Liouville type results for a nonlocal obstacle problem

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Introduction

How to describe the evolution of a population in an environment that possess an inaccessible region ?

$$\begin{aligned} \textit{Environment:}\ & \mathbb{R}^N \setminus K, \ \text{with } K \subset \mathbb{R}^N \text{ compact}, \\ \textit{Density:}\ & u(t,x), \\ \textit{Demographic rate:}\ & f(u(t,x)) \ \text{(non-linear reaction term)}, \\ \textit{Random motion:}\ & \mathcal{D}[u](t,x). \end{aligned}$$

Leads to reaction/diffusion equations of the type:

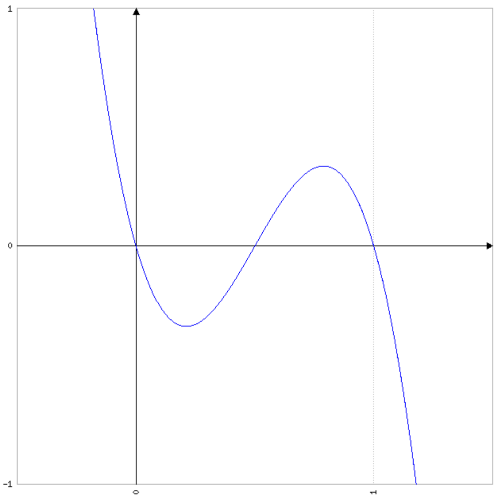
$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{D}[u] + f(u) & \text{in } \mathbb{R} \times \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial K, \end{cases} \tag{1}$$

where ν is the outward unit vector normal to K.

Here, we assume that f is of *bistable* type, that is,

$$\begin{cases} \exists\, \theta \in (0,1), \quad f(0) = f(\theta) = f(1) = 0, \\ f < 0 \text{ in } (0,\theta), \quad f > 0 \text{ in } (\theta, 1), \\ f'(0) < 0, \quad f'(\theta) > 0, \quad f'(1) < 0. \end{cases} \tag{2}$$

The prototypical example is:



$$f(u) := \lambda \overbrace{u(1-u)}^{\text{logistic growth}} \underbrace{(u-\theta)}_{\text{Allee effect}} .$$

λ : growth rate;

Logistic growth: proportional to both the existing population and the resources;

Allee effect: correlation between population density and the mean individual fitness (per capita population growth rate).

The case of Brownian motion: $\mathcal{D}[u] = \Delta u = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2}$.

The equation then becomes

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{R} \times \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial K. \end{cases} \tag{3}$$

Berestycki, Hamel, Matano (2009): $\exists u(t,x)$ with $0 < u(t,x) < 1$ and a classical solution u_∞ to

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ \nabla u_\infty \cdot \nu = 0 & \text{on } \partial K, \\ 0 \leqslant u_\infty \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u_\infty(x) \to 1 & \text{as } |x| \to +\infty, \end{cases} \tag{4}$$

with

$$u(t,x) \underset{t \to +\infty}{\longrightarrow} u_\infty(x), \text{ locally uniformly in } x \in \overline{\mathbb{R}^N \setminus K}.$$

In fact:

$$\boxed{\lim_{t\to-\infty} |u(t,x) - \phi(x_1 + ct)| = 0 \quad \text{unif. in } x,}$$

where

1. $x_1 = x \cdot e_1$ and $e_1 = (1, 0, \cdots, 0)$;
2. (c, ϕ) is the unique (up to shift) increasing solution to:

$$\begin{cases} c\phi' = \phi'' + f(\phi) \ \text{ in } \mathbb{R}, \\ \phi(-\infty) = 0, \ \phi(\infty) = 1. \end{cases} \tag{5}$$

Moreover:

$$\boxed{\lim_{t\to\infty} |u(t,x) - u_\infty(x)\phi(x_1 + ct)| = 0 \quad \text{unif. in } x.}$$

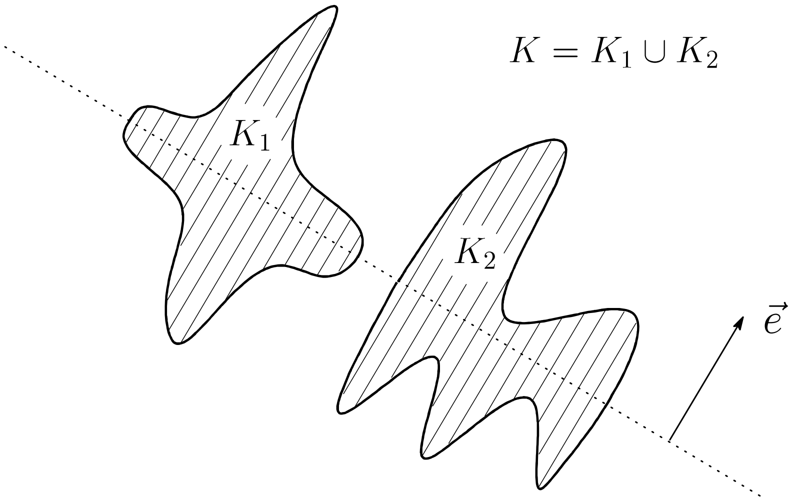
$\rightsquigarrow$ This result is *independent* of the geometry of K.

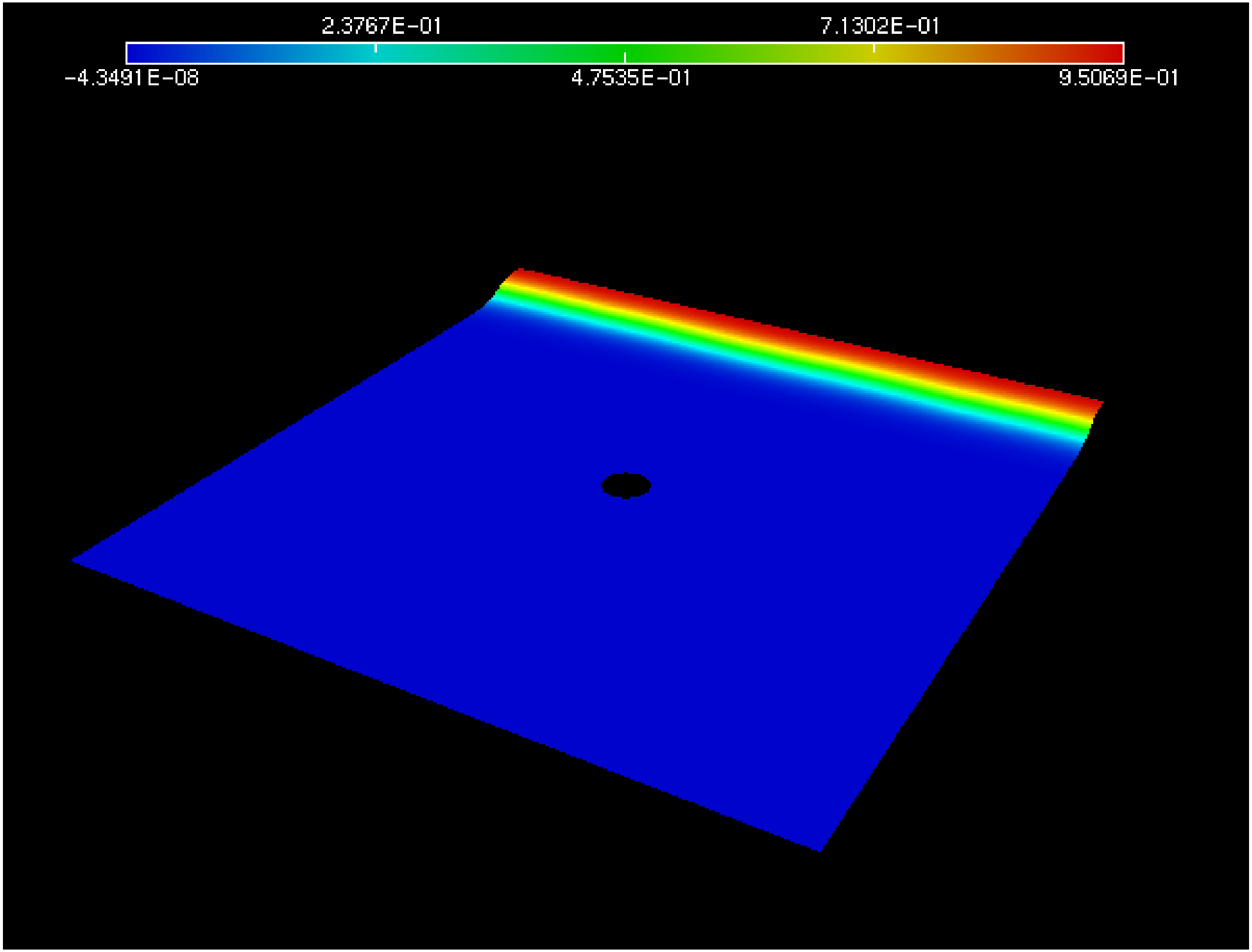
$\rightsquigarrow$ The influence of the geometry of K *is encoded in* u_∞.

Theorem (Berestycki-Hamel-Matano; Liouville-type property)

Let $K \subset \mathbb{R}^N$ be either starshaped or directionally convex. Then, the unique solution $u_\infty(\cdot)$ to (4) is

$$u_\infty \equiv 1 \text{ in } \overline{\mathbb{R}^N \setminus K}.$$

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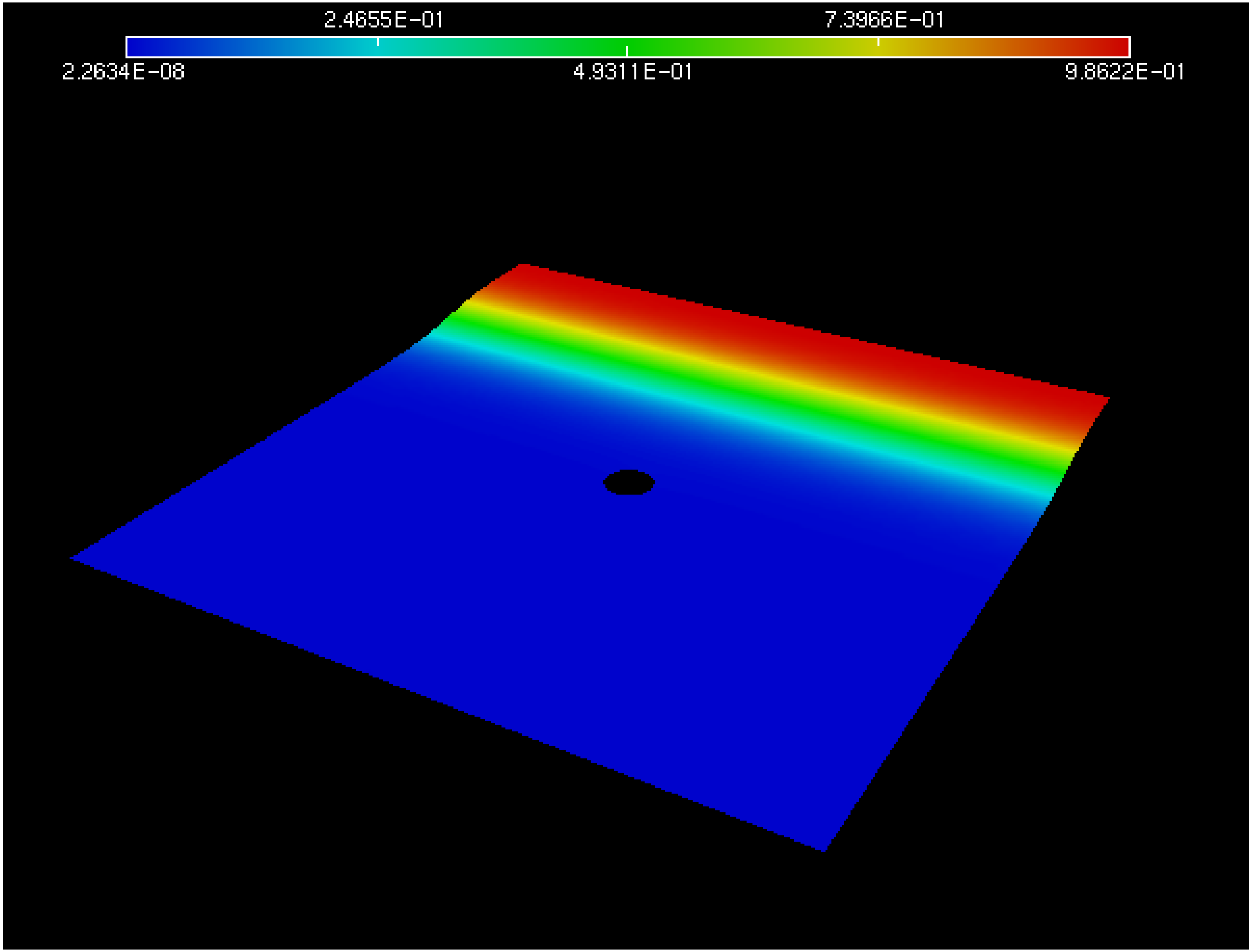
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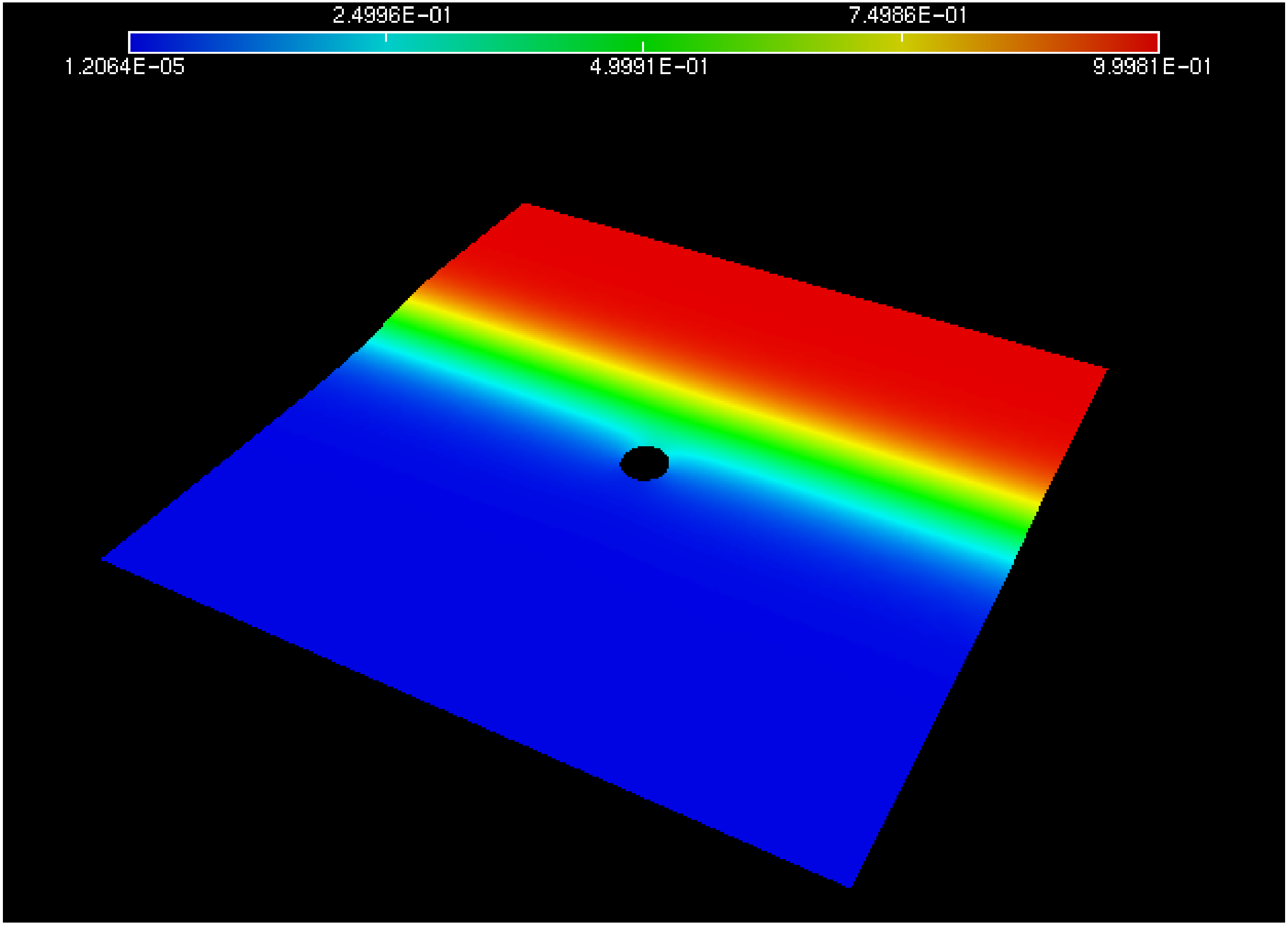
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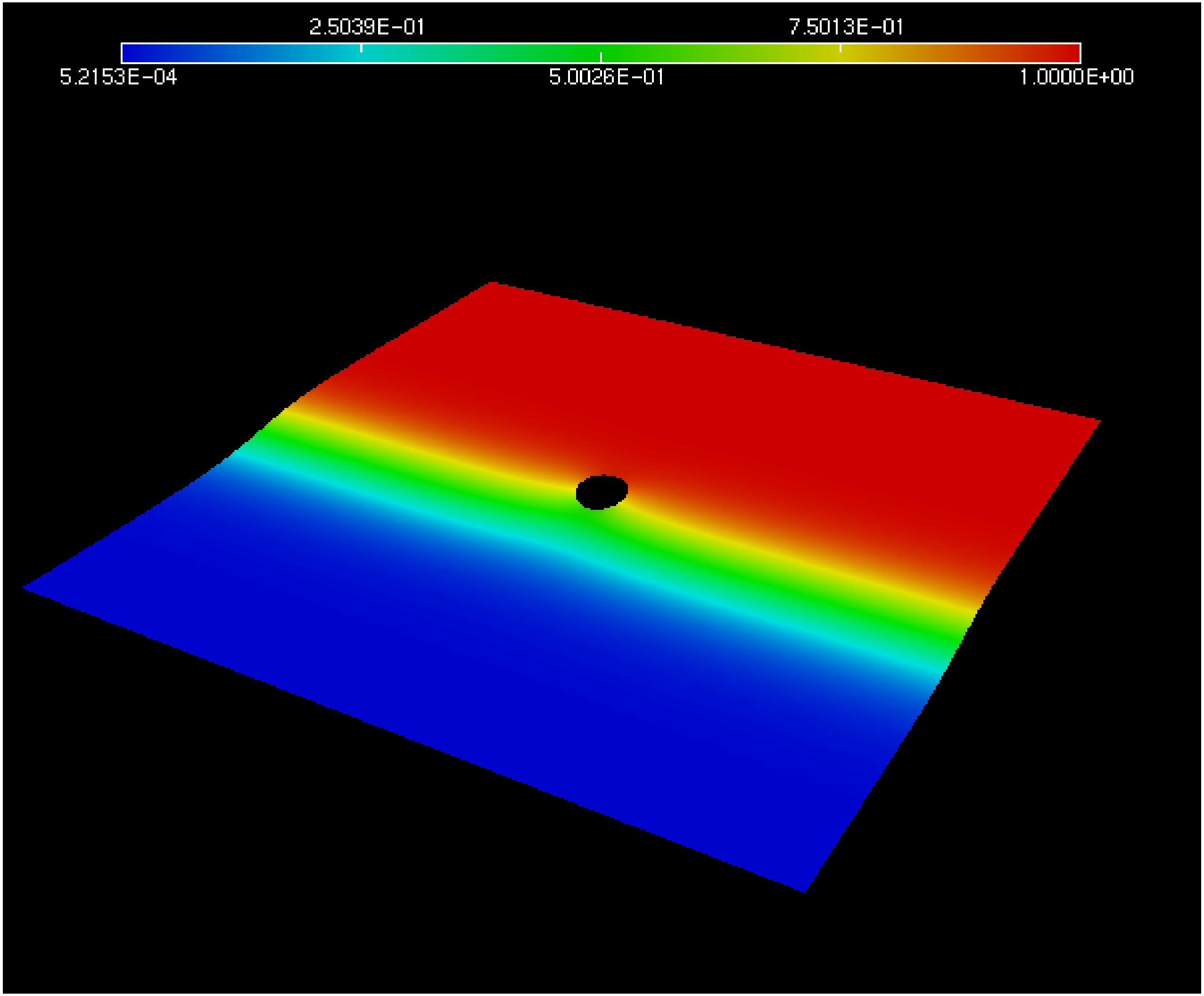
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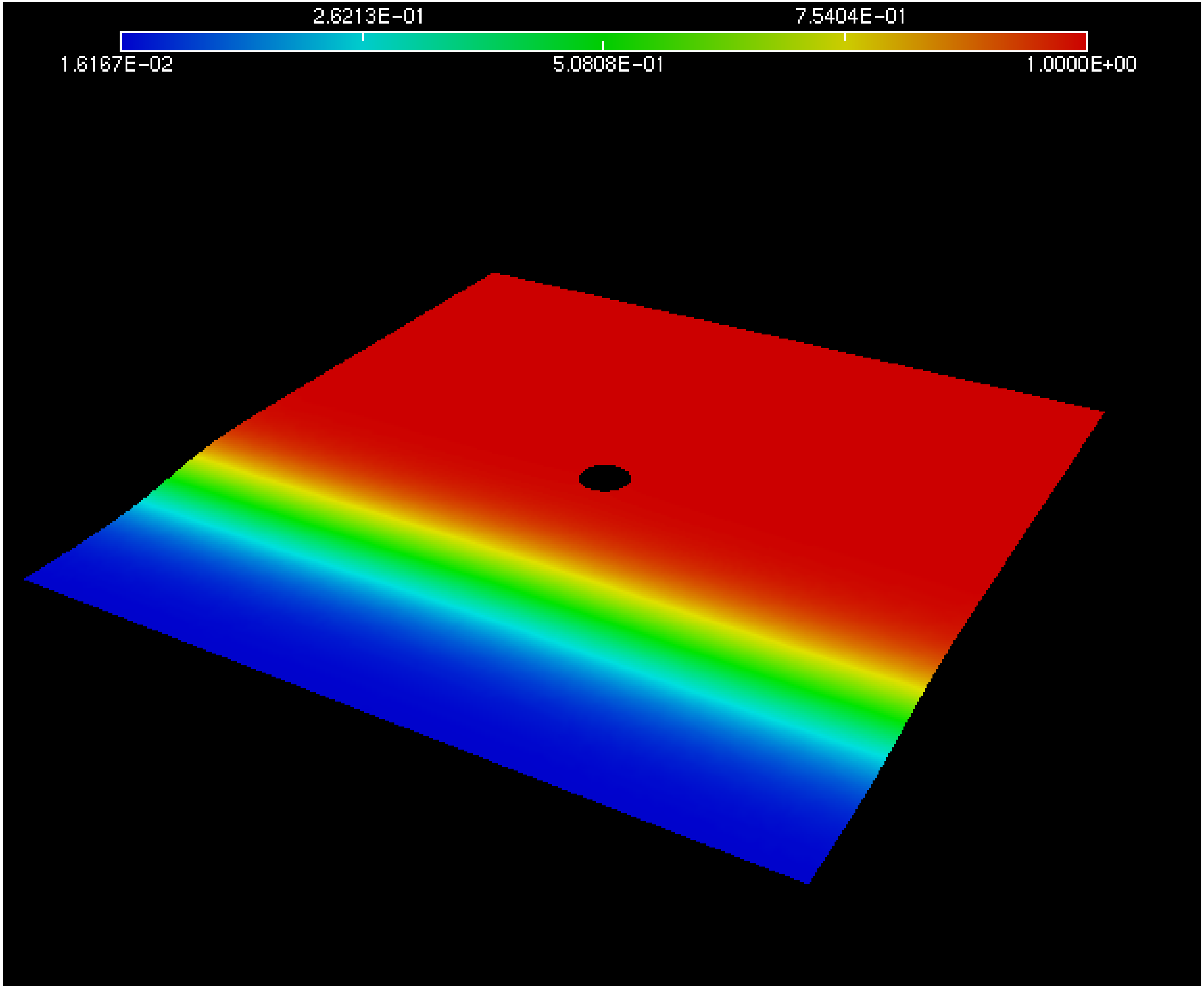
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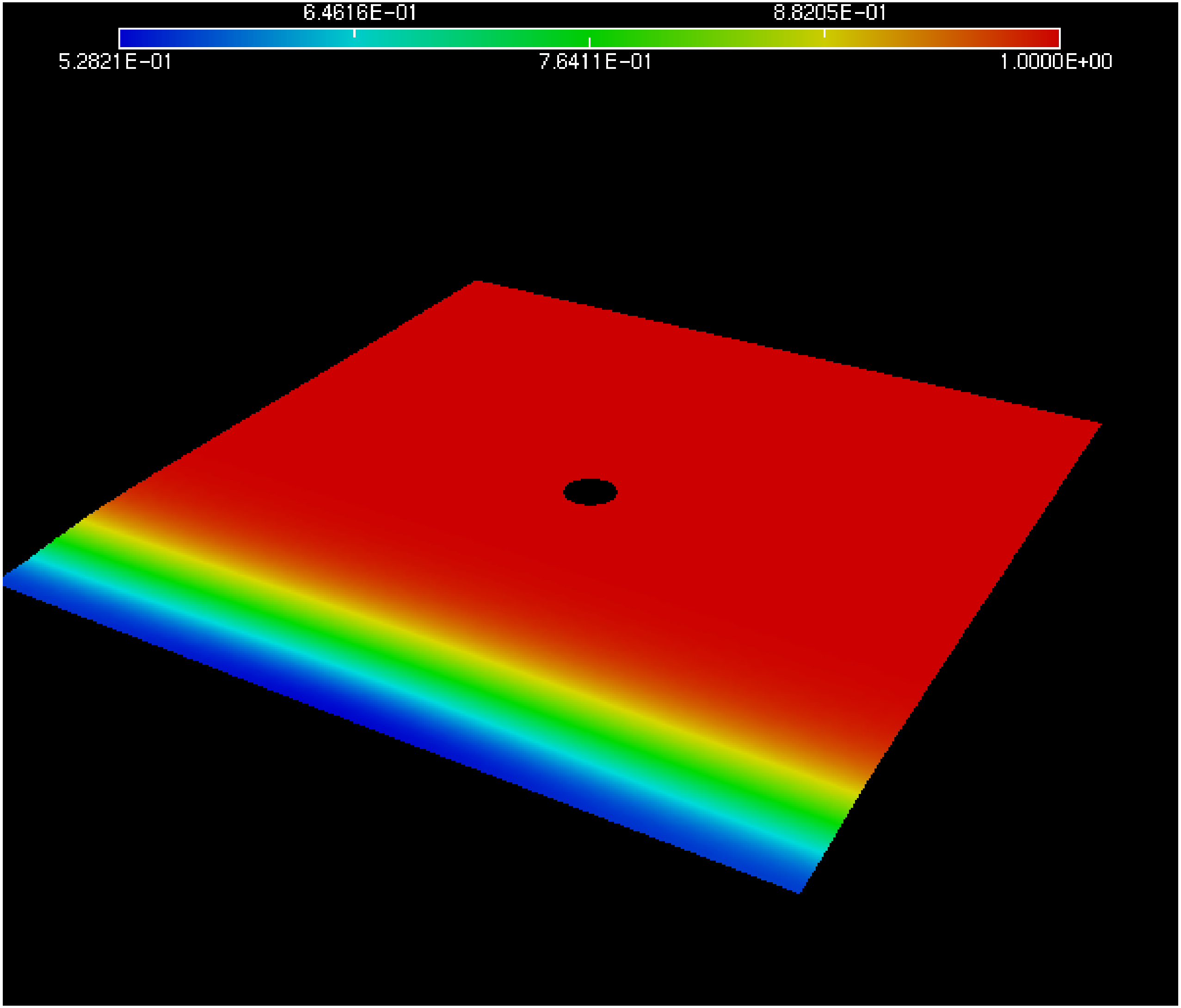
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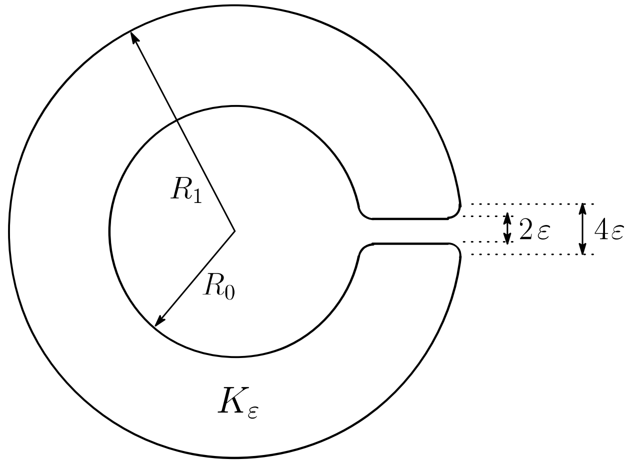
However, the geometric assumptions here ***cannot*** be dropped.

Theorem (Berestycki-Hamel-Matano)

There exist smooth compact simply connected obstacles $K \subset \mathbb{R}^N$ *for which problem (4) admits a solution* $u_\infty(\cdot)$ *satisfying*

$$0 < u_\infty(x) < 1 \text{ for any } x \in \overline{\mathbb{R}^N \setminus K}.$$



What about other random motions?

Of special interest to us are *compound Poisson processes*. The diffusion phenomena are then better described by a *convolution-type* operator such as:

$$Lu(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)(u(y)-u(x))\,\mathrm{d}y.$$

One is lead to:

$$\frac{\partial u}{\partial t} = Lu + f(u) \text{ in } \mathbb{R} \times \mathbb{R}^N \setminus K.$$

⇝ In this talk, we will be (mainly) concerned with:

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ 0 \leqslant u \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty, \end{cases} \tag{6}$$

Remark: *No boundary conditions are required!*

$$L_\varepsilon u(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus K} \widetilde{J}_\varepsilon(x-y)(u(y)-u(x))\,\mathrm{d}y,$$

with $x \in \partial K$, $\widetilde{J}_\varepsilon(z) = \varepsilon^{-N}\widetilde{J}(\varepsilon^{-1}z)$, $\widetilde{J}$ radially symmetric kernel.

Gives:

$$-f(u(x)) = L_\varepsilon u(x)$$

Remark: *No boundary conditions are required!*

$$L_\varepsilon u(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus K} \widetilde{J}_\varepsilon(x-y)(u(y)-u(x))\,\mathrm{d}y,$$

with $x \in \partial K$, $\widetilde{J}_\varepsilon(z) = \varepsilon^{-N} \widetilde{J}(\varepsilon^{-1} z)$, $\widetilde{J}$ radially symmetric kernel.

Gives:

$$-\varepsilon\, f(u(x)) = \frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus K} \widetilde{J}_\varepsilon(x-y)\, \nabla u(c_{x,y}) \cdot (y-x)\,\mathrm{d}y$$

Remark: *No boundary conditions are required!*

$$L_\varepsilon u(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus K} \widetilde{J}_\varepsilon(x-y)(u(y)-u(x))\,\mathrm{d}y,$$

with $x \in \partial K$, $\widetilde{J}_\varepsilon(z) = \varepsilon^{-N} \widetilde{J}(\varepsilon^{-1} z)$, $\widetilde{J}$ radially symmetric kernel.

Gives:

$$-\varepsilon\, f(u(x)) = \int_{\mathbb{R}^N \setminus K} \left(\frac{|x-y|}{\varepsilon} \widetilde{J}_\varepsilon(x-y) \right) \nabla u(c_{x,y}) \cdot \frac{(y-x)}{|x-y|}\,\mathrm{d}y$$

Remark: *No boundary conditions are required!*

$$L_\varepsilon u(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus K} \widetilde{J}_\varepsilon(x-y)(u(y)-u(x))\,\mathrm{d}y,$$

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Gives:

$$-\varepsilon\, f(u(x)) = \int_{\mathbb{R}^N \setminus K} \left(\frac{|x-y|}{\varepsilon} \widetilde{J}_\varepsilon(x-y) \right) \nabla u(c_{x,y}) \cdot \frac{(y-x)}{|x-y|}\,\mathrm{d}y$$

$$\to 0 \qquad\qquad \to \gamma\, \nabla u(x) \cdot \nu$$

II. Notations & Assumptions

Notations & Assumptions

$$\begin{cases} \exists\, \theta \in (0,1), \quad f(0) = f(\theta) = f(1) = 0, \\ f < 0 \text{ in } (0,\theta), \quad f > 0 \text{ in } (\theta,1), \\ f'(0) < 0, \quad f'(\theta) > 0, \quad f'(1) < 0. \end{cases}$$

Moreover, $J \in L^1(\mathbb{R}^N)$ is a non-negative, radially symmetric kernel with unit mass and there exists a function $\phi \in C(\mathbb{R})$ satisfying

$$\begin{cases} J_1 * \phi - \phi + f(\phi) \geqslant 0 \text{ in } \mathbb{R}, \\ \phi \text{ is increasing in } \mathbb{R}, \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \end{cases} \tag{7}$$

where $J_1 \in L^1(\mathbb{R})$ is the non-negative even function with unit mass given for a.e. $x \in \mathbb{R}$ by

$$J_1(x) := \int_{\mathbb{R}^{N-1}} J(x, y_2, \cdots, y_N)\, \mathrm{d}y_2 \cdots \mathrm{d}y_N.$$

III. Main results

Main results

Theorem (B., Coville, Hamel, Valdinoci)

Let $K \subset \mathbb{R}^N$ be a compact convex set. Let $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ be a function satisfying

$$\begin{cases} Lu + f(u) \leqslant 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases}$$

Then, $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$.

Lack of a priori estimates $\rightsquigarrow$ "$u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$".

Theorem

Let $K \subset \mathbb{R}^N$ be a compact convex set and let f be such that

$$\max_{[0,1]} f' < \frac{1}{2}.$$

Let $u : \mathbb{R}^N \setminus K \to [0,1]$ be a measurable function satisfying

$$\begin{cases} Lu + f(u) = 0 & a.e. \ in \ \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & as \ |x| \to +\infty. \end{cases}$$

Then, $u = 1$ a.e. in $\overline{\mathbb{R}^N \setminus K}$.

In this case, if J has *compact support* and $J \in L^2(\mathbb{R}^N)$, then

$$\lim_{|x| \to \infty} u(x) = 1 \iff \sup_{\mathbb{R}^N \setminus K} u = 1.$$

Theorem (B., Coville, Hamel, Valdinoci)

Let $K \subset \mathbb{R}^N$ be a compact convex set with non-empty interior and let $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ be a family of compact sets with $C^{0,1}$ boundary and $K_\varepsilon \to K$ in $C^{0,1}$. Suppose that $J \in BV(\mathbb{R}^N)$ and

$$\max_{[0,1]} f' < \inf_{0<\varepsilon\leqslant 1} \inf_{x\in\mathbb{R}^N\setminus K_\varepsilon} \|J(x-\cdot)\|_{L^1(\mathbb{R}^N\setminus K_\varepsilon)}.$$

For $0<\varepsilon\leqslant 1$, let L_ε be the operator given by

$$L_\varepsilon v(x) := \int_{\mathbb{R}^N\setminus K_\varepsilon} J(x-y)\big(v(y)-v(x)\big)\,\mathrm{d}y.$$

Then, there exists $\varepsilon_0 \in (0,1]$ such that, for all $\varepsilon \in (0,\varepsilon_0]$, $u_\varepsilon \equiv 1$ a.e. is the unique measurable solution of

$$\begin{cases} L_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0 & \text{a.e. in } \mathbb{R}^N \setminus K_\varepsilon, \\ 0 \leqslant u_\varepsilon \leqslant 1 & \text{a.e. in } \mathbb{R}^N \setminus K_\varepsilon, \\ u_\varepsilon(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{8}$$

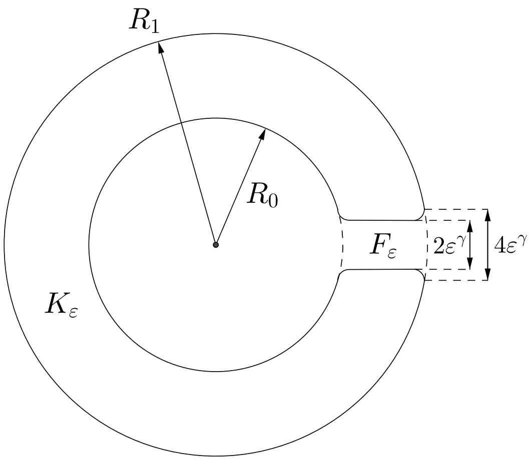
However, the Liouville property may **fail**.

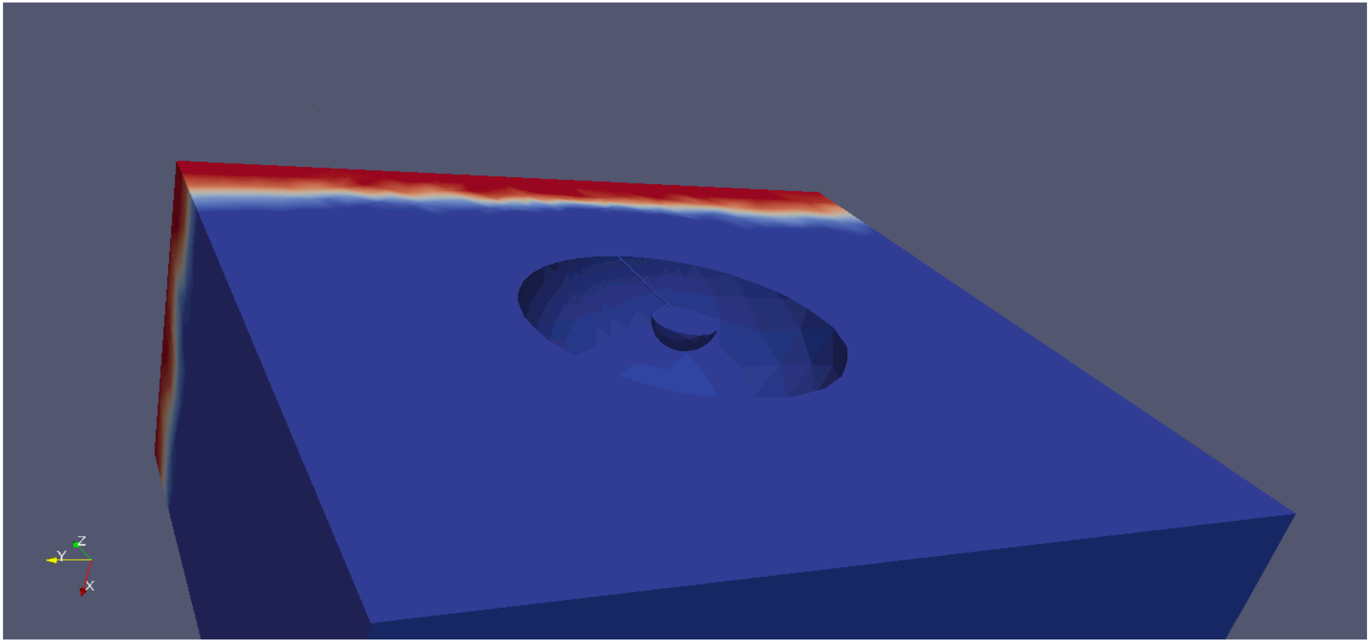
Theorem (B., Coville; Counter-example)

There exist non-trivial (non-starshaped) simply connected obstacles $K \subset \mathbb{R}^N$ *as well as data* f *and* J *for which problem*

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases}$$

has a solution $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ *s.t.* $0 < u < 1$ *in* $\overline{\mathbb{R}^N \setminus K}$.




Z
Y
X

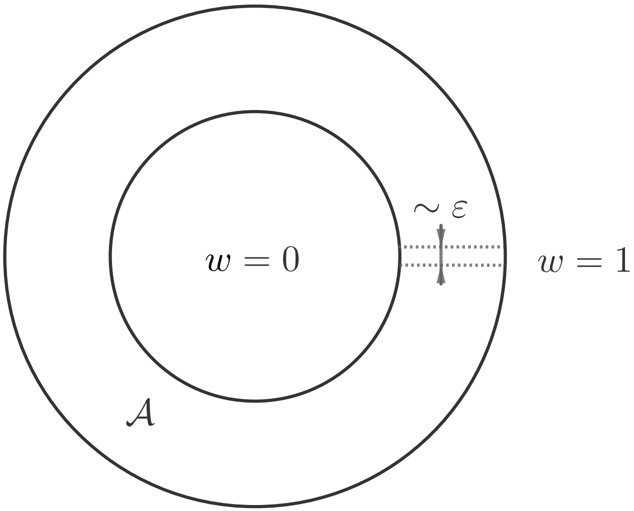
IV. How does the proof(s) work?

Counterexample: how does the proof work?

If $K = \mathcal{A} = \overline{B}_2 \setminus B_1$ with $0 < r_1 < r_2$ and $\mathrm{supp}(J) \subset B_{1/2}$, then

$$w = \begin{cases} 0 & \text{in } B_{r_1}, \\ 1 & \text{in } \mathbb{R}^N \setminus \overline{B}_{r_2}, \end{cases}$$

is a solution.



Pierce a channel of width $\sim \varepsilon \Longrightarrow u_\varepsilon \sim w$?

- Pick: $J_\varepsilon(z) = \varepsilon^{-N} J(z/\varepsilon)$ and $f_\varepsilon(s) = \varepsilon^2 f(s)$.

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- The equation now has the form:

$$\int_{\mathbb{R}^N \setminus K} J_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y + \varepsilon^2 f(u(x)) = 0.$$

- Pick: $J_\varepsilon(z) = \varepsilon^{-N} J(z/\varepsilon)$ and $f_\varepsilon(s) = \varepsilon^2 f(s)$.
- The equation now has the form:

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus K} J_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y + f(u(x)) = 0.$$

- Pick: $J_\varepsilon(z) = \varepsilon^{-N} J(z/\varepsilon)$ and $f_\varepsilon(s) = \varepsilon^2 f(s)$.

- The equation now has the form:

$$\frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus K} J_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y + f(u(x)) = 0.$$

- Energy formulation:

$$\mathcal{E}(u) = \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^N \setminus K} \int_{\mathbb{R}^N \setminus K} J_\varepsilon(x-y)|u(x)-u(y)|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N \setminus K} \int_0^{u(x)} f(s)\mathrm{d}s\mathrm{d}x.$$

- Pick: $J_\varepsilon(z) = \varepsilon^{-N} J(z/\varepsilon)$ and $f_\varepsilon(s) = \varepsilon^2 f(s)$.

- The equation now has the form:

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus K} J_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y + f(u(x)) = 0.$$

- Energy formulation:

$$\mathcal{E}(u) = \frac{1}{4}\int_{\mathbb{R}^N\setminus K}\int_{\mathbb{R}^N\setminus K} \rho_\varepsilon(x-y)\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N\setminus K}\int_0^{u(x)} f(s)\mathrm{d}s\mathrm{d}x.$$

- Pick: $J_\varepsilon(z) = \varepsilon^{-N} J(z/\varepsilon)$ and $f_\varepsilon(s) = \varepsilon^2 f(s)$.

- The equation now has the form:

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus K} J_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y + f(u(x)) = 0.$$

- Energy formulation:

$$\mathcal{E}(u) = \frac{1}{4}\int_{\mathbb{R}^N\setminus K}\int_{\mathbb{R}^N\setminus K} \rho_\varepsilon(x-y)\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y$$
$$+ \int_{\mathbb{R}^N\setminus K}\int_0^{u(x)} f(s)\mathrm{d}s\mathrm{d}x.$$

- Bourgain-Brezis-Mironescu-Ponce $\Rightarrow$ Poincaré-type inequality.

Liouville type result: how does the proof work?

Want to show:

Theorem

Suppose that K is convex. Let $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ be a solution to

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases}$$

Then, $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$.

Strategy:

1. Establish comparison principles (*weak* and *strong* maximum principles);
2. Construction of upper bounds using the 1-dimensional "front" ϕ;
3. Conclude by *reductio ad absurdum*.

Lemma (Weak maximum principle)

Assume that $f' \leqslant -c_1$ *in* $[1-c_0,+\infty)$, *for some* $c_0>0$, $c_1>0$.
Let $H \subset \mathbb{R}^N$ *be an open affine half-space such that* $K \subset H^c = \mathbb{R}^N \setminus H$.
Let $u, v \in L^\infty(\mathbb{R}^N \setminus K)$ *be such that* $u, v \in C\big(\overline{H}\big)$ *and*

$$\begin{cases} Lu + f(u) \leqslant 0 & \text{in } \overline{H}, \\ Lv + f(v) \geqslant 0 & \text{in } \overline{H}. \end{cases} \tag{9}$$

Assume also that

$$u \geqslant 1 - c_0 \quad \text{in } \overline{H}, \tag{10}$$

that

$$\limsup_{|x|\to+\infty} \big(v(x)-u(x)\big) \leqslant 0 \tag{11}$$

and that

$$v \leqslant u \quad \text{a.e. in } H^c \setminus K. \tag{12}$$

Then, $v \leqslant u$ *a.e. in* $\mathbb{R}^N \setminus K$.

Lemma (Strong maximum principle)

Let $H \subset \mathbb{R}^N$ be an open affine half-space such that $K \subset H^c$.
Let $u, v \in L^\infty(\mathbb{R}^N \setminus K) \cap C(\overline{H})$ be such that

$$\begin{cases} Lu + f(u) \leqslant 0 & \text{in } \overline{H}, \\ Lv + f(v) \geqslant 0 & \text{in } \overline{H}, \\ v \leqslant u & \text{in } \mathbb{R}^N \setminus K, \end{cases} \tag{13}$$

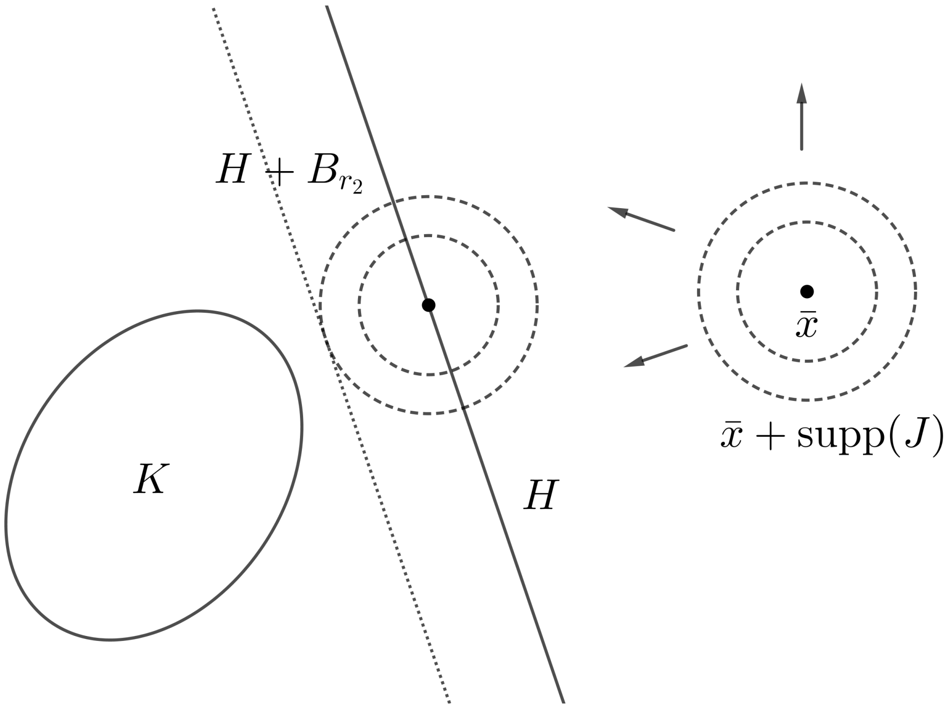
and that there exists $\bar{x} \in \overline{H}$ such that $u(\bar{x}) = v(\bar{x})$. Then,

$$v = u \text{ in } H.$$

Sketch of the proof. Set $w := v - u$ ($\leqslant 0$). Using the assumptions, we may show that

$$0 = \int_{\mathbb{R}^N \setminus K} J(\bar{x} - y) \underbrace{\left(w(y) - w(\bar{x})\right)}_{\leqslant 0} \mathrm{d}y.$$

Therefore: $w = 0$ for a.e. $y \in \left(\bar{x} + \mathrm{supp}(J)\right)$.


$H + B_{r_2}$
$\bar{x}$
$\bar{x} + \mathrm{supp}(J)$
K
H

Construction of lower bounds

- *Strong maximum principle* $\Longrightarrow \gamma := \inf_{\mathbb{R}^N \setminus K} u > 0.$
- We claim that:

$$\boxed{\exists r_0 > 0, \text{ s.t. } \phi(x \cdot e - r_0) \leqslant u(x), \text{ for any } x \in \overline{\mathbb{R}^N \setminus K} \text{ and } e \in \mathbb{S}^{N-1},}$$

where ϕ is as in our assumptions, i.e.

$$\begin{cases} J_1 * \phi - \phi + f(\phi) \geqslant 0 \text{ in } \mathbb{R}, \\ \phi \text{ is increasing in } \mathbb{R}, \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \end{cases}$$

where

$$J_1(x) := \|J(\cdot, y)\|_{L^1(\mathbb{R}^{N-1})}.$$

- For $(r, e) \in \mathbb{R} \times \mathbb{S}^{N-1}$, define:

$$\phi_{r,e}(x) := \phi(x \cdot e - r).$$

- Show that:

$$\int_{\mathbb{R}^N} J(x-y)(\phi_{r,e}(y) - \phi_{r,e}(x))\mathrm{d}y + f(\phi_{r,e}(x)) \geqslant 0,$$

for any $(r, e) \in \mathbb{R} \times \mathbb{S}^{N-1}$, where $\phi_{r,e}(x) := \phi(x \cdot e - r)$.

- Remains to show:

$$-\int_K J(x-y)(\phi_{r,e}(y) - \phi_{r,e}(x))\mathrm{d}y \geqslant 0,$$

in some half-space $H_e \subset \mathbb{R}^N \setminus K$.

• In fact, a half-space $H_e \subset \mathbb{R}^N \setminus K$ and a number $r_0 \in \mathbb{R}$ can be found so that

$$\begin{cases} Lu(x) + f(u(x)) = 0 & \forall x \in \mathbb{R}^N \setminus K, \\ L\phi_{r_0,e}(x) + f(\phi_{r_0,e}(x)) \geqslant 0 & \forall x \in H_e, \\ \phi_{r_0,e} \leqslant u(x) & \forall x \in H_e^c \setminus K. \end{cases}$$

• Conclude using the *weak maximum principle* that

$$\phi_{r_0,e}(x) = \phi(x \cdot e - r_0) \leqslant u(x) \text{ for any } x \in \mathbb{R}^N \setminus K.$$

• Since this is true for any $e \in \mathbb{S}^{N-1}$, we actually have:

$$\phi(|x| - r_0) \leqslant u(x) \text{ for any } x \in \mathbb{R}^N \setminus K.$$

Conclusion of the proof

Suppose, by contradiction, that

$$\inf_{\mathbb{R}^N \setminus K} u < 1.$$

- Recall that $\inf_{\mathbb{R}^N \setminus K} u > 0$, thus:

$$\exists x_0 \in \overline{\mathbb{R}^N \setminus K} \text{ s.t. } u(x_0) = \inf_{\mathbb{R}^N \setminus K} u \in (0, 1].$$

- By **convexity**:

$$\exists e \in \mathbb{S}^{N-1} \text{ s.t. } K \subset H_e^c, \text{ where } H_e := x_0 + \{x \cdot e > 0\}.$$

- By the previous step, the following quantity

$$r_* := \inf \left\{ r \in \mathbb{R}; \phi(x \cdot e - r) \leqslant u(x) \text{ in } \overline{\mathbb{R}^N \setminus K} \right\},$$

is well-defined and $r_* \in [-\infty, r_0]$. We have to show:

$$\boxed{r_* = -\infty.}$$

- Proceed by contradiction and assume $r_* \in \mathbb{R}$. Then,

$$\varphi_{r_*,e}(x) := \phi(x \cdot e - r_*) \leqslant u(x) \text{ in } \overline{\mathbb{R}^N \setminus K}. \tag{14}$$

- Let $H := \{x \cdot e > R_0\}$. Two cases:

$$\textbf{either} \quad \inf_{H^c \setminus K} \left(u - \varphi_{r_*,e}\right) > 0 \quad \textbf{or} \quad \inf_{H^c \setminus K} \left(u - \varphi_{r_*,e}\right) = 0.$$

$\rightsquigarrow$ In the first case, we can find $\varepsilon > 0$ such that $\varphi_{r_*-\varepsilon,e} \leqslant u$ (using the *weak maximum principle*). This contradicts the minimality of r_*.

$\rightsquigarrow$ In the second case, there must be a contact point $\bar{x} \in H^c \setminus K$. In fact $\bar{x} \in \overline{H}_e$. Indeed, if not:

$$u(\bar{x}) = \varphi_{r_*,e}(\bar{x}) < \varphi_{r_*,e}(x_0) \leqslant u(x_0) = \inf_{\mathbb{R}^N \setminus K} u \quad : \quad \text{contradiction.}$$

Thus, it holds:

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{H}_e, \\ L\varphi_{r_*,e} + f(\varphi_{r_*,e}) \geqslant 0 & \text{in } \overline{H}_e, \\ \varphi_{r_*,e} \leqslant u & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \varphi_{r_*,e}(\bar{x}) = u(\bar{x}) & \text{for some } \bar{x} \in \overline{H}_e. \end{cases}$$

Strong maximum principle $\Longrightarrow \varphi_{r_*,e} \equiv u$ in $\overline{H}_e$.

Now let $e^{\perp} \in \mathbb{S}^{N-1}$. Then, it holds:

$$1 = \lim_{t \to \infty} u(x_0 + te^{\perp}) = \lim_{t \to \infty} \varphi_{r_*,e}(x_0 + te^{\perp}) = \varphi_{r_*,e}(x_0) < 1.$$

The proof is thereby complete.

The case of nearly convex obstacles

We consider $C^{0,\alpha}$ deformations, $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$, of a convex obstacle K.

$$\begin{cases} L_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0 & \text{a.e. in } \mathbb{R}^N \setminus K_\varepsilon, \\ 0 \leqslant u_\varepsilon \leqslant 1 & \text{a.e. in } \mathbb{R}^N \setminus K_\varepsilon, \\ u_\varepsilon(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{15}$$

- $u_\varepsilon \in [\theta, 1] \implies u_\varepsilon \equiv 1$;

- $\lim\limits_{|x|\to\infty} u_\varepsilon(x) = 1$ *uniformly* in ε;

- Use the conditions on J and f' to deduce that the u_ε's converge in $C^{0,\alpha}$ to a function u_0 satisfying

$$\begin{cases} Lu_0 + f(u_0) = 0 & \text{a.e. in } \mathbb{R}^N \setminus K, \\ 0 \leqslant u_0 \leqslant 1 & \text{a.e. in } \mathbb{R}^N \setminus K, \\ u_0(x) \to 1 & \text{as } |x| \to +\infty. \end{cases}$$

K convex $\Rightarrow u_0 \equiv 1$ in $\overline{\mathbb{R}^N \setminus K}$.

• Conclude with a contradiction argument:

IF $u_{\varepsilon_j}(x_j) := \inf_{\mathbb{R}^N \setminus K_{\varepsilon_j}} u_{\varepsilon_j} < 1$ for some $\varepsilon_j \to 0$,

THEN $u_{\varepsilon_j}(x_j) < \theta$.

But: $(x_j)_{j \geqslant 0}$ is bounded, so (up to a subsequence) $x_j \to x_0$.

Therefore: $1 > \theta > u_{\varepsilon_j}(x_j) \to u_0(x_0) = 1$: contradiction.

Concluding remarks

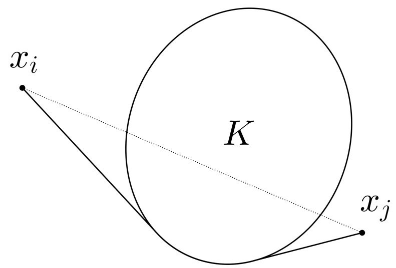
Most of our results adapt to operators of the form:

$$L_{\mathrm{g}}u(x) := \int_{\mathbb{R}^N\setminus K} J(d_{\mathrm{g}}(x,y))(u(y)-u(x))\mathrm{d}y,$$

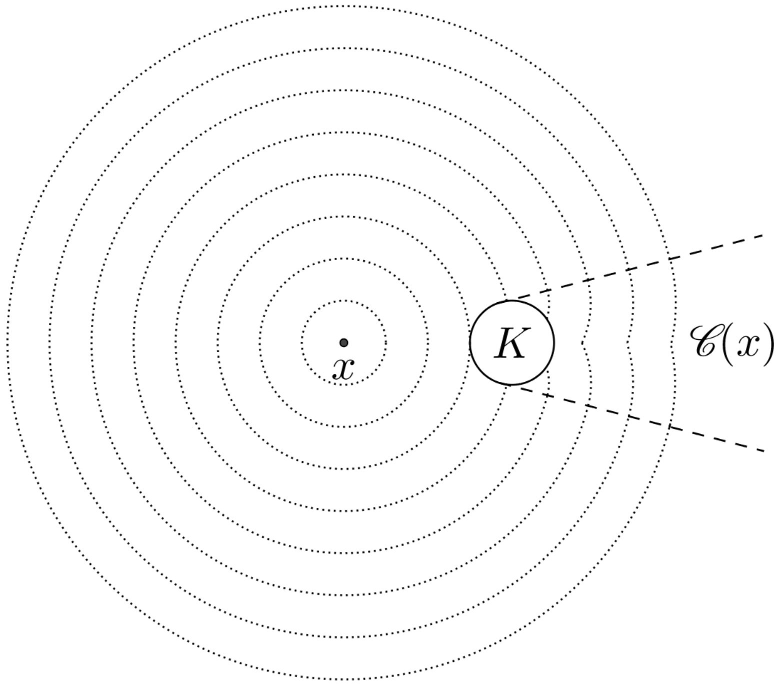
where:

1. $d_{\mathrm{g}}(\cdot,\cdot)$ is the *geodesic distance in* $\overline{\mathbb{R}^N \setminus K}$,
2. $J \in L^1_{\mathrm{loc}}(0,\infty)$ is such that $J(|z|)$ satisfies the previous assumptions and

$$\sup_{x\in\mathbb{R}^N\setminus K} \int_{\mathbb{R}^N\setminus K} J(d_{\mathrm{g}}(x,y))\mathrm{d}y < \infty.$$



The geometry of K impacts the level sets of $J(d_{\mathrm{g}}(x,\cdot))$.



Thank you for your attention!

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