# Risque de couverture des options dans un marché financier avec actif suivant un modèle exponentiel-Lévy. Minimisation de la CVaR

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## Outline

# Aim and motivation

**Main objective**

Construct new stochastic methods in order to approximate risk measures like the VaR and the CVaR in incomplete markets for options driven by stocks which behave like a stochastic process with jumps - exponential-Lévy process model.

# Statement of the problem

## Option pricing

- Classical models for financial prices - **Black and Scholes model** - the stock price is described by a geometric Brownian motion
  - **Advantages**: risk neutral approach via uniqueness of the equivalent martingale measure that implies that the derivative price is the unique arbitrage-free contingent claim.
  - **Drawbacks**: model inconsistent with real data, asset price process has jumps or spikes, the empirical distributions of asset returns exhibit fat tails or skewness behaviors ($\neq$ normal variable), volatility smile ($\neq$ constant volatility as in B&S).
- **Lévy processes**
  - **Advantages**: well adapted to capture the financial behavior.
  - **Drawbacks**: mathematical problems become more complicated, incomplete markets (infinitely many equivalent martingale measures...).

# Some observations

## Context

- We consider models with jumps and assume that the logarithm of the price follows a Lévy process.
- We are in the following situation: incomplete markets, under the assumption of no arbitrage which implies :
  - there are infinitely many martingale measures
  - there is an interval of arbitrage-free prices - one way to proceed super-hedging but is too costly
  - we use a relaxation problem with a particular equivalent martingale measure.

# Problem formulation

- Exponential-Lévy model for the stock price:

$$S_t = S_0 e^{X_t}, \quad t \geq 0.$$

- Value-at-Risk (VaR) of $L$ for a given confidence level $a$

$$\mathrm{VaR}_a(L) = \inf \left\{ x \in \mathbb{R} \mid \mathbb{P}(L \leq x) > a \right\}$$

- **Conditional Value-at-Risk (CVaR)** of $L$ for a confidence level $a$

$$\mathrm{CVaR}_a(L) \quad = \frac{1}{1-a} \int_a^1 \mathrm{VaR}_x(L) dx$$

1. Monetary loss on $T$: $L = S_0 - S_T$;
2. Loss in terms of interest rates on $T$: $L = -\log(S_T / S_0) = -X_T$.

# Lévy processes in finance

**Exponential-Lévy model for the stock price**: $S_t = S_0 e^{X_t}, \quad t \geq 0$, where $S_0 > 0$ is a constant and $(X_t)_{t \in [0,T]}$ is a one-dimensional Lévy process with characteristic triplet $(b, \sigma^2, \nu)$.

**Definition Lévy process** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $(X_t)_{t \geq 0}$ is said to be a Lévy process if:

- The paths of $X$ are $\mathbb{P}$-a.s. right continuous and with left limits.
- Independent increments: for $0 \leq s \leq t$, $(X_t - X_s) \amalg \{X_u, u \leq s\}$.
- Stationary increments: for $0 \leq s \leq t$, $X_t - X_s \overset{(d)}{=} X_{t-s}$.
- $\mathbb{P}(X_0 = 0) = 1$.

**Characteristic function - Lévy Khintchine** $(X_t)_{t \geq 0}$ Lévy process with characteristic triplet $(b, \sigma^2, \nu)$ where $b \in \mathbb{R}, \sigma > 0$ and $\nu$ is a Lévy measure (non-neg. measure on $\mathbb{R} \backslash \{0\}$ s.t. $\int_{\mathbb{R} \backslash \{0\}} 1 \wedge |x|^2 \nu(\mathrm{d}x) < \infty$).

**Characteristic function**:

$$\Phi_t(u) = \mathbb{E}\left[e^{iuX_t}\right] = e^{t\psi(u)}, \; u \in \mathbb{R}$$

where $\psi$ is the characteristic exponent given by:

$$\psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux \mathbb{1}_{|x|<1}\right) \nu(\mathrm{d}x).$$

# Self-financing strategy

A **self-financing strategy** $(V_0, Y)$ is given by an initial capital $V_0 \geq 0$ and a predictable process $Y$ s.t. the discounted value process $(V_t)_{t\geq 0}$

$$V_t = V_0 + \int_0^t Y_s \mathrm{d}S_s, \forall t \in [0, T] \text{ is well-defined.}$$

A self-financing strategy $(V_0, Y)$ is admissible if $V_t \geq 0 \forall t \in [0, T]$.
"Ideal context" In a complete market the equivalent martingale measure $\mathbb{P}^*$ is unique: For a contingent claim and its payoff at maturity $T$ is a non-negative r.v. $\mathcal{F}_T-$measurable $H \in L^1$. Define $U_0 = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H] < \infty$. $U_0$ is the smallest amount $V_0$ s.t. there exists an admissible strategy $(V_0, Y)$ whose discounted value process satisfies $V_T \geq H$.
Complete market: $U_0 = \mathbb{E}^*[H]$ the unique arbitrage-free price,
Incomplete markets more than one equivalent measure $\Longrightarrow$ range of arbitrage-free prices.
Need other ideas to find good criteria

- super-hedging too expensive
- weaken the constraints: we fix a smaller amount $\tilde{V}_0 < U_0$.

# Risk measure Conditional Value-at-Risk

Let $\mathcal{A} = \{(V_0, Y) | (V_0, Y) \text{ self-financing and admissible}\}$.
**Definition CVaR** For a given strategy $(V_0, Y)$ and a fixed confidence level $a \in (0, 1)$ the **CVaR at $a$%** level is defined as the expected return of the portfolio in the worst $a$% cases, that is

$$CVaR_a(L) = \frac{1}{1-a} \int_a^1 VaR_x(L) \mathrm{d}x,$$

where $L$ is the terminal hedging risk given by

$$L(V_0, Y) = H - V_0 - \int_0^T Y_t \mathrm{d}S_t,$$

($H$ is the option payoff, $S_t$ is the asset price).
The Value-at-Risk (VaR) is the $a$-quantile $q_a^L$ of $L$

$$VaR_a(L) = q_a^L = \inf\{x \in \mathbb{R} : \mathbb{P}(L \geq x) > a\}.$$

# Statement of the problem

Difficulty in incomplete markets more than one equivalent measure $\Longrightarrow$ range of arbitrage-free prices.
Need other ideas to find good criteria weaken the constraints: we fix a smaller amount $\tilde{V}_0 < U_0$.
For $\tilde{V}_0$ fixed (the wealth constraint) define

$$\mathcal{A}_{\tilde{V}_0} = \{(V_0, Y) | (V_0, Y) \in \mathcal{A}, V_0 \leq \tilde{V}_0\}.$$

**Aim** Solve the optimization problem

$$\min_{(V_0, Y) \in \mathcal{A}_{\tilde{V}_0}} CVaR(L(V_0, y)).$$

**Complex problem**: the objective function is defined via a quantile - the numerical models involve ordering of the position values.

**Solution: Rockafellar and Uryasev** give an equivalent formula for CVaR as a convex function.

# Rockafellar and Uryasev Theorem

## Theorem (Rockafellar and Uryasev (2002))

*Let $a \in (0,1)$ and $L$ be the r.v. describing the loss. The function*

$$z \mapsto f(z) = z + \frac{1}{1-a}\,\mathbb{E}[(L-z)_+],$$

*is finite and convexe (and continuous), and*

$$\mathrm{CVaR}_a(L) = \min_{z\in\mathbb{R}} f(z),$$

$$\mathrm{VaR}_a(L) = \min\{y : y \in \operatorname*{argmin}_{z\in\mathbb{R}} f(z)\}.$$

$$\mathrm{CVaR}_a(S_0 - S_T) = \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-a}\mathbb{E}\left[(S_0 - S_T - z)_+\right] \right\}.$$

$$\mathrm{CVaR}_a(-X_T) = \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-a}\mathbb{E}\left[(-X_T - z)_+\right] \right\}.$$

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# Principle of the method

$$\begin{aligned}\mathrm{CVaR}_a(S_0 - S_T) &= \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-a}\mathbb{E}\left[(S_0 - S_T - z)_+\right] \right\} \\ &= \min_{z<S_0} \left\{ z + \frac{1}{1-a}\mathbb{E}\left[(S_0 - S_T - z)_+\right] \right\} \\ &= \max_{w\in\mathbb{R}} \left\{ S_0(1-e^w) + \frac{1}{1-a}\mathbb{E}\left[(S_0 e^w - S_T)_+\right] \right\}, (w = \log(1 - \frac{z}{S_0}\end{aligned}$$

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$$f_0(w) := \mathbb{E}\left[(S_0 e^w - S_T)_+\right], \text{ for } w \in \mathbb{R}.$$

1. Explicit $\hat{f}_0$ the Fourier transform of $f_0$ (express it with respect to $\Phi_T$, the characteristic function of $X_T$)

$$\hat{f}_0(u) = \int_{\mathbb{R}} e^{iuw} f_0(w)dw.$$

2. Obtain $f_0$ by inverting (numerically) $\hat{f}_0$

$$f_0(w) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iuw}\hat{f}_0(u)du.$$

$f_0$ is not integrable.

# Expression of $\hat{f}_\alpha$ and Fourier inversion

The Fourier transform of $f_\alpha$ can be expressed explicitly with respect to the characteristic function of $X_T$.

## Proposition

*Let $\alpha > 1$ be s.t $\mathbb{E}\left[e^{-(\alpha-1)X_T}\right] < \infty$. The Fourier transform of $f_\alpha(w)$ is well defined and it is given by*

$$\hat{f}_\alpha(u) = \frac{S_0 \Phi_T(u + i(\alpha - 1))}{\alpha(\alpha - 1) - u^2 - iu(2\alpha - 1)},$$

*where $\Phi_T$ is the characteristic function of $X_T$.*

$$\text{CVaR}_a(S_0 - S_T) = \max_{w \in \mathbb{R}} \left\{ S_0(1 - e^w) + \frac{e^{\alpha w}}{(1-a)\pi} \text{Re}\left( \int_0^\infty e^{-iuw} \hat{f}_\alpha(u) du \right) \right\}.$$

Furthermore,

$$\text{VaR}_a(S_0 - S_T) = S_0 \left(1 - e^{w_{max}}\right).$$

# Statement of the problem - notations

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- Exp-Lévy model for the asset price

$$S_t = S_0 e^{X_t}, \quad t \in [0, T],$$

where $X$ Lévy process with triplet $(b, \sigma^2, \nu)$.

- $\mathcal{P} = \{\text{risk neutral measures}\}$.
- Arbitrage free hypothesis: $\mathcal{P} \neq \emptyset$.
This is verified if $\sigma \neq 0$ or $\nu \neq 0$.

# Main steps

- R & U proved that the CVaR is equivalent to:

$$\mathrm{CVaR}_a(L(V_0, Y)) = \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-a}\mathbb{E}[(L(V_0, Y) - z)_+] \right\}.$$

- Define the function

$$f(z) = z + \frac{1}{1-a} \min_{(V_0,Y)\in\mathcal{A}_{\tilde{V}_0}} \mathbb{E}[(L(V_0, Y) - z)_+],$$

- New form of the optimization problem

$$\min_{(V_0,Y)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}_a(L(V_0, Y)) = \min_{z\in\mathbb{R}} f(z).$$

# The steps to succeed

- **Compute $f$**: $\forall z \in \mathbb{R}$ we explicit the formula of $f$ by finding

$$(V_0^*(z), Y^*(z)) \in \text{argmin}_{(V_0, Y) \in \mathcal{A}_{\tilde{V}_0}} \mathbb{E}[L(V_0, Y) - z)_+].$$

- **Minimize the** CVaR: solve the one dimensional optimization problem

$$z^* \in \text{argmin}_{z \in \mathbb{R}} f(z).$$

**To conclude** the strategy $(V_0^*(z^*), Y^*(z^*))$ is optimal

$$\min_{(V_0, Y) \in \mathcal{A}_{\tilde{V}_0}} \text{CVaR}_a(L(V_0, Y)) = \text{CVaR}_a(L(V_0^*(z^*), Y^*(z^*)).$$

# Dual problem

**Aim** Optimise $\mathbb{E}[L(V_0, Y) - z)_+]$ in $\mathcal{A}_{\tilde{V}_0}$ for all $z \Longrightarrow$ numerical valuation of the function $f(z)$

Technic we rewrite the problem in the form of hypothesis testing one. Define

$$\mathcal{R} = \{\varphi : \Omega \to [0,1] | \varphi \text{ is } \mathcal{F}_T - \text{measurable}\}.$$

## Theorem

*For all $z \in \mathbb{R}$, there exists a solution $\varphi^*(z) \in \mathcal{R}$ to the optimisation problem :*

$$\begin{aligned} \varphi^*(z) \in \operatorname{argmin}_{\varphi \in \mathcal{R}} & \, \mathbb{E}[(1 - \varphi(z))(H - z)_+] \\ s.t. \sup_{\mathbb{P}^* \in \mathcal{P}} & \, \mathbb{E}^*[\varphi(z)(H - z)_+] \leq \tilde{V}_0. \end{aligned} \tag{5.1}$$

*Moreover, we have*

$$\begin{aligned} \min_{(V_0, Y) \in \mathcal{A}_{\tilde{V}_0}} \mathbb{E}[(L(V_0, Y) - z)_+] \;=\; & \min_{\varphi(z) \in \mathcal{R}} \mathbb{E}[(1 - \varphi(z))(H - z)_+] \\ & s.t. \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[\varphi(z)(H - z)_+] \leq \tilde{V}_0. \end{aligned} \tag{5.2}$$

# Dual problem - treatment

**Still** obtaining numerical results from this formulation not possible.

**Set of interest (Neyman Pearson lemma)**

$$Z_{\mathbb{P}^*} = \left\{ \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} : \mathbb{P}^* \in \mathcal{P} \right\}.$$

**Classical results** show that we have the structure of the optimal randomised test $\varphi^*(z)$ for the problem (5.1) when the set $Z_{\mathbb{P}^*}$ is compact.

**And...** $Z_{\mathbb{P}}^*$ is compact only if $\mathcal{P}$ reduces to a singleton.

**Our problem** The compactness of $Z_{\mathbb{P}^*}$ is not verified by our process exponential-Lévy process !

# Dual problem - treatment

## Proposition

*Let* $(X_t)_{t\in[0,T]}$ *be a Lévy process under* $\mathbb{P}$ *with characteristic triplet* $(b,\sigma^2,\nu)$*. The set of densities* $Z_{\mathbb{P}^*} = \left\{ \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} : \mathbb{P}^* \in \mathcal{P} \right\}$ *is given by*

$$Z_{\mathbb{P}^*} = \Bigg\{ e^{U_T^{(\theta,\phi)}} : \theta \in \mathbb{R} \text{ and } \phi : \mathbb{R} \to \mathbb{R} \text{ s.t. } \int_{\mathbb{R}} \left( e^{\phi(x)/2} - 1 \right)^2 \nu(\mathrm{d}x) < \infty$$
$$\text{and } \; b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left( e^{\phi(x)}(e^x - 1) - x\mathbb{1}_{|x|\le 1} \right) \nu(\mathrm{d}x) = 0 \Bigg\}$$

*where* $\left(U_T^{(\theta,\phi)}\right)_{t\in[0,T]}$ *is a Lévy process under* $\mathbb{P}$ *with characteristic triplet*

$$\left( -\frac{\sigma^2\theta^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|\le 1})\nu \circ \phi^{-1}(\mathrm{d}x), \sigma^2\theta^2, \nu \circ \phi^{-1} \right).$$

# Compacteness criterion

## From the Proposition

- For $\nu \neq 0$ the set $Z_{\mathbb{P}^*}$ is infinite as we can find many functions $\phi$ satisfying this condition.
- **Particular case $\nu = 0$** then $Z_{\mathbb{P}^*}$ is reduced to a singleton thus compact.

### Proposition

*Let $(X_t)_{t\in[0,T]}$ be a Lévy process under $\mathbb{P}$ with characteristic triplet $(b, \sigma^2, \nu)$. The set of densities $Z_{\mathbb{P}^*} = \left\{ \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} : \mathbb{P}^* \in \mathcal{P} \right\}$ is compact if and only if $\nu = 0$. In this case, $Z_{\mathbb{P}^*}$ is reduced to a singleton*

$$Z_{\mathbb{P}^*} = \left\{ \exp\left( -\frac{\sigma^2\theta^2}{2} T + \sigma\theta B_T \right) \right\}.$$

*with $\theta = (-b - \frac{\sigma^2}{2})/\sigma^2$ and $(B_t)_{t\in[0,T]}$ a standard Brownian motion.*

# Approximation of the dual problem

**Relax the constraint** $\sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[\varphi(z)(H-z)_+] \leq \tilde{V}_0$.

We fix a measure - the Esscher measure $\mathbb{P}^+$ defined by

$$\frac{\mathrm{d}\mathbb{P}^+}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]}$$

where $\theta$ is solution to

$$b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{\theta x}(e^x - 1) - x\mathbb{1}_{|x|\leq 1}\right) \nu(\mathrm{d}x) = 0.$$

**With respect to previous results** the Esscher measure corresponds to the choice $\phi(x) = \theta x$. The approximation of our problem writes

$$\hat{\varphi}(z) = \mathrm{argmin}_{\varphi(z)\in\mathcal{R}} \mathbb{E}[(1-\varphi(z))(H-z)_+]$$

$$\text{s.t. } \mathbb{E}^+[\varphi(z)(H-z)_+] \leq \tilde{V}_0.$$

# Solution of the approximated problem

**Theorem**

*The solution to this new optimisation problem is*

$$\hat{\varphi}(z) = \mathbb{1}_{\{e^{-\theta X_T} > c(z)\}}$$

*where*

$$c(z) = \inf \left\{ u \geq 0 : \mathbb{E}^+ \left[ (H - z)_+ \mathbb{1}_{\{e^{-\theta X_T} > u\}} \right] \leq \tilde{V}_0 \right\}.$$

**Approximation of $f(z)$**

$$\hat{f}(z) = z + \frac{1}{1-a} \mathbb{E}[(1 - \hat{\varphi}(z))(H - z)_+].$$

**The approximated value of the CVaR minimisation**

$$\min_{(V_0, Y) \in \mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(L(V_0, Y)) = \min_{z \in \mathbb{R}} \hat{f}(z).$$

# Example - hedging of a call option

**Call option** strike $K$, maturity $T$. Payoff ( $r$ risk-free interest rate)

$$(S_T - Ke^{-rt})_+.$$

## Theorem

*Let $(X_t)_{t\in[0,T]}$ be a Lévy process under $\mathbb{P}$ with characteristic triplet $(b,\sigma^2,\nu)$ and let $\mathbb{P}^+$ denote the Esscher martingale measure with parameter $\theta$. The solution to the approximated problem of the* CVaR *minimisation is given, with respect to the sign of $\theta$ by*

- *If $\theta < 0$*

$$\min_{(V_0,Y)\in\mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(H - V_T) = \min_{z\in(0,z_c)} \Big\{ z + \frac{1}{1-a}\Big(\mathbb{E}[(S_T - K(z))_+] - \mathbb{E}[(S_T - c_1(z))_+] - (c_1(z) - K(z))\mathbb{P}(S_T > c_1(z))\Big)\Big\}$$

*where $K(z) = Ke^{-rT} + z$, $z_c > 0$ s.t. $\mathbb{E}^+[(S_T - K(z_c))_+] = \tilde{V}_0$ and*

$$c_1(z) = \inf\Big\{u \geq K(z), \mathbb{E}^+[(S_T - u)_+] + (u - K(z))\mathbb{P}^+(S_T > u) \leq \tilde{V}_0\Big\}.$$

# Example - hedging of a call option

## Theorem

- *If* $\theta > 0$

$$\min_{(V_0,Y)\in\mathcal{A}_{\tilde{V}_0}} \widehat{\mathrm{CVaR}}_a(H - V_T) = \min_{z\in(0,z_c)} \Big\{ z + \frac{1}{1-a}\Big(\mathbb{E}[(S_T - c_1(z))_+]$$

$$+(c_1(z) - K(z))\mathbb{P}(S_T > c_1(z))\Big)\Big\}$$

*where* $K(z) = Ke^{-rT} + z$, $z_c > 0$ *s.t.* $\mathbb{E}^+[(S_T - K(z_c))_+] = \tilde{V}_0$ *and*

$$c_1(z) = \inf\Big\{ u \geq K(z), \mathbb{E}^+[(S_T - K(z))_+] - \mathbb{E}^+[(S_T - u)_+]$$

$$-(u - K(z))\mathbb{P}^+(S_T > u) \leq \tilde{V}_0 \Big\}.$$

**Essential** Objective function expressed only with:

$$\mathbb{E}[(S_T - x)_+], \mathbb{P}(S_T > x), \mathbb{E}^+[(S_T - x)_+], \mathbb{P}^+(S_T > x)$$

can be numerically computed by Fast Fourier Transform (FFT).

# Particular example - the Merton model

**The process** discounted price at time $t$ is $S_t = S_0 e^{X_t}$ where

$$X_t = (\gamma - r)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i,$$

where $\gamma \in \mathbb{R}$, $(B_t)_{t\in[0,T]}$ a standard Brownian motion, $(N_t)_{t\in[0,T]}$ a Poisson process with intensity $\lambda$ and $(Y_i)_{i\geq 1}$ are i.i.d. Gaussian random variables $\mathcal{N}(m, \sigma^2)$.

**Characteristic function** $\Phi_t(u) = \mathbb{E}[e^{iuX_t}] = e^{t\psi(u)}$ where

$$\psi(u) = i(\gamma - r)u - \frac{\sigma^2}{2}u^2 + \lambda\left(\exp\left(imu - \frac{\delta^2}{2}u^2\right) - 1\right)$$

**Existence of the Esscher measure** depends on the existence of a real solution to

$$(\gamma - r) + \sigma^2\theta + \frac{\sigma^2}{2} + \lambda\left(e^{m(\theta+1)+\frac{\delta^2}{2}(\theta+1)^2} - e^{m\theta+\frac{\delta^2}{2}\theta^2}\right) = 0.$$

# Numerical example - the Merton model

| Market | Merton | FFT |
|---|---|---|
| $S_0 = 100$ | $\gamma = 0.1$ | $N = 4096$ |
| $r = 0.02$ | $\sigma = 0.3$ | $\Delta v = 0.25$ |
| $T = 0.5$ | $\lambda = 1$ | |
| $K = 110$ | $m = -0.1$ | $\Delta k = \frac{2\pi}{N\Delta v}$ |
| | $\delta = 0.2$ | $k_1 = -\frac{N}{2}\Delta k$ |

**Numerical result** By solving with these parameters (Newton-Raphson method) we obtain the value

$$\theta = -0.352$$

so we have the corresponding Esscher measure thus

- we can provide analytic expression of the characteristic function under an equivalent measure
- this analytic form can be inverted numerically by FFT
- we can compute the expressions in the theorem.

# Numerical example - the Merton model

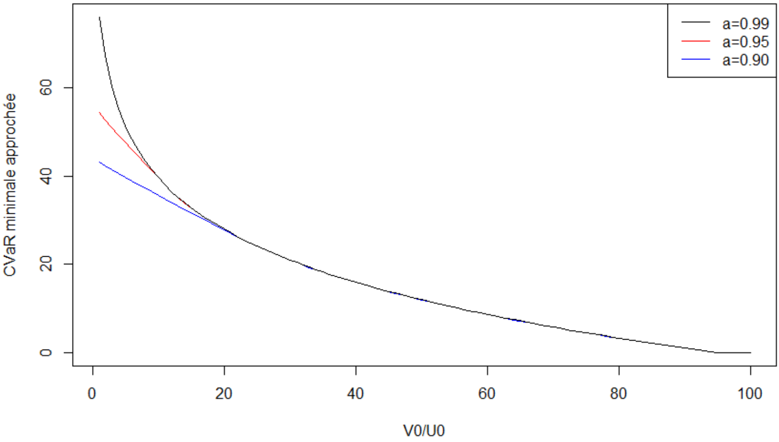


Figure : Approximation of the Minimal CVaR with respect to the proportion of the initial capital allowed $\tilde{V}_0/U_0$, Merton model.

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