufficient conditions for directional regularity by generalized differentiation

Papers

The results of this talk are mainly based on the papers:

- M. Durea, M. Panţiruc, R. Strugariu, *Minimal time function with respect to a set of directions. Basic properties and applications*, Optimization Methods and Software, 31 (2016), 535–561.
- M. Durea, M. Panţiruc, R. Strugariu, *A new type of directional regularity for mappings and applications to optimization*, SIAM Journal on Optimization, 27 (2017), 1204–1229.
- R. Cibulka, M. Durea, M. Panţiruc, R. Strugariu, *On the stability of the directional regularity*, submitted.

Directional regularity: definition and basic properties

Notations

- X, Y are normed vector spaces over the real field $\mathbb{R}$.
- $B(x, \varepsilon)$ and $B[x, \varepsilon]$ are the open and closed balls with center $x \in X$ and radius $\varepsilon > 0$, respectively.
- We use the symbols B_X, $\mathbb{B}_X$ and S_X for the open and the closed balls, and the sphere of center 0 and radius 1, respectively.
- For a set $A \subset X$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ its topological interior, closure and boundary, respectively.
- The cone generated by A is designated by $\operatorname{cone} A$, and the convex hull of A is $\operatorname{conv} A$.

Notations

- The polar of a set $A \subset X$ is

$$A^{\circ} := \{x^* \in X^* \mid \langle x^*, u\rangle \geq -1, \forall u \in A\}. \quad (1)$$

If A is a cone, then its polar (denoted A^+) becomes

$$A^{+} = \{x^* \in X^* \mid \langle x^*, u\rangle \geq 0, \forall u \in A\}, \quad (2)$$

and is called the positive dual cone of A.

- Given two sets $A, B \subset X$, one defines the distance between A and B by

$$d(A,B) := \inf\{\|a-b\| \mid a \in A, b \in B\}.$$

If $x \in X$ and $A \subset X$, then the distance from x to A is $d(x,A) := d(\{x\},A)$. As usual, $d(A,\emptyset) := \infty$. The excess from A to B is defined as

$$e(A,B) := \sup\{d(a,B) \mid a \in A\},$$

under the convention $e(\emptyset,B) = 0$ if $B \neq \emptyset$ and $e(A,\emptyset) = +\infty$ for any A.

Multifunctions

- Let $F : X \rightrightarrows Y$ be a multifunction and $A \subset X$. The domain, the graph of F and the image of A through F are denoted respectively by

$$\operatorname{Dom} F := \{x \in X \mid F(x) \neq \emptyset\},$$
$$\operatorname{Gr} F := \{(x, y) \in X \times Y \mid y \in F(x)\},$$
$$F(A) := \bigcup_{x \in A} F(x).$$

- The inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by

$$F^{-1}(y) := \{x \in X \mid y \in F(x)\}.$$

Linear openness, metric regularity, Aubin property: classical case

Let $F : X \rightrightarrows Y$ be a multifunction and $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$.

- F is said to be open at linear rate $L > 0$ around $(\bar{x}, \bar{y})$ if there exist a positive number $\varepsilon > 0$ and two neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $\rho \in (0, \varepsilon)$ and every $(x, y) \in \operatorname{Gr} F \cap [U \times V]$,

$$B(y, \rho L) \subset F(B(x, \rho)).$$

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$$B(y, \rho L) \subset F(B(x, \rho)).$$

  The modulus of openness of F around $(\overline{x}, \overline{y})$, denoted by $\operatorname{sur} F(\overline{x}, \overline{y})$, is the supremum of $L > 0$ such that F is open at linear rate L around $(\overline{x}, \overline{y})$.

Linear openness, metric regularity, Aubin property: classical case

- F is said to be metrically regular around $(\bar{x}, \bar{y})$ with constant $L > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $(x, y) \in U \times V$,

$$d(x, F^{-1}(y)) \leq Ld(y, F(x)).$$

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  The modulus of regularity of F around $(\bar{x}, \bar{y})$, denoted by $\operatorname{reg} F(\bar{x}, \bar{y})$, is the infimum of $L > 0$ such that F is metrically regular around $(\bar{x}, \bar{y})$ with the constant L.

Linear openness, metric regularity, Aubin property: classical case

- F is said to have the Aubin property around $(\overline{x}, \overline{y})$ with constant $L > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $x, u \in U$,

$$e(F(x) \cap V, F(u)) \leq Ld(x, u).$$

Linear openness, metric regularity, Aubin property: classical case

- F is said to have the Aubin property around $(\bar{x}, \bar{y})$ with constant $L > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $x, u \in U$,
$$e(F(x) \cap V, F(u)) \leq Ld(x, u).$$
The modulus of the Aubin property of F around $(\bar{x}, \bar{y})$, denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$, is the infimum of $L > 0$ such that F has the Aubin property around $(\bar{x}, \bar{y})$ with the constant L.

Linear openness, metric regularity, Aubin property: links

Proposition 1

Let $F : X \rightrightarrows Y$ be a multifunction and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Then F is open at linear rate around $(\overline{x},\overline{y})$ iff F^{-1} has the Aubin property around $(\overline{y},\overline{x})$ iff F is metrically regular around $(\overline{x},\overline{y})$. Moreover, in every of the previous situations,

$$\operatorname{reg} F(\overline{x},\overline{y}) = (\operatorname{sur} F(\overline{x},\overline{y}))^{-1} = \operatorname{lip} F^{-1}(\overline{y},\overline{x}). \tag{3}$$

Minimal time function

We study a minimal time function which prove to be adequate to deal with the directional phenomena in variational analysis and optimization.

- Let $\Omega \subset X$ and $M \subset S_X$ be nonempty sets. Then the function

$$T_M(x,\Omega) := \inf\{t \geq 0 \mid \exists u \in M : x + tu \in \Omega\} \tag{4}$$

is called the directional minimal time function with respect to M.

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- When $M := \{u\}$, then we denote $T_{\{u\}}(x, \Omega)$ by $T_u(x, \Omega)$. This case was analyzed by Nam and Zălinescu.
- For two sets $A, B \subset X$, we consider the directional excess from A to B with respect to M as

$$e_M(A, B) := \sup_{x \in A} T_M(x, B).$$

Remark that $e_M(A, B) = \infty$ if $A \not\subset B - \operatorname{cone} M$. If $M = S_X$, $e_M(A, B)$ becomes the usual excess from A to B.

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Remark that $e_M(A,B) = \infty$ if $A \not\subset B - \operatorname{cone} M$. If $M = S_X$, $e_M(A,B)$ becomes the usual excess from A to B.

- Moreover, we denote in what follows $T_M(x,\{u\})$ by $T_M(x,u)$.

Minimal time function: basic properties

Proposition 2

(i) The domain of the directional minimal time function with respect to M is given by

$$\operatorname{dom} T_M(\cdot,\Omega) = \Omega - \operatorname{cone} M.$$

(ii) One has

$$T_M(x,\Omega) = \inf_{u\in M} T_u(x,\Omega).$$

(iii) One has, for any $x \in X$ and $\Omega \subset X$,

$$d(x,\Omega) \le T_M(x,\Omega).$$

If $M = S_X$, then $\operatorname{dom} T_M(\cdot,\Omega) = X$ and

$$T_M(x,\Omega) = d(x,\Omega), \quad \forall x \in X.$$

Directional regularity: definitions

Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $L \subset S_X$, $M \subset S_Y$.

- One says that F is directionally linearly open around $(\overline{x}, \overline{y})$ with respect to L and M with modulus $\alpha > 0$ if there are $\varepsilon > 0$ and some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $r \in (0, \varepsilon]$ and every $(x, y) \in [U \times V] \cap \operatorname{Gr} F$,

$$B(y, \alpha r) \cap [y - \operatorname{cone} M] \subset F(B(x, r) \cap [x + \operatorname{cone} L]). \tag{5}$$

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  The modulus of directional openness of F around $(\bar{x}, \bar{y})$ with respect to L and M, denoted by $\operatorname{dirsur}_{L \times M} F(\bar{x}, \bar{y})$, is the supremum of $\alpha > 0$ such that F is directionally linearly open around $(\bar{x}, \bar{y})$ with respect to L and M with modulus α.

Directional regularity: definitions

- One says that F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to L and M with modulus $\alpha > 0$ if there are $\varepsilon > 0$ and some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $(x, y) \in U \times V$ such that $T_M(y, F(x)) < \varepsilon$,

$$T_L(x, F^{-1}(y)) \leq \alpha \cdot T_M(y, F(x)). \tag{6}$$

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The modulus of directional regularity of F around $(\overline{x}, \overline{y})$ with respect to L and M, denoted by $\operatorname{dirreg}_{L\times M} F(\overline{x}, \overline{y})$, is the infimum of $\alpha > 0$ such that F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to L and M with modulus α.

Directional regularity: definitions

- One says that F is directionally Aubin continuous around $(\overline{x}, \overline{y})$ with respect to L and M with modulus $\alpha > 0$ if there are some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $x, u \in U$,

$$e_M(F(x) \cap V, F(u)) \leq \alpha T_L(u, x). \tag{7}$$

Directional regularity: definitions

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The modulus of the directional Aubin property of F around $(\overline{x}, \overline{y})$ with respect to L and M, denoted by $\operatorname{dirlip}_{L\times M} F(\overline{x}, \overline{y})$, is the infimum of $\alpha > 0$ such that F has the directional Aubin property around $(\overline{x}, \overline{y})$ with respect to L and M with modulus α.

Directional regularity: links

- Observe that, when one takes $L = S_X$, $M = S_Y$, the previous concepts reduce to the usual metric regularity, linear openness and Aubin property around the reference point.

Directional regularity: links

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Proposition 3

Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $L \subset S_X$, $M \subset S_Y$, and $\alpha > 0$. Then F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to L and M with modulus α iff F is directionally linearly open around $(\overline{x}, \overline{y})$ with respect to L and M with modulus α^{-1} iff F^{-1} is directionally Aubin continuous around $(\overline{y}, \overline{x})$ with respect to M and L with modulus α.
Moreover, in every of the previous situations,

$$\operatorname{dirreg}_{L\times M} F(\overline{x}, \overline{y}) = (\operatorname{dirsur}_{L\times M} F(\overline{x}, \overline{y}))^{-1} = \operatorname{dirlip}_{M\times L} F^{-1}(\overline{y}, \overline{x}). \qquad (8)$$

Directional regularity with respect to a variable

For a set-valued mapping $F : X \times Y \rightrightarrows Z$, one may speak about the directional regularities with respect to one variable, uniformly for the other.

- We use the notation $F_y := F(\cdot, y)$, we consider nonempty sets $L \subset S_X$, $M \subset S_Z$, and we say that F is directionally metrically regular relative to x uniformly in y around $(\bar{x}, \bar{y}, \bar{z}) \in \operatorname{Gr} F$ with respect to L and M with modulus $\alpha > 0$ if there are $\varepsilon > 0$ and neighborhoods U of $\bar{x}$, V of $\bar{y}$, and W of $\bar{z}$ such that, for every $y \in V$, and every $(x, z) \in U \times W$ such that $T_M(z, F_y(x)) < \varepsilon$,

$$T_L(x, F_y^{-1}(z)) \leq \alpha \cdot T_M(z, F_y(x)). \tag{9}$$

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$$T_L(x, F_y^{-1}(z)) \leq \alpha \cdot T_M(z, F_y(x)). \tag{9}$$

  The modulus of directional regularity of F relative to x uniformly in y around $(\bar{x}, \bar{y}, \bar{z})$ with respect to L and M, denoted by $\widehat{\operatorname{dirreg}}^{x}_{L \times M} F(\bar{x}, \bar{y}, \bar{z})$, is defined as the infimum of $\alpha > 0$ such that the above property holds.

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  The modulus of directional regularity of F relative to x uniformly in y around $(\overline{x}, \overline{y}, \overline{z})$ with respect to L and M, denoted by $\widehat{\operatorname{dirreg}}^{x}_{L\times M} F(\overline{x}, \overline{y}, \overline{z})$, is defined as the infimum of $\alpha > 0$ such that the above property holds.
- Analogously, one may define the other two regularity properties relative to one variable, uniformly for the other, and the regularity moduli are denoted by $\widehat{\operatorname{dirsur}}^{x}_{L\times M} F(\overline{x}, \overline{y}, \overline{z})$ and $\widehat{\operatorname{dirlip}}^{x}_{L\times M} F(\overline{x}, \overline{y}, \overline{z})$.

Directional regularity: some examples

1. Consider $X := Y := \mathbb{R}$, and take $f : [0,\infty) \to [0,\infty)$ a strictly increasing function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Define $L := \{-1\}$, $M := \{-1, 1\}$, and the multifunction $F : X \rightrightarrows Y$ given by

$$F(x) := \begin{cases} [0, f(x)], & \text{if } x \geq 0 \\ \emptyset, & \text{if } x < 0. \end{cases}$$

Then the multifunction F is directionally Aubin continuous with respect to L and M, around any $(x,y) \in \operatorname{Gr} F$, with any modulus $\alpha > 0$, but F is not Aubin continuous around $(0,0)$, for instance.

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2. Consider $X := Y := \mathbb{R}$, $L := \{-1\}$, $M := \{-1, 1\}$, and take $f : [0,\infty) \to \mathbb{R}$ with $f(0) = 0$ be a Lipschitz function with modulus $\alpha > 0$. Then the multifunction $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$F(x) := \begin{cases} \{f(x)\}, & \text{if } x \geq 0 \\ \emptyset, & \text{if } x < 0 \end{cases}$$

   is directionally Aubin continuous with respect to L and M around $(0,0)$, with modulus α, but is not Aubin continuous around $(0,0)$.

Directional regularity: comparisons

Next, we compare our concepts with other related ones. First, recall the notion of the directional metric regularity in a given direction by Huynh and Théra.

Definition 1

A set-valued mapping $F : X \rightrightarrows Y$ from a metric space (X, ϱ) to a normed space $(Y, \|\cdot\|)$ is said to be directionally metrically regular at $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ in a direction $w \in Y$ *with a constant* $\kappa > 0$ if there exist $\varepsilon > 0$ and $\delta > 0$ such that, for every $(x, y) \in B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)$ satisfying $d(y, F(x)) < \varepsilon$ and $y \in F(x) + \operatorname{cone} B(w, \delta)$,

$$d(x, F^{-1}(y)) \leq \kappa\, d(y, F(x)). \tag{10}$$

Directional regularity: comparisons

Proposition 4

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed spaces, $F : X \rightrightarrows Y$ be a set-valued mapping, and $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. If F is directionally metrically regular at $(\bar{x}, \bar{y})$ in a direction $w \in Y$ with a constant $\kappa > 0$, then there is a nonempty closed $M \subset S_Y$ with $\operatorname{cone} M$ being convex such that F is directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to S_X and M with the constant κ.

Directional regularity: comparisons

The converse of the previous result does not hold in general, as the next example shows.

Example 2

Consider $X := Y := \mathbb{R}$, $M := \{1\}$, $F : X \rightrightarrows Y$ given by

$$F(x) := \begin{cases} \mathbb{R}, & \text{for } x \leq 0 \text{ or } x \geq 1 \\ (-\infty, x^2] \cup [\sqrt{x}, +\infty), & \text{for } x \in (0,1), \end{cases}$$

and $(\bar{x}, \bar{y}) := (0,0)$. Then F is directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to S_X and M with the constant $c = 1$. But, for any direction $w \in \mathbb{R}$, F is not directionally metrically regular at $(\bar{x}, \bar{y})$ in the direction w.

Directional regularity: comparisons

Another related concept is the regularity along a subspace by Dmitruk and Kruger.

Definition 3

A set-valued mapping $F : X \rightrightarrows Y$ from a normed space X to a metric space Y is called metrically regular along a closed subspace H *of* X *around* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *with a constant* $\kappa > 0$ if there exists $\varepsilon > 0$ such that, for every $(x,y) \in B(\overline{x}, \varepsilon \times B(\overline{y}, \varepsilon)$,

$$\inf\{\|h\| : \ h \in H \text{ and } x + h \in F^{-1}(y)\} \leq \kappa\, d(y, F(x)). \tag{11}$$

Lemma 4

Let H *be a closed subspace of a normed space* X*. Then, for every* $\Omega \subset X$ *and every* $x \in X$,

$$d_H(x,\Omega) := \inf\{\|h\| : \ h \in H \text{ and } x + h \in \Omega\} = T_{S_H}(x,\Omega). \tag{12}$$

Directional regularity: comparisons

Corollary 5

Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces X and Y with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, H be a closed subspace of X, and $\kappa > 0$. If F is metrically regular along H around $(\overline{x}, \overline{y})$ with the constant κ, then F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to S_H and S_Y with the constant κ. Conversely, if F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to S_H and S_Y with the constant κ, then there is $\varepsilon > 0$ such that for all $(x, y) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon)$ with $d(y, F(x)) < \varepsilon$ inequality (11) holds.

A generalized directional Ekeland Variational Principle

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Theorem 5 (generalized Ekeland principle)

Let X be a Banach space and $A \subset X$ be a closed set. Let $M \subset S_X$ be a closed set such that $\operatorname{cone} M$ *is convex, $\Omega \subset X$ a compact subset of X with $\Omega \cap A = \emptyset$ and $f : A \to \mathbb{R} \cup \{\infty\}$ be a bounded from below lower semicontinuous function. Then, for every $x_0 \in \operatorname{dom} f \cap A$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that*

$$f(x_\varepsilon) \leq f(x_0) - \varepsilon T_M(x_\varepsilon, \Omega \cup \{x_0\}) \tag{13}$$

and

$$f(x_\varepsilon) < f(x) + \varepsilon T_M(x, \Omega \cup \{x_\varepsilon\}), \ \forall x \in A \setminus \{x_\varepsilon\}. \tag{14}$$

Directional EVP: some remarks

- If one takes $\Omega = \emptyset$, then $T_M(x, \Omega \cup \{u\})$ reduces to $T_M(x, u)$.

Directional EVP: some remarks

- If one takes $\Omega = \emptyset$, then $T_M(x, \Omega \cup \{u\})$ reduces to $T_M(x, u)$.
- Moreover, if cone M is convex, then T_M has the properties of a generalized extended-valued quasi-metric:
  (i) $T_M(x, u) = 0$ iff $x = u$;
  (ii) $T_M(x, u) \leq T_M(x, v) + T_M(v, u)$, for all $x, v, u \in X$.

Directional EVP: some remarks

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Corollary 6

Let X be a Banach space and $A \subset X$ be a closed set. Let $M \subset S_X$ be a closed set such that cone M *is convex, and $f : A \to \mathbb{R} \cup \{\infty\}$ be a bounded from below lower semicontinuous function. Then, for every $x_0 \in \operatorname{dom} f \cap A$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that*

$$f(x_\varepsilon) \le f(x_0) - \varepsilon T_M(x_\varepsilon, x_0) \tag{15}$$

and

$$f(x_\varepsilon) < f(x) + \varepsilon T_M(x, x_\varepsilon), \ \forall x \in A \setminus \{x_\varepsilon\}. \tag{16}$$

Directional EVP on product spaces

Corollary 7

Let X, Y be Banach spaces and $A \subset X \times Y$ be a closed set. Let $L \subset S_X$ and $M \subset S_Y$ be closed sets such that cone L *and* cone M *are convex, and $f : A \to \mathbb{R} \cup \{\infty\}$ be a bounded from below lower semicontinuous function. Then, for every $(x_0, y_0) \in \operatorname{dom} f \cap A$ and every $\varepsilon > 0$, there exists $(x_\varepsilon, y_\varepsilon) \in A$ such that*

$$f(x_\varepsilon, y_\varepsilon) \leq f(x_0, y_0) - \varepsilon \left[T_L(x_\varepsilon, x_0) + T_M(y_\varepsilon, y_0)\right] \tag{17}$$

and

$$f(x_\varepsilon, y_\varepsilon) < f(x, y) + \varepsilon \left[T_L(x, x_\varepsilon) + T_M(y, y_\varepsilon)\right], \ \forall (x, y) \in A \setminus \{(x_\varepsilon, y_\varepsilon)\}. \tag{18}$$

Minimal time function on product spaces

Lemma 6

Let $(X_1, \|\cdot\|), \ldots, (X_n, \|\cdot\|)$ be normed spaces and positive constants $\alpha_1, \ldots, \alpha_n$ be given. Consider nonempty closed subsets L_i of S_{X_i} for $i = 1, \ldots, n$. Define the equivalent norm $\|\cdot\|_{\widetilde{X}}$ on $\widetilde{X} := X_1 \times ... \times X_n$ for each $(u_1, ..., u_n) \in \tilde{X}$ by $\|(u_1, ..., u_n)\|_{\widetilde{X}} := \max\{\alpha_1\|u_1\|, ..., \alpha_n\|u_n\|\}$. Then there exists $\widetilde{L} \subset S_{\widetilde{X}}$ such that $\operatorname{cone}\widetilde{L} = \operatorname{cone} L_1 \times ... \times \operatorname{cone} L_n$ and, for each $(u_1, ..., u_n), (u_1', ..., u_n') \in \widetilde{X}$,

$$T_{\widetilde{L}}((u_1, ..., u_n), (u_1', ..., u_n')) = \max\{\alpha_1 T_{L_1}(u_1, u_1'), ..., \alpha_n T_{L_n}(u_n, u_n')\}. \qquad (19)$$

Directional EVP on product spaces: another variant

Corollary 8

Let X, Y be Banach spaces and $A \subset X \times Y$ be a closed set. Let $L \subset S_X$ and $M \subset S_Y$ be closed sets such that cone L *and* cone M *are convex, and $f : A \to \mathbb{R} \cup \{\infty\}$ be a bounded from below lower semicontinuous function. Then, for every $(x_0, y_0) \in \operatorname{dom} f \cap A$ and every $\varepsilon > 0$, there exists $(x_\varepsilon, y_\varepsilon) \in A$ such that*

$$f(x_\varepsilon, y_\varepsilon) \le f(x_0, y_0) - \varepsilon \max\{T_L(x_\varepsilon, x_0), T_M(y_\varepsilon, y_0)\} \qquad (20)$$

and

$$f(x_\varepsilon, y_\varepsilon) < f(x, y) + \varepsilon \max\{T_L(x, x_\varepsilon), T_M(y, y_\varepsilon)\}, \ \forall (x, y) \in A \setminus \{(x_\varepsilon, y_\varepsilon)\}. \qquad (21)$$

Ioffe-type criteria for directional regularity

Ioffe-type criteria for directional regularity: single-valued case

Proposition 7

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider a nonempty closed subset L of S_X such that $\operatorname{cone} L$ is convex, a nonempty closed subset M of S_Y, a point $\overline{x} \in X$, and a mapping $g : X \to Y$ such that there is a neighborhood U of $\bar{x}$ such that the set $D := U \cap \operatorname{Dom} g$ is closed and g is continuous on D. Then $\operatorname{dirsur}_{L\times M} g(\overline{x})$ equals to the supremum of $c > 0$ for which there is $r > 0$ such that for all $(x, y) \in (B[\overline{x}, r] \cap \operatorname{Dom} g) \times B[g(\overline{x}), r]$, with $0 < T_M(y, g(x)) < +\infty$, there is a point $x' \in \operatorname{Dom} g$ satisfying

$$cT_L(x, x') < T_M(y, g(x)) - T_M(y, g(x')). \tag{22}$$

Ioffe-type criteria for directional regularity: set-valued case

Proposition 8

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider nonempty closed subsets L of S_X and M of S_Y such that $\operatorname{cone} L$ is convex, a point $(\bar{x}, \bar{y}) \in X \times Y$, and a set-valued mapping $F : X \rightrightarrows Y$ the graph of which is locally closed near $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. Then $\operatorname{dirsur}_{L\times M} F(\bar{x}, \bar{y})$ equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $(x, v) \in (B[\bar{x}, r] \times B[\bar{y}, r]) \cap \operatorname{Gr} F$ and any $y \in B[\bar{y}, r]$, with $0 < T_M(y, v) < +\infty$, there is a pair $(x', v') \in \operatorname{Gr} F$ such that

$$c \max\{T_L(x, x'), \alpha\|v - v'\|\} < T_M(y, v) - T_M(y, v'). \tag{23}$$

Idea of the proof. Apply Proposition 7 to $g := p_Y\,|_{\operatorname{Gr} F}$, where p_Y is the canonical projection from $X \times Y$ onto Y.

Stability of the directional regularity

Compositions of multifunctions. Composition stability around a point

Given metric spaces (X, ϱ), (Y, ϱ), and (Z, ϱ), a composition of set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ is the mapping $G \circ F : X \rightrightarrows Z$ defined by

$$(G \circ F)(x) := \bigcup_{y \in F(x)} G(y), \quad x \in X.$$

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A product of set-valued mappings $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$ is the mapping $(F_1,F_2) : X \rightrightarrows Y \times Z$ defined by

$$(F_1,F_2)(x) := F_1(x) \times F_2(x), \quad x \in X.$$

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Definition 9

Let (X,ϱ), (Y,ϱ), and (Z,ϱ) be metric spaces and $(\overline{x},\overline{y},\overline{z}) \in X \times Y \times Z$ be fixed. Consider set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ such that $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{y})$. We say that the pair F, G is composition-stable around $(\overline{x},\overline{y},\overline{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\overline{x},\delta)$ and every $z \in (G \circ F)(x) \cap B(\overline{z},\delta)$, there exists $y \in F(x) \cap B(\overline{y},\varepsilon)$ such that $z \in G(y)$.

Directional openness stability at composition

Theorem 10

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$, and $(W, \|\cdot\|)$ be Banach spaces and $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in X \times Y \times Z \times W$ be fixed. Consider nonempty closed subsets L of S_X, M of S_Y, N of S_Z, and P of S_W such that $\operatorname{cone} L$, $\operatorname{cone} M$, $\operatorname{cone} N$, and $\operatorname{cone} P$ are convex, set-valued mappings $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, and $G : Y \times Z \rightrightarrows W$ such that F_1 has a locally closed graph near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$, F_2 has a locally closed graph near $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$, and G has a locally closed graph near $(\overline{y}, \overline{z}, \overline{w}) \in \operatorname{Gr} G$. Define the mapping $\mathcal{E}_{G,(F_1,F_2)} : X \times Y \times Z \rightrightarrows W$ by

$$\mathcal{E}_{G,(F_1,F_2)}(x,y,z) := \begin{cases} G(y,z), & \text{if } (y,z) \in (F_1,F_2)(x), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{24}$$

Directional openness stability at composition: continued

Theorem 10: continued

Then

$$\operatorname{dirsur}_{L\times M\times N\times P}\mathcal{E}_{G,(F_1,F_2)}(\overline{x},\overline{y},\overline{z},\overline{w}) \geq \operatorname{dirsur}_{L\times M}F_1(\overline{x},\overline{y})\cdot\widehat{\operatorname{dirsur}}^{y}_{-M\times P}G(\overline{y},\overline{z},\overline{w}) - \operatorname{dirlip}_{-L\times N}F_2(\overline{x},\overline{z})\cdot\widehat{\operatorname{dirlip}}^{z}_{-N\times P}G(\overline{y},\overline{z},\overline{w}). \tag{25}$$

If, in addition, the pair $(F_1,F_2),G$ *is composition-stable around* $(\overline{x},(\overline{y},\overline{z}),\overline{w})$, *then*

$$\operatorname{dirsur}_{L\times P}(G\circ(F_1,F_2))(\overline{x},\overline{w}) \geq \operatorname{dirsur}_{L\times M}F_1(\overline{x},\overline{y})\cdot\widehat{\operatorname{dirsur}}^{y}_{-M\times P}G(\overline{y},\overline{z},\overline{w}) - \operatorname{dirlip}_{-L\times N}F_2(\overline{x},\overline{z})\cdot\widehat{\operatorname{dirlip}}^{z}_{-N\times P}G(\overline{y},\overline{z},\overline{w}). \tag{26}$$

Sum stability around a point

Definition 11

Let (X,ϱ) *and* (Y,ϱ) *be metric spaces and* $(\overline{x},\overline{y},\overline{z}) \in X \times Y \times Y$ *be fixed. Consider set-valued mappings* $F : X \rightrightarrows Y$ *and* $G : X \rightrightarrows Y$ *such that* $\overline{y} \in F(\overline{x})$ *and* $\overline{z} \in G(\overline{x})$. *We say that the pair* F, G *is sum-stable around* $(\overline{x},\overline{y},\overline{z})$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for every* $x \in B(\overline{x},\delta)$ *and every* $w \in (F+G)(x) \cap B(\overline{y}+\overline{z},\delta)$, *there exist* $y \in F(x) \cap B(\overline{y},\varepsilon)$ *and* $z \in G(x) \cap B(\overline{z},\varepsilon)$ *such that* $w = y + z$.

Sum stability around a point

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Let (X, ϱ) and (Y, ϱ) be metric spaces and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ be fixed. Consider set-valued mappings $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ such that $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$. We say that the pair F, G is sum-stable around $(\overline{x}, \overline{y}, \overline{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\overline{x}, \delta)$ and every $w \in (F + G)(x) \cap B(\overline{y} + \overline{z}, \delta)$, there exist $y \in F(x) \cap B(\overline{y}, \varepsilon)$ and $z \in G(x) \cap B(\overline{z}, \varepsilon)$ such that $w = y + z$.

Remark 1

Observe that, if one takes in Definition 9 $F : X \rightrightarrows Y \times Y$, $F := (F_1, F_2)$, where $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ are two multifunctions, $G := g$, where $g : Y \times Y \to Y$ is given by $g(y, z) := y + z$, for each $(y, z) \in Y \times Y$, and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ such that $(\overline{y}, \overline{z}) \in F_1(\overline{x}) \times F_2(\overline{x})$, then the composition-stability of the pair F, G around $(\overline{x}, (\overline{y}, \overline{z}), \overline{y} + \overline{z})$ is just the sum-stability of F_1, F_2 around $(\overline{x}, \overline{y}, \overline{z})$.

Directional openness stability at summation

Corollary 9

Let $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces and* $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ *be fixed. Consider nonempty closed subsets* L *of* S_X *and* M *of* S_Y *such that* $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex, set-valued mappings* $F_1, F_2 : X \rightrightarrows Y$ *such that* F_1 *has a locally closed graph near* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$ *and* F_2 *has a locally closed graph near* $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$*. Define the mapping* $\mathcal{E}_{F_1,F_2} : X \times Y \times Y \rightrightarrows Y$ *by*

$$\mathcal{E}_{F_1,F_2}(x,y,z) := \begin{cases} y+z, & \text{if } (y,z) \in (F_1,F_2)(x), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{27}$$

Then

$$\operatorname{dirsur}_{L\times M\times M\times M} \mathcal{E}_{F_1,F_2}(\overline{x},\overline{y},\overline{z},\overline{y}+\overline{z}) \geq \operatorname{dirsur}_{L\times M} F_1(\overline{x},\overline{y}) - \operatorname{dirlip}_{-L\times M} F_2(\overline{x},\overline{z}). \tag{28}$$

Directional openness stability at summation

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Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ be fixed. Consider nonempty closed subsets L of S_X and M of S_Y such that $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex, set-valued mappings $F_1, F_2 : X \rightrightarrows Y$ such that F_1 has a locally closed graph near $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$ and F_2 has a locally closed graph near $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$. Define the mapping $\mathcal{E}_{F_1,F_2} : X \times Y \times Y \rightrightarrows Y$ by*

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Then

$$\operatorname{dirsur}_{L \times M \times M \times M} \mathcal{E}_{F_1,F_2}(\overline{x}, \overline{y}, \overline{z}, \overline{y} + \overline{z}) \geq \operatorname{dirsur}_{L \times M} F_1(\overline{x}, \overline{y}) - \operatorname{dirlip}_{-L \times M} F_2(\overline{x}, \overline{z}). \tag{28}$$

If, in addition, the pair F_1, F_2 is sum-stable around $(\overline{x}, \overline{y}, \overline{z})$, then

$$\operatorname{dirsur}_{L \times M}(F_1 + F_2)(\overline{x}, \overline{y} + \overline{z}) \geq \operatorname{dirsur}_{L \times M} F_1(\overline{x}, \overline{y}) - \operatorname{dirlip}_{-L \times M} F_2(\overline{x}, \overline{z}). \tag{29}$$

Single-valued perturbation

Corollary 10

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be fixed. Consider nonempty closed subsets L of S_X and M of S_Y such that $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex, a set-valued mapping $F : X \rightrightarrows Y$ the graph of which is locally closed near $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$, and a single-valued mapping $f : X \to Y$ which is continuous at $\bar{x}$. Then*

$$\operatorname{dirsur}_{L\times M}(f+F)(\bar{x}, f(\bar{x})+\bar{y}) \geq \operatorname{dirsur}_{L\times M} F(\bar{x}, \bar{y}) - \operatorname{dirlip}_{-L\times M} f(\bar{x}). \quad (30)$$

Necessary and sufficient conditions for directional regularity by generalized differentiation

Fréchet-type generalized differentiation objects

- Let S be a non-empty subset of X and let $x \in S$. The Fréchet normal cone to S at x is

$$\widehat{N}(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \right\}, \tag{31}$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

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where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

- Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be finite at $\overline{x} \in X$; the Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}, \tag{32}$$

where $\operatorname{epi} f$ denotes the epigraph of f.

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where $\operatorname{epi} f$ denotes the epigraph of f.

- If $F : X \rightrightarrows Y$ is a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, then its Fréchet coderivative at $(\overline{x}, \overline{y})$ is the set-valued mapping $\widehat{D}^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}. \tag{33}$$

Subdifferential formulae for $T_M(\cdot,\Omega)$

Proposition 12

Let X be a normed vector space, $M \subset S_X$ and $\Omega \subset X$.
(i) If $\bar{x} \in \Omega$, then

$$\widehat{\partial} T_M(\cdot,\Omega)(\bar{x}) = M^{\circ} \cap \widehat{N}(\Omega,\bar{x}). \tag{34}$$

(ii) Suppose that one of the sets Ω and M is compact and the other one is closed. Take $\bar{x} \in (\Omega - \operatorname{cone} M) \setminus \Omega$. Then for every $u \in M$ and $\omega \in \Omega$ with $\bar{x} + T_M(\bar{x},\Omega)u = \omega$, one has

$$\widehat{\partial} T_M(\cdot,\Omega)(\bar{x}) \subset \{x^* \in X^* \mid \langle x^*, u\rangle = -1\} \cap \widehat{N}(\Omega,\omega). \tag{35}$$

Asplund spaces

- A Banach space X is Asplund if every convex continuous function $f : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.

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- The class of Asplund spaces includes all Banach spaces having Fréchet smooth bump functions (in particular, spaces with Fréchet smooth renorms, hence every reflexive Banach space) and all spaces with separable duals.
- Important property: the dual unit ball is weak* sequentially compact.
- Characterization: X is Asplund iff every separable closed subspace of X has a separable dual.

Extremal system

- Let $S_1, ..., S_p$ be nonempty subsets of X with $p \geq 2$, and let $\overline{x}$ be a common point of them. We say that $\overline{x}$ is a local extremal point for the system $\{S_1, ..., S_p\}$ if there exist the sequences $(a_{in}) \subset X$, $i = 1, ..., p$, and a neighborhood U of $\overline{x}$ such that $a_{in} \to 0$ for $n \to \infty$ and

$$\bigcap_{i=1}^{p} (S_i - a_{in}) \cap U = \emptyset \text{ for } n \text{ sufficiently large.} \tag{36}$$

In this case $\{S_1, ..., S_p, \overline{x}\}$ is called an extremal system in X.

Extremal principle holds iff the space is Asplund

- Let $\{S_1, ..., S_p, \overline{x}\}$ be an extremal system in X. Then $\{S_1, ..., S_p, \overline{x}\}$ satisfies the Approximate Extremal Principle (AEP) if for every $\varepsilon > 0$ there exist $x_i \in S_i \cap B[\overline{x}, \varepsilon]$ and $x_i^* \in \widehat{N}(S_i, x_i) + \varepsilon \mathbb{B}_{X^*}$, $i = 1, ..., p$, such that

$$x_1^* + ... + x_p^* = 0, \quad \|x_1^*\| + ... + \left\|x_p^*\right\| = 1. \tag{37}$$

holds.

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$$x_1^* + ... + x_p^* = 0, \quad \|x_1^*\| + ... + \left\|x_p^*\right\| = 1. \tag{37}$$

  holds.
- We say that the extremal principle holds in the space X if it holds for every extremal system $\{S_1, ..., S_p, \overline{x}\}$ in X, where all the sets S_i are locally closed at $\overline{x}$.

Extremal principle holds iff the space is Asplund

- Let $\{S_1, ..., S_p, \overline{x}\}$ be an extremal system in X. Then $\{S_1, ..., S_p, \overline{x}\}$ satisfies the Approximate Extremal Principle (AEP) if for every $\varepsilon > 0$ there exist $x_i \in S_i \cap B[\overline{x}, \varepsilon]$ and $x_i^* \in \widehat{N}(S_i, x_i) + \varepsilon \mathbb{B}_{X^*}$, $i = 1, ..., p$, such that

$$x_1^* + ... + x_p^* = 0, \quad \|x_1^*\| + ... + \left\|x_p^*\right\| = 1. \tag{37}$$

  holds.
- We say that the extremal principle holds in the space X if it holds for every extremal system $\{S_1, ..., S_p, \overline{x}\}$ in X, where all the sets S_i are locally closed at $\overline{x}$.

Theorem 13

Let X be a Banach space. Then X is Asplund iff the AEP holds in X.

Another characterization: approximate calculus rule for the Fréchet subdifferential

Theorem 14

Let X be a Banach space and $\overline{x} \in X$. Then X is Asplund iff for every $\varphi_1, \varphi_2 : X \to \mathbb{R} \cup \{\infty\}$ such that φ_1 is Lipschitz continuous around $\overline{x} \in \operatorname{dom} \varphi_1 \cap \operatorname{dom} \varphi_2$ and φ_2 is lsc around $\overline{x}$ and for every $\gamma > 0$, the next relation holds

$$\begin{aligned}\widehat{\partial}(\varphi_1 + \varphi_2)(\overline{x}) \subset \bigcup \{ & \widehat{\partial}\varphi_1(x_1) + \widehat{\partial}\varphi_2(x_2) \mid x_i \in B[\overline{x}, \gamma], \\ & |\varphi_i(x_i) - \varphi_i(\overline{x})| \leq \gamma,\ i = 1, 2\} + \gamma \mathbb{B}_{X^*}.\end{aligned} \tag{38}$$

Necessary conditions for directional regularity

Proposition 15

Let X, Y be normed vector spaces, and $L \subset S_X$, $M \subset S_Y$. Consider a multifunction $F : X \rightrightarrows Y$, and take $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$.
(i) If F is directionally Aubin continuous around $(\overline{x}, \overline{y})$ with respect to L and M with modulus $\ell > 0$, then there is $r > 0$ such that for every $w \in L$, every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, and every $(x^*, y^*) \in \operatorname{Gr} \widehat{D}^* F(x, y)$, there exists $u \in M$ such that

$$\langle -x^*, w\rangle \leq \ell \cdot |\langle y^*, u\rangle| . \tag{39}$$

(ii) If F is directionally linearly open around $(\overline{x}, \overline{y})$ with respect to L and M with modulus $c > 0$, then there exists $r > 0$, such that for every $u \in M$, every $y^* \in Y^*$, every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, and every $x^* \in \widehat{D}^* F(x, y)(y^*)$, there exists $w \in L$ such that

$$c \cdot \langle y^*, u\rangle \leq |\langle x^*, w\rangle| . \tag{40}$$

Sufficient conditions for directional regularity: finite dimensional spaces

Theorem 16

Let X, Y be finite dimensional spaces, and the closed sets $L \subset S_X, M \subset S_Y$, such that cone L *and* cone M *are convex. Consider a closed-graph multifunction $F : X \rightrightarrows Y$, and take $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that there exists $c > 0, r > 0$, such that for every $u \in M$ and every $y^* \in Y^*$ such that $\langle y^*, u \rangle = 1$, every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, every $x^* \in \widehat{D}^* F(x, y)(y^*)$, there exists $w \in L$ such that*

$$c \leq - \langle x^*, w \rangle . \tag{41}$$

Then for every $a \in (0, c)$, the multifunction F is directional linearly open around $(\overline{x}, \overline{y})$ with respect to L and M with modulus a.

Sufficient conditions for directional regularity: finite dimensional spaces

Idea of the proof.

- Choose appropriate $\varepsilon > 0$, $(\widetilde{x}, \widetilde{y}) \in \operatorname{Gr} F$ close to $(\overline{x}, \overline{y})$, take arbitrary $\rho \in (0, \varepsilon)$ and $v \in B(\widetilde{y}, \rho a) \cap [\widetilde{y} - \operatorname{cone} M]$.

Sufficient conditions for directional regularity: finite dimensional spaces

Idea of the proof.

- Choose appropriate $\varepsilon > 0$, $(\widetilde{x}, \widetilde{y}) \in \operatorname{Gr} F$ close to $(\overline{x}, \overline{y})$, take arbitrary $\rho \in (0, \varepsilon)$ and $v \in B(\widetilde{y}, \rho a) \cap [\widetilde{y} - \operatorname{cone} M]$.
- Apply directional Ekeland Variational Principle (i.e., Corollary 7) for $-L$ and M, and for the function $f : \operatorname{Gr} F \to \mathbb{R}$,

$$f(x, y) := T_M(v, y) = T_{-M}(y, v),$$

to get $(u_b, v_b) \in \operatorname{Gr} F$ with certain properties.

Sufficient conditions for directional regularity: finite dimensional spaces

Idea of the proof.

- Choose appropriate $\varepsilon > 0$, $(\widetilde{x}, \widetilde{y}) \in \operatorname{Gr} F$ close to $(\overline{x}, \overline{y})$, take arbitrary $\rho \in (0, \varepsilon)$ and $v \in B(\widetilde{y}, \rho a) \cap [\widetilde{y} - \operatorname{cone} M]$.
- Apply directional Ekeland Variational Principle (i.e., Corollary 7) for $-L$ and M, and for the function $f : \operatorname{Gr} F \to \mathbb{R}$,

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- Observe that if $v = v_b$, the conclusion holds.

Sufficient conditions for directional regularity: finite dimensional spaces

Idea of the proof.

- Choose appropriate $\varepsilon > 0$, $(\tilde{x}, \tilde{y}) \in \operatorname{Gr} F$ close to $(\bar{x}, \bar{y})$, take arbitrary $\rho \in (0, \varepsilon)$ and $v \in B(\tilde{y}, \rho a) \cap [\tilde{y} - \operatorname{cone} M]$.
- Apply directional Ekeland Variational Principle (i.e., Corollary 7) for $-L$ and M, and for the function $f : \operatorname{Gr} F \to \mathbb{R}$,

$$f(x, y) := T_M(v, y) = T_{-M}(y, v),$$

  to get $(u_b, v_b) \in \operatorname{Gr} F$ with certain properties.
- Observe that if $v = v_b$, the conclusion holds.
- Suppose $v \neq v_b$, and finalize proof, by way of contradiction, using the weak approximate calculus rule for the Fréchet subdifferential of the function

$$(x, y) \mapsto T_M(v, y) + b\,[T_{-L}(x, u_b) + T_M(y, v_b)] + \delta_{\operatorname{Gr} F}(x, y). \qquad \square$$

Sufficient conditions for directional regularity: infinite dimensions, epigraphical multifunction

- If $F : X \rightrightarrows Y$ is a multifunction between normed vector spaces, and $K \subset Y$ is a cone, we denote by $\widetilde{F}$ the epigraphical multifunction associated to F, i.e., $\widetilde{F}$: $X \rightrightarrows Y$,

$$\widetilde{F}(x) := F(x) + K, \forall x \in X.$$

Sufficient conditions for directional regularity: infinite dimensions, epigraphical multifunction

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$$\widetilde{F}(x) := F(x) + K, \forall x \in X. \tag{42}$$

- In this case, one may use that if $(\overline{x}, \overline{y}) \in \operatorname{Gr} \widetilde{F}$ and $\widehat{D}^* \widetilde{F}(\overline{x}, \overline{y})(y^*) \neq \emptyset$, then $y^* \in K^+$.

Sufficient conditions for directional regularity: infinite dimensions, epigraphical multifunction

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- In this case, one may use that if $(\overline{x}, \overline{y}) \in \operatorname{Gr} \widetilde{F}$ and $\widehat{D}^* \widetilde{F}(\overline{x}, \overline{y})(y^*) \neq \emptyset$, then $y^* \in K^+$.
- The main difficulty is that, one cannot apply the approximate sum rule for the Fréchet subdifferential, and the proof becomes much more involved.

Sufficient conditions for directional regularity: infinite dimensions, epigraphical multifunction

Theorem 17

Let X, Y be Asplund spaces, K be a closed convex cone with nonempty interior, $F : X \rightrightarrows Y$ be a multifunction such that $\widetilde{F}$ has closed graph, and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that there exist $r > 0$, $c > 0$ such that for every $u \in M := K \cap S_Y$, every $y^* \in Y^*$, every $(x, y) \in \operatorname{Gr} \widetilde{F} \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, and every $x^* \in \widehat{D}^* \widetilde{F}(x, y)(y^*)$,

$$\langle y^*, u \rangle \cdot c \leq \|x^*\| . \tag{43}$$

Then for every $a \in (0, c)$, the multifunction $\widetilde{F}$ is directionally linearly open around $(\overline{x}, \overline{y})$ with respect to $L := S_X$ and M with modulus a.

Sufficient conditions for directional regularity: proof

Proof.

- Take $a \in (0, c)$, $b \in \left(\frac{a}{a+1}, \frac{c}{c+1}\right)$ and $\tau > 0$ such that

$$\frac{a}{a+1} < b + \tau < \frac{c}{c+1}, \qquad (44)$$
$$b^{-1} a \tau < 2^{-1} r.$$

- Choose $(\widetilde{x}, \widetilde{y}) \in \operatorname{Gr} \widetilde{F} \cap \left[B(\overline{x}, 2^{-1}r) \times B(\overline{y}, 2^{-1}r)\right]$. We will prove that for every $\rho \in (0, \tau)$, one has

$$B(\widetilde{y}, \rho a) \cap [\widetilde{y} - \operatorname{cone} M] \subset \widetilde{F}(B(\widetilde{x}, \rho)). \qquad (45)$$

- Take $\rho \in (0, \tau)$ and $v \in B(\widetilde{y}, \rho a) \cap [\widetilde{y} - \operatorname{cone} M]$. Consider the function $f : \operatorname{Gr} \widetilde{F} \to \mathbb{R}$,

$$f(x, y) := T_M(v, y) = T_{-M}(y, v).$$

Sufficient conditions for directional regularity: proof

- Remark that $(\widetilde{x}, \widetilde{y}) \in \operatorname{dom} f$. Apply to f the (classical) Ekeland Variational Principle to get $(u_b, v_b) \in \operatorname{Gr} \widetilde{F}$ such that

$$T_M(v, v_b) \leq T_M(v, \widetilde{y}) - b\left[\|u_b - \widetilde{x}\| + \|v_b - \widetilde{y}\|\right] \tag{46}$$

and

$$T_M(v, v_b) \leq T_M(v, y) + b\left[\|x - u_b\| + \|y - v_b\|\right], \quad \forall (x, y) \in \operatorname{Gr} \widetilde{F}. \tag{47}$$

- Since
$$T_M(v, \widetilde{y}) = \|v - \widetilde{y}\|,$$
we have from (46) that $T_M(v, v_b)$ is finite and, consequently, $v \in v_b - \operatorname{cone} M$ and $T_M(v, v_b) = \|v - v_b\|$.
- Show that $(u_b, v_b) \in \operatorname{Gr} \widetilde{F} \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$.

Sufficient conditions for directional regularity: proof

- If $v_b = v$, then

$$b \|\widetilde{x} - u_b\| \leq (1-b) \|\widetilde{y} - v\| < (1-b)a\rho < b\rho,$$

  hence $u_b \in B(\widetilde{x}, \rho)$ and $v \in F(B(\widetilde{x}, \rho))$, which is exactly the conclusion.
- Next, prove that $v_b = v$ is the only possibility. For this, suppose that $v \neq v_b$ and consider the function

$$h : X \times Y \to \mathbb{R} \cup \{\infty\}, \quad h(x,y) := T_M(v,y) + b\left[\|x - u_b\| + \|y - v_b\|\right].$$

  From (47), we have that the pair (u_b, v_b) is a minimum point for h on $\operatorname{Gr} \widetilde{F}$.

Sufficient conditions for directional regularity: proof

- We consider the extremal system $(\Omega_1, \Omega_2, (u_b, v_b, h(u_b, v_b)))$, where

$$\Omega_1 := \operatorname{epi} h,\ \Omega_2 := \operatorname{Gr} \widetilde{F} \times \{h(u_b, v_b)\},$$

and apply the Approximate Extremal Principle to get, for any $\varepsilon > 0$, the existence of $(x_i, y_i, \alpha_i) \in \Omega_i$, $i = 1, 2$, $(x_1^*, y_1^*, -\lambda_1) \in \widehat{N}(\operatorname{epi} h, (x_1, y_1, \alpha_1))$, $(-x_2^*, -y_2^*, \lambda_2) \in \widehat{N}(\operatorname{Gr} \widetilde{F}, (x_2, y_2)) \times \mathbb{R}$ such that

$$\|(x_i, y_i, \alpha_i) - (u_b, v_b, h(u_b, v_b))\| < \varepsilon, \quad i = 1, 2, \tag{48}$$

$$1 - \varepsilon \le \|(x_i^*, y_i^*, -\lambda_i)\| \le 1 + \varepsilon, \quad i = 1, 2, \tag{49}$$

$$\|(x_1^*, y_1^*, -\lambda_1) + (-x_2^*, -y_2^*, \lambda_2)\| \le \varepsilon. \tag{50}$$

- Observe that $\alpha_2 = \|v - v_b\| > 0$. Since $(x_1^*, y_1^*, -\lambda_1) \in \widehat{N}(\operatorname{epi} h, (x_1, y_1, \alpha_1))$, for any $\gamma > 0$, there exists $\delta > 0$ such that, for any $(x, y, \alpha) \in \operatorname{epi} h \cap B((x_1, y_1, \alpha_1), \delta)$,

$$\langle x_1^*, x - x_1 \rangle + \langle y_1^*, y - y_1 \rangle - \lambda_1(\alpha - \alpha_1) \le \gamma \left(\|x - x_1\| + \|y - y_1\| + |\alpha - \alpha_1| \right). \tag{51}$$

Sufficient conditions for directional regularity: proof

- Because $(x_1, y_1, \alpha_1) \in \operatorname{epi} h$, it means that $(x_1, y_1) \in \operatorname{dom} h$, i.e., $y_1 \in v + \operatorname{cone} M$. Moreover, there is $\theta \geq 0$ such that $\alpha_1 = h(x_1, y_1) + \theta$.
- We want to prove that $\lambda_1 > 0$. Take $y := y_1$, $x \in B(x_1, \min\{2^{-1}\delta, 2^{-1}b^{-1}\delta\})$, and $\alpha := h(x, y_1) + \theta$. Then

$$|\alpha - \alpha_1| = |h(x, y_1) - h(x_1, y_1)| \leq b\,\|x - x_1\| < \frac{\delta}{2},$$

hence

$$\|(x, y_1, \alpha) - (x_1, y_1, \alpha_1)\| = \|x - x_1\| + |\alpha - \alpha_1| < \delta.$$

It means that $(x, y_1, \alpha) \in \operatorname{epi} h \cap B((x_1, y_1, \alpha_1), \delta)$, hence (51) holds.

Sufficient conditions for directional regularity: proof

- Therefore, for any $x \in B(x_1, \min\{2^{-1}\delta, 2^{-1}b^{-1}\delta\})$, we have

$$\langle x_1^*, x - x_1 \rangle - \lambda_1(\alpha - \alpha_1) \leq \gamma \left(\|x - x_1\| + |\alpha - \alpha_1| \right),$$
$$\langle x_1^*, x - x_1 \rangle \leq \lambda_1(h(x, y_1) - h(x_1, y_1)) + \gamma \left(\|x - x_1\| + |h(x, y_1) - h(x_1, y_1)| \right)$$
$$\leq (\lambda_1 b + \gamma + \gamma b) \|x - x_1\|.$$

- It follows that $\|x_1^*\| \leq \lambda_1 b + \gamma + \gamma b$ for any $\gamma > 0$ and, therefore, $\|x_1^*\| \leq \lambda_1 b$. We know hence that $\lambda_1 \geq 0$. Suppose, by contradiction, that $\lambda_1 = 0$. Then $x_1^* = 0$, hence, by (49),

$$1 - \varepsilon \leq \|(x_1^*, y_1^*, -\lambda_1)\| = \|y_1^*\| \leq 1 + \varepsilon.$$

But since $\|y_1^* - y_2^*\| \leq \varepsilon$ from (50), it follows that

$$\|y_2^*\| \geq \|y_1^*\| - \|y_1^* - y_2^*\| \geq 1 - 2\varepsilon.$$

Also, since $x_1^* = 0$, we also have from (50) that $\|x_2^*\| \leq \varepsilon$.

Sufficient conditions for directional regularity: proof

- Recall that, since $(-x_2^*, -y_2^*) \in \widehat{N}(\operatorname{Gr} \widetilde{F}, (x_2, y_2))$, i.e., $-x_2^* \in \widehat{D}^* \widetilde{F}(x_2, y_2)(y_2^*)$, we have that $y_2^* \in K^+$. Also, since $(u_b, v_b) \in \operatorname{Gr} \widetilde{F} \cap [B(\bar{x}, r) \times B(\bar{y}, r)]$, we may suppose, using (48), that the ε for which the Approximate Extremal Principle was applied is sufficiently small such that $(x_2, y_2) \in \operatorname{Gr} \widetilde{F} \cap [B(\bar{x}, r) \times B(\bar{y}, r)]$. Also, since $\operatorname{int} K \neq \emptyset$, we choose ε small enough such that there exists $u_0 \in M$ for which $B[u_0, \sqrt{\varepsilon}] \subset K$.
- Using the assumption made, we have that for every $\zeta \in \mathbb{B}_Y$, $\dfrac{u_0 + \sqrt{\varepsilon}\zeta}{\|u_0 + \sqrt{\varepsilon}\zeta\|} \in M$, hence

$$c^{-1}\varepsilon \geq c^{-1} \|x_2^*\| \geq \left\langle y_2^*, \frac{u_0 + \sqrt{\varepsilon}\zeta}{\|u_0 + \sqrt{\varepsilon}\zeta\|} \right\rangle \geq \frac{\sqrt{\varepsilon}\,\langle y_2^*, \zeta \rangle}{1 + \sqrt{\varepsilon}}, \quad \forall \zeta \in \mathbb{B}_Y.$$

It follows that

$$c^{-1}\varepsilon \geq \frac{\sqrt{\varepsilon}\,\|y_2^*\|}{1 + \sqrt{\varepsilon}} \geq \frac{\sqrt{\varepsilon}(1 - 2\varepsilon)}{1 + \sqrt{\varepsilon}}.$$

Sufficient conditions for directional regularity: proof

- Hence,

$$\sqrt{\varepsilon} \geq c\frac{(1-2\varepsilon)}{1+\sqrt{\varepsilon}}$$

  for any ε sufficiently small, a contradiction. It follows that $\lambda_1 > 0$.
- We prove next that $\alpha_1 = h(x_1, y_1)$. Suppose, by contradiction, that $\alpha_1 > h(x_1, y_1)$. From (51) applied for $\gamma \in (0, \lambda_1)$, $(x, y) := (x_1, y_1)$ and $\alpha \in (h(x_1, y_1), \alpha_1)$ arbitrarily close to α_1, we get that

$$\lambda_1(\alpha_1 - \alpha) < \gamma(\alpha_1 - \alpha),$$

  which give us the contradiction $\lambda_1 < \gamma$.

Sufficient conditions for directional regularity: proof

- In conclusion, by denoting $(x_0^*, y_0^*) := \frac{1}{\lambda_1}(x_1^*, y_1^*)$, we have that $(x_0^*, y_0^*) \in \widehat{\partial} h(x_1, y_1)$. Moreover, since h is the sum of three convex functions (notice that, in our case, due to [2, Proposition 3.1, (i)], $T_M(v, \cdot)$ is convex), two of which are Lipschitz, it follows that its Fréchet subdifferential coincides with the Fenchel subdifferential (denoted by ∂), whence

$$\begin{aligned}\partial h(x_1, y_1) &\subset \{0\} \times \partial T_{-M}(\cdot, v)(y_1) + b(\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}) \\ &\subset \{0\} \times \{y^* \in Y^* \mid \langle y^*, y_1 - v\rangle = \|y_1 - v\|\} + b(\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}).\end{aligned}$$

- We deduce that $x_0^* \in b\mathbb{B}_{X^*}$. Moreover (for y_1 sufficiently close to v_b, hence different from v),

$$\frac{y_1 - v}{\|y_1 - v\|} =: \overline{u} \in M$$

and there exists $y_3^* \in \mathbb{B}_{Y^*}$ such that $\langle y_0^* + by_3^*, \overline{u}\rangle = 1$, hence

$$\langle y_0^*, \overline{u}\rangle = 1 - b\,\langle y_3^*, \overline{u}\rangle \geq 1 - b.$$

Sufficient conditions for directional regularity: proof

- Furthermore, using (50), it follows that there is $(x_4^*, y_4^*) \in \mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}$ such that

$$-(x_0^*, y_0^*) - \frac{\varepsilon}{\lambda_1}(x_4^*, y_4^*) = -\frac{1}{\lambda_1}(x_2^*, y_2^*) \in \widehat{N}(\operatorname{Gr} \widetilde{F}, (x_2, y_2)),$$

  hence

$$-x_0^* - \frac{\varepsilon}{\lambda_1} x_4^* \in \widehat{D}^* \widetilde{F}(x_2, y_2) \left(y_0^* + \frac{\varepsilon}{\lambda_1} y_4^* \right).$$

- Using the assumption made in the formulation of the theorem, it follows that

$$c\left(1 - b - \frac{\varepsilon}{\lambda_1}\right) \le c \left\langle y_0^* + \frac{\varepsilon}{\lambda_1} y_4^*, \overline{u} \right\rangle \le \left\| -x_0^* - \frac{\varepsilon}{\lambda_1} x_4^* \right\| \le b + \frac{\varepsilon}{\lambda_1}.$$

  This will obviously provide us a contradiction, if we show that we can choose ε in such a way such that $\frac{\varepsilon}{\lambda_1} < \tau$, where τ is such that (44) is satisfied.

Sufficient conditions for directional regularity: proof

- Observe that, in view of the fact $\frac{1}{\lambda_1}(x_1^*, y_1^*) \in \widehat{\partial} h(x_1, y_1)$, we have that for any $\gamma > 0$, there exists $\delta > 0$ such that, for any $y \in B(y_1, \delta)$,

$$\left\langle \frac{y_1^*}{\lambda_1}, y - y_1 \right\rangle \le \gamma \left\| y - y_1 \right\| + h(x_1, y) - h(x_1, y_1),$$

hence, in particular, for any $y \in B(y_1, \delta) \cap [y_1 + \operatorname{cone} M]$,

$$\left\langle \frac{y_1^*}{\lambda_1}, y - y_1 \right\rangle \le (1 + b + \gamma) \left\| y - y_1 \right\|,$$

and

$$\langle y_1^*, u \rangle \le \lambda_1 (1 + b), \quad \forall u \in M. \tag{52}$$

- Using also (50), we find that

$$\langle y_2^*, u \rangle = \langle y_2^* - y_1^*, u \rangle + \langle y_1^*, u \rangle \le \varepsilon + \lambda_1 (1 + b), \quad \forall u \in M.$$

Moreover, by a similar argument, we obtain that

$$\| x_1^* \| \le \lambda_1 b < \lambda_1.$$

Sufficient conditions for directional regularity: proof

- Remark that we may suppose, without loosing the generality, that the $\varepsilon > 0$ for which the Approximate Extremal Principle was applied is sufficiently small such that

$$D(u_0, \sqrt{\varepsilon}) \subset K \text{ and}$$
$$\max\left\{(1-\sqrt{\varepsilon}-3\varepsilon)^{-1}(1+b)\left(1+\sqrt{\varepsilon}\right)\sqrt{\varepsilon}, (1-\varepsilon)^{-1}\varepsilon\right\} < \tau.$$

- Suppose first that $\left\|y_1^*\right\| \leq \lambda_1$. Hence, by (49), we have (since the dual of the sum norm is the max norm) that

$$1-\varepsilon \leq \left\|(x_1^*, y_1^*, -\lambda_1)\right\| = \max\left\{\left\|x_1^*\right\|, \left\|y_1^*\right\|, \lambda_1\right\} = \lambda_1,$$

and then

$$\frac{\varepsilon}{\lambda_1} \leq \frac{\varepsilon}{1-\varepsilon} < \tau.$$

Sufficient conditions for directional regularity: proof

- Suppose now $\|y_1^*\| > \lambda_1$, hence by (49) we have that $\|y_1^*\| \geq 1 - \varepsilon$, from which we deduce (as above) that $\|y_2^*\| \geq 1 - 2\varepsilon$. Then, for arbitrary $\zeta \in \mathbb{B}_Y$, denote

$$z := \frac{u_0 + \sqrt{\varepsilon}\zeta}{\|u_0 + \sqrt{\varepsilon}\zeta\|} \in M \tag{53}$$

and observe (using that $\langle y_2^*, u_0 \rangle \geq 0$) that

$$\langle y_2^*, z \rangle = \frac{\langle y_2^*, u_0 + \sqrt{\varepsilon}\zeta \rangle}{\|u_0 + \sqrt{\varepsilon}\zeta\|} \geq \frac{\langle y_2^*, \sqrt{\varepsilon}\zeta \rangle}{\|u_0 + \sqrt{\varepsilon}\zeta\|} \geq \frac{\sqrt{\varepsilon}\,\langle y_2^*, \zeta \rangle}{1 + \sqrt{\varepsilon}}.$$

- Since ζ was arbitrarily chosen from $\mathbb{B}_Y$, it follows, using also (52), that

$$\lambda_1(1+b) + \varepsilon \geq \frac{\sqrt{\varepsilon}\,\|y_2^*\|}{1+\sqrt{\varepsilon}} \geq \frac{\sqrt{\varepsilon}\,(1-2\varepsilon)}{1+\sqrt{\varepsilon}},$$

and hence

$$\frac{\varepsilon}{\lambda_1} \leq \frac{(1+b)\left(1+\sqrt{\varepsilon}\right)}{(1-\sqrt{\varepsilon}-3\varepsilon)}\sqrt{\varepsilon} < \tau,$$

as needed. □

Sufficient conditions for directional regularity: general case

Make use of Fréchet $\varepsilon-$subdifferential and the $\varepsilon-$support of a function at a point and two results of Fabian to get the following statement.

Theorem 18

Let X, Y be Asplund spaces, $M \subset S_Y$ be a closed set, $F : X \rightrightarrows Y$ be a closed graph multifunction, and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that there exist $r > 0, c > 0$ such that for every $u \in M$, every $y^* \in Y^*$, every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, and every $x^* \in \widehat{D}^* F(x, y)(y^*)$,

$$\langle y^*, u \rangle \cdot c \leq \|x^*\| . \tag{54}$$

Then for every $a \in (0, c)$, the multifunction F is directionally linearly open around $(\overline{x}, \overline{y})$ with respect to $L := S_X$ and M with modulus a, provided that one of the following assumptions hold:
(i) $\operatorname{int} \operatorname{cone} M \cup \{0\}$ contains lines.
(ii) M is a compact set and either X is finite dimensional or F is proper.

References

- J.P. Aubin, H. Frankowska, *Set-valued Analysis*, Birkäuser, Basel, 1990.
- M. Bounkhel, *Directional Lipschitzness of minimal time functions in Hausdorff topological vector spaces*, Set-Valued and Variational Analysis 22 (2014), 221–245.
- M. Bounkhel, *On subdifferentials of a minimal time function in Hausdorff topological vector spaces*, Applicable Analysis 93 (2014), 1761–1791.
- R. Cibulka, M. Durea, M. Panţiruc, R. Strugariu, *On the stability of the directional regularity*, submitted.
- G. Colombo, P.R. Wolenski, *The subgradient formula for the minimal time function in the case of constant dynamics in Hilbert space*, Journal of Global Optimization 28 (2004), 269–282.
- G. Colombo, P.R. Wolenski, *Variational analysis for a class of minimal time functions in Hilbert spaces*, Journal of Convex Analysis 11 (2004), 335–361.

References

- A.V. Dmitruk, A.Y. Kruger, *Extensions of metric regularity*, Optimization, 58 (2009), 561–584
- M. Durea, M. Panţiruc, R. Strugariu, *Minimal time function with respect to a set of directions: basic properties and applications*, Optimization Methods and Software, 31 (2016), 535-561.
- M. Durea, M. Panţiruc, R. Strugariu, *A new type of directional regularity for mappings and applications to optimization*, SIAM Journal on Optimization, 27 (2017), 1204–1229.
- M. Durea, R. Strugariu, *On some Fermat rules for set-valued optimization problems*, Optimization, 60 (2011), 575–591.
- M. Fabian, *Subdifferentials, local ε-supports and Asplund spaces*, Journal of the London Mathematical Society, 34 (1986), 568–576.

References

- M. Fabian, *Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss*, Acta Universitatis Carolinae – Mathematica et Physica, 30 (1989), 51–56.
- V.N. Huynh, M. Théra, *Directional metric regularity of multifunctions*, Mathematics of Operations Research, 40 (2015), 969–991.
- A.D. Ioffe, *On subdifferentiability spaces*, Annals of the New York Academy of Sciences, 410 (1983), 107–119.
- B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation*, Vol. I: Basic Theory, Vol. II: Applications, Springer, Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), Vol. 330 and 331, Berlin, 2006.
- N.M. Nam, C. Zălinescu, *Variational analysis of directional minimal time functions and applications to location problems*, Set-Valued and Variational Analysis, 21 (2013), 405–430.

Thank you!