On fluid-structure interactions with the Coulomb friction law boundary condition

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Joint work with
Jorge San Martín and Takéo Takahashi

Outline

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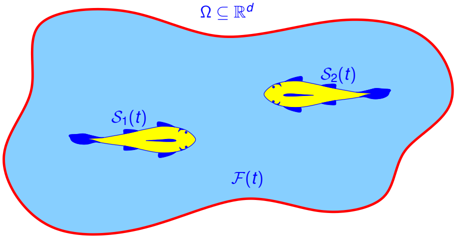
The general setting of the problem



Ω: bounded regular domain

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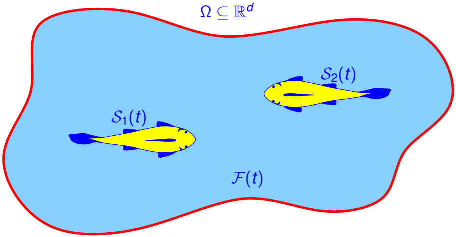

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$\mathcal{S}_i(t)$: rigid or deformable solids

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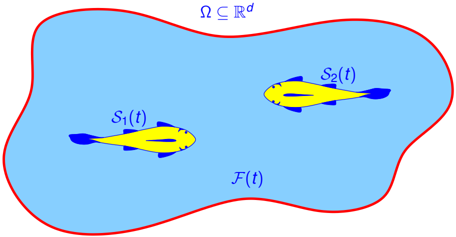

Ω:	bounded regular domain
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$\mathcal{F}(t)$:	$= \Omega \setminus \bigcup_i \overline{\mathcal{S}_i(t)}$ fluid

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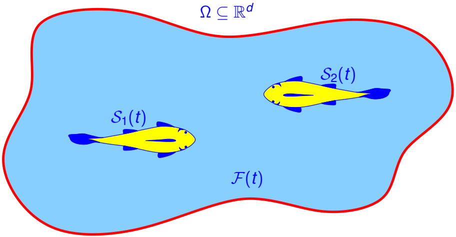

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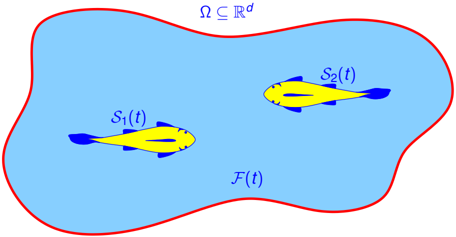

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  Model of the solids $\leftrightarrow$ Newton's laws

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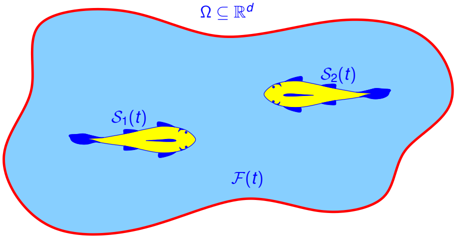

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- The fluid is viscous incompressible
- Model of fluid part $\leftrightarrow$ Navier-Stokes equations

  Model of the solids $\leftrightarrow$ Newton's laws
- ρ_F the density of fluid ($\rho_F = 1$)

  ρ_{S_i} the densities of solids ($=$ constants)

The fluid-solid system

Unknowns

- Fluid part: the Eulerian velocity $\mathbf{u}$ and the pressure p

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Equations of the fluid

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = 0 \quad & \text{in } \mathcal{F}(t), \\ \operatorname{div} \mathbf{u} = 0 \quad & \text{in } \mathcal{F}(t), \end{aligned}$$

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Equations of the solids $\forall i$

$$m_i \boldsymbol{h}_i''(t) = -\int_{\partial \mathcal{S}_i(t)} \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\Gamma,$$
$$(J_i \omega_i)'(t) = -\int_{\partial \mathcal{S}_i(t)} (\mathbf{x} - \boldsymbol{h}_i) \times \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\Gamma,$$
$$R_i'(t) = \mathbb{A}(\boldsymbol{\omega}_i(t)) \; R_i(t),$$

The fluid-solid system

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$$R_i'(t) = \mathbb{A}(\omega_i(t)) \; R_i(t),$$

Notations

μ	dynamic viscosity of the fluid;
m_i, J_i	mass and the moment of inertia of the solids;

Cauchy stress tensor:

$$\boldsymbol{\sigma}(\mathbf{u}, p) = -p\mathbf{Id} + 2\mu D(\mathbf{u}),$$

where

$$D(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^*\right).$$

$$\mathbb{A}(\boldsymbol{\omega}_i) = \begin{bmatrix} 0 & -\omega_{i,3} & \omega_{i,2} \\ \omega_{i,3} & 0 & -\omega_{i,1} \\ -\omega_{i,2} & \omega_{i,1} & 0 \end{bmatrix}.$$

Outline

No-slip boundary condition

Dirichlet condition

$$
\begin{aligned}
\mathbf{u}(t,\mathbf{x}) &= \boldsymbol{h}_i'(t) + \boldsymbol{\omega}_i(t) \times (\mathbf{x} - \boldsymbol{h}_i(t)) \quad \text{on } \partial\mathcal{S}_i(t),\\
\mathbf{u}(t,\mathbf{x}) &= \mathbf{0} \quad \text{on } \partial\Omega.
\end{aligned}
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There are not collisions between solids!

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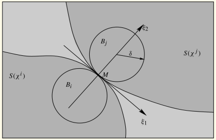
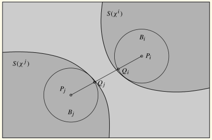
Reference

J. A. San Martín, V. Starovoitov, and M. Tucsnak,
Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid,
Arch. Ration. Mech. Anal., 161 (2002), pp. 113–147.

Proposition 4.1. *Suppose that* $i, j \in \{1, \dots, N\}$, $i \neq j$, *are such that* $\partial S(\chi^i) \cap \partial S(\chi^j) \neq \emptyset$. *Then, for any* $\boldsymbol{u} \in K(\chi)$, *there exists a rigid velocity field* $\boldsymbol{w}$ *such that* $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in S(\chi^i) \cup S(\chi^j)$.





Principal reasons for the lack of collisions

- The no-slip boundary condition
- Regularity of boundaries
- H^1-regularity of solution ($\operatorname{div} \mathbf{u} = 0$)

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D. Gérard–Varet and M. Hillairet,
Regularity issues in the problem of fluid structure interaction,
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D. Gérard–Varet and M. Hillairet,
Existence of Weak Solutions up to collision for Viscous Fluid-Solid Systems with Slip,
Comm. Pure Appl. Math., Vol. 67 (2014), pp. 2022–2075.

D. Gérard–Varet, M. Hillairet, and G. Wang,
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The body cannot reach the boundary of the cavity in finite time, under assumptions on the shapes of the cavity and the body.

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A rigid ball moving into a viscous incompressible fluid over a fixed horizontal plane.
They show that the rigid ball never touches the plane.

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One $C^{1,\alpha}$-rigid body falling over a flat surface and they show that a collision is possible in finite time if and only if $\alpha < 1/2$ (with Dirichlet boundary conditions).

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Instead of a no-slip boundary condition, the authors take the Navier condition.

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Using Navier type boundary condition, they prove one can again recover collisions.

Outline

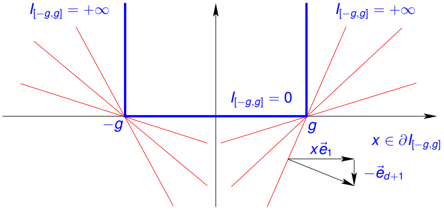
Coulomb boundary condition

Coulomb coupling condition

$$\begin{aligned} \mathbf{u}_n &= 0 && \text{on } \partial\Omega, \\ -\mathbf{u}_\tau &\in \partial I_{\overline{B}(0,g)}((\sigma(\mathbf{u},p)\mathbf{n})_\tau) && \text{on } \partial\Omega, \end{aligned}$$

where

$$I_{\overline{B}(0,g)}(\mathbf{x}) = \begin{cases} 0 & \text{if } |\mathbf{x}| \leq g \\ +\infty & \text{if } |\mathbf{x}| > g \end{cases}.$$



- $I_{\overline{B}(0,g)}$ is the characteristic function of the closed ball $\overline{B}(0,g)$;
- $g > 0$ is a constant characterizing the roughness of boundary.

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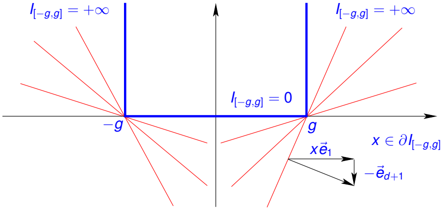

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- The subdifferential of $I_{\overline{B}(0,g)}$ is given by

$$\partial I_{\overline{B}(0,g)}(\mathbf{x}) = \begin{cases} \{0\} & \text{if } |\mathbf{x}| < g \\ \{\alpha\mathbf{x}\,;\ \alpha \ge 0\} & \text{if } |\mathbf{x}| = g \\ \emptyset & \text{if } |\mathbf{x}| > g \end{cases}.$$

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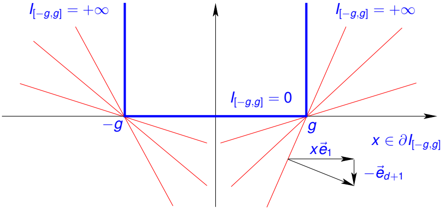

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- Recall that

$$\mathbf{y} \in \partial F(\mathbf{x}) \iff F(\mathbf{x}+\mathbf{h}) \ge F(\mathbf{x}) + \mathbf{y}\cdot\mathbf{h} \quad \forall \mathbf{h} \in \mathbb{R}^d.$$

Coulomb boundary condition

Using convex theory

$$-\mathbf{u}_\tau \in \partial I_{\overline{B}(0,g)}((\sigma(\mathbf{u},p)\mathbf{n})_\tau) \iff -(\sigma(\mathbf{u},p)\mathbf{n})_\tau \in \partial I^*_{\overline{B}(0,g)}(\mathbf{u}_\tau),$$

where $I^*_{\overline{B}(0,g)}$ is the conjugate function of $I_{\overline{B}(0,g)}$:

$$\begin{aligned} I^*_{\overline{B}(0,g)}(\mathbf{y}) &= \boxed{\sup_{\mathbf{x}\in\mathbb{R}^d} \left\{\mathbf{y}\cdot\mathbf{x} - I_{\overline{B}(0,g)}(\mathbf{x})\right\}} \\ &= \sup_{\mathbf{x}\in\overline{B}(0,g)} \mathbf{y}\cdot\mathbf{x} = \sup_{\mathbf{x}\in\overline{B}(0,1)} g\mathbf{y}\cdot\mathbf{x} \\ &= g|\mathbf{y}| \qquad \forall \mathbf{y}\in\mathbb{R}^d. \end{aligned}$$

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Thus,

$$(\sigma(\mathbf{u},p)\mathbf{n})_\tau\cdot\mathbf{y} \ge g|\mathbf{u}_\tau| - g|\mathbf{u}_\tau+\mathbf{y}| \quad \textbf{on } \partial\Omega, \forall \mathbf{y}\in\mathbb{R}^d.$$

On the moving interfaces: rigid case

Primal formulation on the moving interfaces $\partial\mathcal{S}_i(t)$

$$(\mathbf{u}_F)_n = (\mathbf{u}_{R_i})_n \quad \text{on } \partial\mathcal{S}_i(t),$$

$$-\Big((\mathbf{u}_F)_\tau - (\mathbf{u}_{R_i})_\tau\Big) \in \partial I_{\overline{B}(0,g)}((\sigma(\mathbf{u}_F, p_F)\mathbf{n})_\tau) \quad \text{on } \partial\mathcal{S}_i(t),$$

where

$$\mathbf{u}_{R_i}(t,\mathbf{x}) := \boldsymbol{h}_i'(t) + \boldsymbol{\omega}_i(t) \times (\mathbf{x} - \boldsymbol{h}_i(t)).$$

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where

$$\mathbf{u}_{R_i}(t, \mathbf{x}) := \boldsymbol{h}_i'(t) + \boldsymbol{\omega}_i(t) \times (\mathbf{x} - \boldsymbol{h}_i(t)).$$

Dual formulation on the moving interfaces $\partial\mathcal{S}_i(t)$

$$-\Big((\mathbf{u}_F)_\tau - (\mathbf{u}_{R_i})_\tau\Big) \in \partial I_{\overline{B}(0,g)}((\sigma(\mathbf{u}_F, p_F)\mathbf{n})_\tau)$$
$$\iff -(\sigma(\mathbf{u}_F, p_F)\mathbf{n})_\tau \in \partial I^*_{\overline{B}(0,g)}\Big((\mathbf{u}_F)_\tau - (\mathbf{u}_{R_i})_\tau\Big).$$

Thus,

$$(\sigma(\mathbf{u}_F, p_F)\mathbf{n})_\tau \cdot \mathbf{y} \geq g\Big|(\mathbf{u}_F)_\tau - (\mathbf{u}_{R_i})_\tau\Big| - g\Big|(\mathbf{u}_F)_\tau - (\mathbf{u}_{R_i})_\tau + \mathbf{y}\Big| \quad \forall \mathbf{y} \in \mathbb{R}^d.$$

Outline

Navier-Stokes system without solids

Complete system

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, p) &= 0 && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u}_n &= 0 && \text{on } \partial\Omega, \\ (\sigma(\mathbf{u}, p)\mathbf{n})_\tau \cdot \mathbf{y} \geq g|\mathbf{u}_\tau| - g|\mathbf{u}_\tau + \mathbf{y}| && && \text{on } \partial\Omega, \ \forall \mathbf{y} \in \mathbb{R}^d, \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}^0(\mathbf{x}) && \forall \mathbf{x} \in \Omega. \end{aligned}$$

Definition of weak solution

A weak solution $\mathbf{u}$ *of the Navier–Stokes system with the Coulomb friction law is a function*

$$\mathbf{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$$

such that

$$- \int_\Omega \mathbf{u}^0 \cdot \mathbf{v}(0, \cdot)\, d\mathbf{x} - \int_{(0,T)\times\Omega} \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} + [(\mathbf{u} \cdot \nabla)\mathbf{v}] \cdot \mathbf{u} \right) d\mathbf{x}\, dt$$

$$+ \int_0^T a(\mathbf{u}, \mathbf{v})\, dt + \int_0^T J(\mathbf{u} + \mathbf{v})\, dt - \int_0^T J(\mathbf{u})\, dt \geq 0$$

holds true for all $\mathbf{v} \in \mathscr{C}_c^1([0, T); V)$.

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$$-\int_\Omega \mathbf{u}^0 \cdot \mathbf{v}(0,\cdot)\, d\mathbf{x} - \int_{(0,T)\times\Omega} \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} + [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u} \right) d\mathbf{x}\, dt$$

$$+ \int_0^T a(\mathbf{u},\mathbf{v})\, dt + \int_0^T J(\mathbf{u}+\mathbf{v})\, dt - \int_0^T J(\mathbf{u})\, dt \geq 0$$

holds true for all $\mathbf{v} \in \mathscr{C}_c^1([0,T);V)$.

Notations

$$a(\mathbf{u},\mathbf{v}) = 2\mu \int_\Omega D(\mathbf{u}) : D(\mathbf{v})\, d\mathbf{x},$$

$$J(\mathbf{v}) = \int_{\partial\Omega} g|\mathbf{v}|\, d\Gamma,$$

Functional spaces

$$H = \{\mathbf{v} \in L^2(\Omega)^d : \ \mathrm{div}\, \mathbf{v} = 0,\ \mathbf{v}_n = 0 \ \text{ on } \partial\Omega\},$$

$$V = \{\mathbf{v} \in H^1(\Omega)^d : \ \mathrm{div}\, \mathbf{v} = 0,\ \mathbf{v}_n = 0 \ \text{ on } \partial\Omega\}.$$

Outline

THEOREM (Existence and Uniqueness)

If $\mathbf{u}^0 \in H$, then ***there exists at least one weak solution*** *of the Navier–Stokes system with the Coulomb friction law. Moreover, we have*

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(0, T; V') \qquad \text{if } d = 2,$$

$$\frac{\partial \mathbf{u}}{\partial t} \in L^{4/3}(0, T; V') \qquad \text{if } d = 3,$$

and for almost every $t \in [0, T]$, we have

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\Omega)^d} + \int_0^t a(\mathbf{u}, \mathbf{u})\, ds + \int_0^t J(\mathbf{u})\, ds \leq \frac{1}{2}\|\mathbf{u}(0)\|^2_{L^2(\Omega)^d}.$$

Additionally, if $d = 2$, we have that ***the solution is unique*** *and that*

$$\mathbf{u} \in \mathscr{C}^0([0, T]; H).$$

Navier-Stokes system without solids

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Complete proof

L. BĂLILESCU, J. SAN MARTÍN, AND T. TAKAHASHI,
On the Navier-Stokes system with the Coulomb friction law boundary condition,
Z. Angew. Math. Phys., 68, (2017), pp. Art.3, 25.

Outline

Sketch of the proof [1]

For any $\varepsilon > 0$ and $m \in \mathbb{N}^*$, we introduce a (ε, m)–regularized problem:

- We first define

$$J_\varepsilon(\mathbf{v}) = \int_{\partial\Omega} g\, j_\varepsilon(\mathbf{v})\, d\Gamma,$$

with $j_\varepsilon(\mathbf{x})$ a $\mathscr{C}^1$-convex regularized version of $|\mathbf{x}|$ such that: $j_\varepsilon(\mathbf{0}) = 0$,

$$\nabla j_\varepsilon(\mathbf{x}) \cdot \mathbf{x} \geq 0, \qquad |\nabla j_\varepsilon(\mathbf{x})| \leq 1, \qquad \big|j_\varepsilon(\mathbf{x}) - |\mathbf{x}|\big| \leq \varepsilon \qquad \forall \mathbf{x} \in \mathbb{R}^d.$$

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- We then use the Galerkin method: given an orthonormal basis $\{\mathbf{v}_j\}$ of H, we find the approximate solution of our problem as the function $\mathbf{u}_{\varepsilon,m}(t,\cdot) \in V_m = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, satisfying the equation:

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for all $\mathbf{v} \in V_m$, with the initial condition $\mathbf{u}_{\varepsilon,m}(0,\cdot)$ being the orthogonal projection of $\mathbf{u}^0$ onto V_m.

Sketch of the proof [2]

One can easily deduce that

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Then, taking $\varepsilon = \frac{1}{m}$ and passing to the limit as $m \to \infty$, we get

$$\mathbf{u}_{\varepsilon,m} \rightharpoonup \mathbf{u} \quad \text{weakly* in } L^\infty(0,T;H) \cap L^2(0,T;V),$$
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Using compactness results, one can deduce

$$\mathbf{u}_{\varepsilon,m} \to \mathbf{u} \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)).$$

Sketch of the proof [3]

Integrating over $[0, T]$ the equation satisfied by $\mathbf{u}_{\varepsilon,m}$ and using

$$\nabla j_\varepsilon(\mathbf{u}_{\varepsilon,m}) \cdot (\mathbf{v} + \mathbf{u}_{\varepsilon,m} - \mathbf{u}_{\varepsilon,m}) \leq j_\varepsilon(\mathbf{v} + \mathbf{u}_{\varepsilon,m}) - j_\varepsilon(\mathbf{u}_{\varepsilon,m}),$$

for any $\mathbf{v} \in \mathscr{C}^1_c([0, T); V_m)$, we get

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Fixing $\mathbf{v} \in \mathscr{C}_c^1([0, T); V_m)$, we pass to the limit in all terms, we obtain

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Then, for any $\mathbf{v} \in \mathscr{C}_c^1([0, T); V)$, using the orthogonal projection of $\mathbf{v}$ on V_m, as test function, and due to its strong convergence to $\mathbf{v}$ in $\mathscr{C}_c^1([0, T); V)$, we conclude the existence of a weak solution.

□

Outline

Mixed formulation

Spaces:

$$V_0 = \left\{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v}_n = 0 \text{ on } \partial\Omega \right\}, \qquad M = \left\{ q \in L^2(\Omega) : \int_\Omega q \, d\mathbf{x} = 0 \right\}$$

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Notation

$\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \longleftrightarrow$ material derivative associated with $\mathbf{u}$.

The characteristic function

The material derivative is given by

$$\frac{d\mathbf{u}}{dt}(\mathbf{x}, t_0) = \frac{d}{dt}\Big[\mathbf{u}\big(t, \psi(t; t_0, \mathbf{x})\big)\Big]_{|t=t_0},$$

where the characteristic function $\psi : [0, T]^2 \times \Omega \to \Omega$ is defined as solution of

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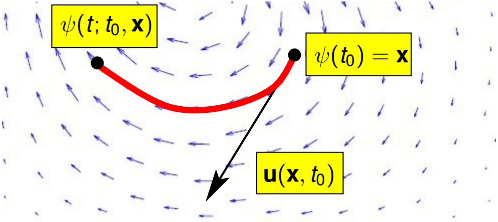
$\psi(t; t_0, \mathbf{x})$: the trajectory of a particle which at time t_0 will be in $\mathbf{x}$.



Fully discrete formulation

Let $h > 0$ and $\mathcal{T}_h$ a quasi-uniform triangulation of Ω. For $N \in \mathbb{N}^*$, we denote

$$\Delta t = T/N \quad \text{and} \quad t^k = k\Delta t \qquad \text{for } k \in \{0, \cdots, N\}.$$

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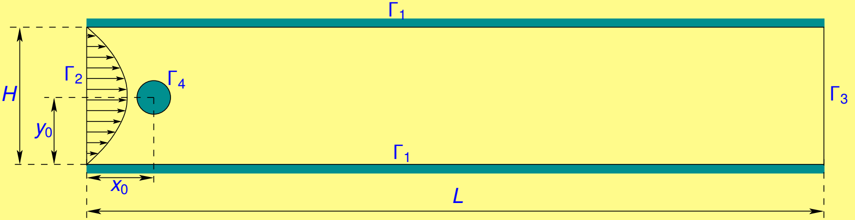
$\overline{\mathbf{X}}_h^k$ is an approximation of the exact characteristic function $\psi(t^k; t^{k+1}, \cdot)$:

$$\overline{\mathbf{X}}_h^k(\mathbf{x}) = \psi_h^k(t_k; t_{k+1}, \mathbf{x}) \qquad \forall \mathbf{x} \in \Omega,$$

where ψ_h^k is defined as the solution of

$$
\begin{cases}
\dfrac{d}{dt}\psi_h^k(t; t_{k+1}, \mathbf{x}) = \mathbf{u}_h^k(\psi_h^k(t; t_{k+1}, \mathbf{x})) & \forall t \in [t_k, t_{k+1}], \\
\psi_h^k(t_{k+1}; t_{k+1}, \mathbf{x}) = \mathbf{x}.
\end{cases}
$$

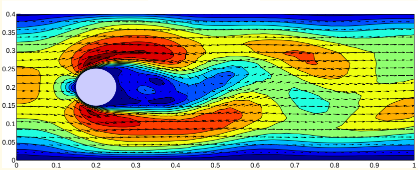
We consider the fluid flow after a cylindrical obstacle in a horizontal channel:



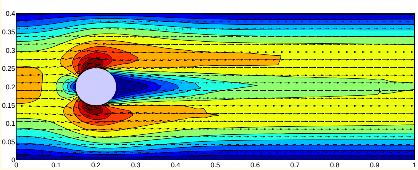
- Γ_1 Homogeneous Dirichlet boundary condition, modelling the contact with an infinitely adherent wall.
- Γ_2 Inlet Dirichlet boundary condition, where the inlet velocity field is given by a parabolic profile.
- Γ_3 Outlet boundary condition.
- Γ_4 Special wall where we study the Coulomb law effect.

Horizontal channel with a cylindrical obstacle

a) Zero Dirichlet boundary condition



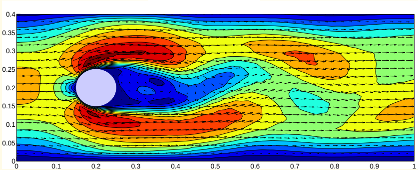
b) Neumann boundary condition



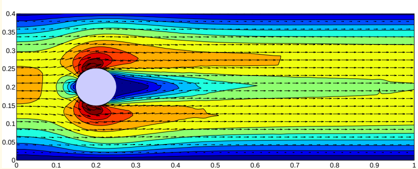
Velocity field at $t = 2s$, obtained as the solution of Navier–Stokes equation with the four boundary conditions on the ball boundary (with $Re = 100$).

Horizontal channel with a cylindrical obstacle

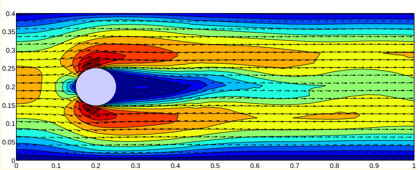
a) Zero Dirichlet boundary condition



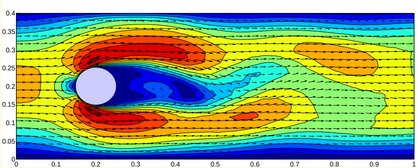
b) Neumann boundary condition



c) Coulomb boundary condition with $g = 0.07$

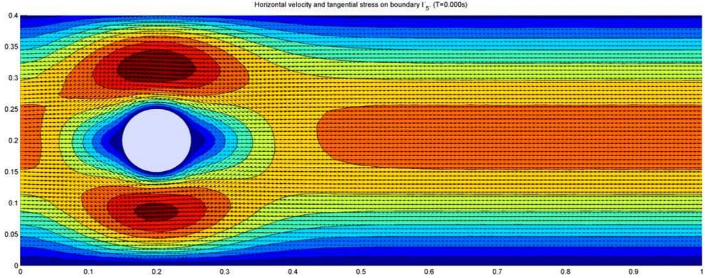


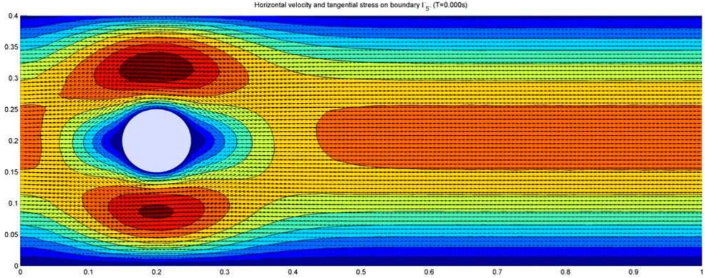
d) Coulomb boundary condition with $g = 0.20$



Velocity field at $t = 2s$, obtained as the solution of Navier–Stokes equation with the four boundary conditions on the ball boundary (with $Re = 100$).

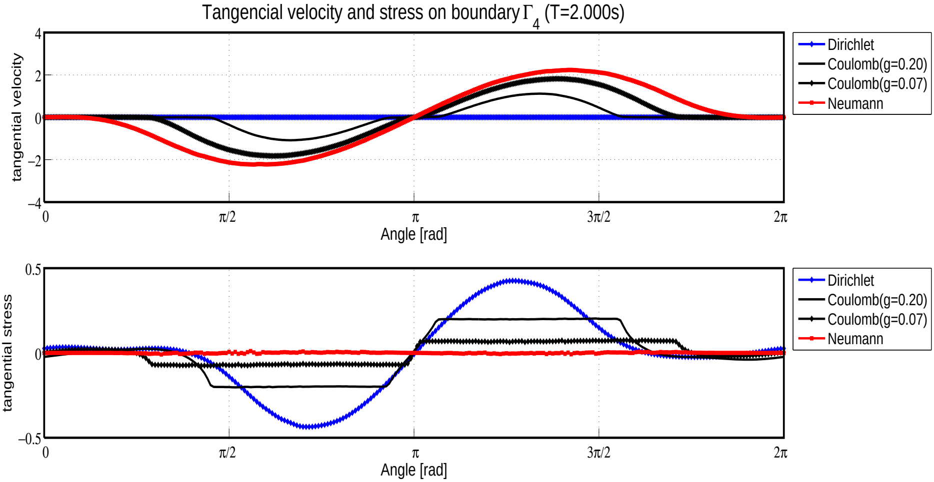
Dirichlet vs Coulomb (g=0.20)





Tangential velocity and stress

Tangential velocity $\mathbf{u}_\tau$ and tangential stress $(\sigma\mathbf{n})_\tau$ on the boundary of the obstacle.



Outline

Equation of the fluid $\mathcal{F}(t)$

$$
\begin{aligned}
\frac{\partial \mathbf{u}_F}{\partial t} + (\mathbf{u}_F \cdot \nabla)\mathbf{u}_F - \operatorname{div} \sigma(\mathbf{u}_F, p_F) &= 0, \\
\operatorname{div} \mathbf{u}_F &= 0,
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Fluid-rigid structure interaction

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Equation of the rigid

$$\begin{aligned} m\boldsymbol{h}''(t) &= -\int_{\partial \mathcal{S}(t)} \boldsymbol{\sigma}(\mathbf{u}_F, p_F)\mathbf{n}\, d\Gamma, \quad t > 0, \\ (J\omega)'(t) &= -\int_{\partial \mathcal{S}(t)} (\mathbf{x} - \boldsymbol{h}) \times \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n}\, d\Gamma, \quad t > 0, \\ R'(t) &= \mathbb{A}(\omega(t))R(t), \quad t > 0, \end{aligned}$$

Fluid-rigid structure interaction

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Boundary condition on $\partial\Omega$

$$
\begin{aligned}
(\mathbf{u}_F)_n &= 0, \\
(\sigma(\mathbf{u}_F, p_F)\mathbf{n})_\tau \cdot \boldsymbol{y} \geq g|(\mathbf{u}_F)_\tau| - g|(\mathbf{u}_F)_\tau + \boldsymbol{y}|, &\qquad \forall \boldsymbol{y} \in \mathbb{R}^d,
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Boundary condition on $\partial\mathcal{S}(t)$

$$(\mathbf{u}_F)_n = (\mathbf{u}_R)_n,$$

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Initial conditions

$$\boldsymbol{h}(0) = \mathbf{0}, \qquad R(0) = I_3,$$

$$\boldsymbol{h}'(0) = \ell^0, \qquad \omega(0) = \omega^0,$$

$$\mathbf{u}_F(0, \mathbf{x}) = \mathbf{u}_F^0(x) \quad \mathbf{x} \in \mathcal{F}^0.$$

Weak formulation

Definition of weak solution

A weak solution is a triplet $(\boldsymbol{h}, R, \mathbf{u})$ *with the following properties:*

$$(\boldsymbol{h}, R) \in W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)), \quad \mathcal{S}(t) \Subset \Omega \quad (t \in (0,T)),$$
$$\mathbf{u} \in L^\infty(0,T;V_n^0(\Omega)), \quad \mathbf{u}(t,\cdot) \in H_R(\mathcal{S}(t)) \quad \text{a.e. in } (0,T),$$
$$\mathbf{u}_F \in L^2(0,T;H^1(\Omega)), \quad \mathbf{u}_R(t,\mathbf{x}) = \boldsymbol{h}'(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \boldsymbol{h}(t)),$$

such that $R'(t) = \mathbb{A}(\boldsymbol{\omega}(t))R(t)$, $t > 0$, $\boldsymbol{h}(0) = \mathbf{0}$, $R(0) = I_3$ *hold true, and for any* $\mathbf{v} \in \mathcal{T}_T$*:*

$$\begin{aligned}
&-\int_{\mathcal{F}^0} \mathbf{u}_F^0 \cdot \mathbf{v}_F(0,\cdot)\, d\mathbf{x} - \int_{\mathcal{S}^0} \rho_{\mathcal{S}} \mathbf{u}_R^0 \cdot \mathbf{v}_R(0,\cdot)\, d\mathbf{x} \\
&\quad - \int_0^T \int_{\mathcal{F}(t)} \mathbf{u}_F \cdot \left[\frac{\partial \mathbf{v}_F}{\partial t} + [(\mathbf{u}_F \cdot \nabla)\mathbf{v}_F]\right] d\mathbf{x}\, dt - \int_0^T \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \mathbf{u}_R \cdot \frac{\partial \mathbf{v}_R}{\partial t}\, d\mathbf{x}\, dt \\
&\quad + 2\mu \int_0^T \int_{\mathcal{F}(t)} D(\mathbf{u}_F) : D(\mathbf{v}_F)\, d\mathbf{x}\, dt + \int_0^T \int_{\partial\Omega} \Big[g|(\mathbf{u}_F + \mathbf{v}_F)_\tau| - g|(\mathbf{u}_F)_\tau|\Big] d\Gamma\, dt \\
&\quad + \int_0^T \int_{\partial\mathcal{S}(t)} g\,|(\mathbf{u}_F - \mathbf{u}_R + \mathbf{v}_F - \mathbf{v}_R)_\tau|\, d\Gamma\, dt - \int_0^T \int_{\partial\mathcal{S}(t)} g\,|(\mathbf{u}_F - \mathbf{u}_R)_\tau|\, d\Gamma\, dt \geq 0
\end{aligned}$$

and for almost every $t \in (0,T)$

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{F}} |\mathbf{u}_F(t,\cdot)|^2\, d\mathbf{x} + \frac{\rho_S}{2}\int_{\mathcal{S}} |\mathbf{u}_R(t,\cdot)|^2\, d\mathbf{x} + 2\mu \int_0^t \int_{\mathcal{F}(t)} |D(\mathbf{u}_F)|^2\, d\mathbf{x}\, dt \\
&\quad + \int_0^t \int_{\partial\Omega} g|\mathbf{u}_F|\, d\Gamma\, dt + \int_0^t \int_{\partial\mathcal{S}(t)} g|\mathbf{u}_F - \mathbf{u}_R|\, d\Gamma\, dt \\
&\qquad \leq \frac{1}{2}\int_{\mathcal{F}^0} |\mathbf{u}_F^0|^2\, d\mathbf{x} + \frac{\rho_S}{2}\int_{\mathcal{S}^0} |\mathbf{u}_R^0|^2\, d\mathbf{x}.
\end{aligned}$$

Weak formulation

Functional spaces

$$H_R(\mathcal{S}) = \left\{ \mathbf{v} \in V_n^0(\Omega) : \exists \mathbf{v}_R \in \mathcal{R},\ \exists \mathbf{v}_F \in H^1(\Omega),\ \mathbf{v} = \mathbf{v}_F \text{in } \Omega \setminus \overline{\mathcal{S}},\ \mathbf{v} = \mathbf{v}_R \text{ in } \mathcal{S} \right\},$$

$$\mathcal{T}_T := \left\{ \mathbf{v} \in \mathcal{C}_c^0([0,T); V_n^0(\Omega)) : \exists \mathbf{v}_R \in \mathcal{C}^1([0,T];\mathcal{R}), \exists \mathbf{v}_F \in \mathcal{C}^1([0,T]; H^1(\Omega)) \right.$$
$$\left. \text{and} \quad \mathbf{v} = \mathbf{v}_F \text{in } \mathcal{F}(t),\ \mathbf{v} = \mathbf{v}_R \text{in } \mathcal{S}(t) \right\},$$

where

$$V_n^0(A) = \left\{ \mathbf{v} \in L^2(A) : \ \text{div}\, \mathbf{v} = 0 \text{in } A,\ \mathbf{v}_n = 0 \text{on } \partial A \right\}$$

and $\mathcal{R}$ is the space of rigid velocities:

$$\mathcal{R} := \left\{ \boldsymbol{\ell} + \boldsymbol{\omega} \times \mathbf{x} : \boldsymbol{\ell}, \boldsymbol{\omega} \in \mathbb{R}^3 \right\}.$$

Outline

Main result

THEOREM (EXISTENCE)

Assume $\mathcal{S}^0 \Subset \Omega$, $\partial\mathcal{S}^0$ *and* $\partial\Omega$ *are of class* $\mathcal{C}^{1,1}$, $\mathbf{u}_F^0 \in V_n^0(\Omega)$, $\mathbf{u}_R^0 \in \mathcal{R}$ *with*

$$(\mathbf{u}_F^0)_n = (\mathbf{u}_R^0)_n \quad \text{on } \partial\mathcal{S}^0.$$

Then, there exist $T \in (0, \infty]$ *and a weak solution in* $(0, T)$.

Moreover, we can choose T *such that one of the following alternatives holds true:*

1. $T = \infty$;
2. $\lim_{t \to T} dist(\mathcal{S}(t), \partial\Omega) = 0$.

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Complete proof

L. Bălilescu, J. San Martín, and T. Takahashi,
Fluid-rigid structure interaction system with Coulomb's law,
SIAM J. Math. Anal., 49 (2017), no. 6, pp. 4625–4657.

Outline

1. Existence of collisions with the Coulomb friction law.

Perspectives

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2. Multiples rigid solids.

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4. Multiple deformable body.

Perspectives

1. Existence of collisions with the Coulomb friction law.
2. Multiples rigid solids.
3. Deformable solid.
4. Multiple deformable body.
5. Navier-Stokes + elastic or viscoelastic body.

Thank you!

Thank you!

Thank you!

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