Quasimartingales functionals and their Doob-Meyer decompositions

Iulian Cîmpean

Simion Stoilow Institute of Mathematics of the Romanian Academy

Joint work with **Lucian Beznea**

Atelier de travail en Stochastique et EDP

14, 15 Septembre 2018, Bucarest, Roumanie

Motivation

Semimartingale = martingale + BV process ((pre)locally)

Statement

i) Knowing that a process is a semimartingale is very nice

ii) In most of the cases i) is not enough!

We need

$$Z_t = M_t + A_t, t \geq 0$$

$$Z_t = M_t + A_t, t \geq 0$$

1. Ito calculus. $f \in C^2(\mathbb{R})$

$$f(Z_t) - f(Z_0) = \underbrace{\int_0^t f'(Z_s) dM_s}_{\text{martingale}} + \underbrace{\int_0^t f'(Z_s) dA_s + \frac{1}{2} \int_0^t f''(Z_s) d[M]_s}_{\text{BV process}}$$

If we are interested in expectation

$$\mathbb{E}\{f(Z_t)\} - \mathbb{E}\{f(Z_0)\} = \mathbb{E}\{\int_0^t f'(Z_s) dA_s\} + \frac{1}{2}\mathbb{E}\{\int_0^t f''(Z_s) d[M]_s\}$$

2. Skorohod decomposition. Let D a domain in $\mathbb{R}^d$ and $X(t), t > 0$ be the diffusion associated to (the closure of)

$$\mathcal{E}(u,v) = \int_D \nabla u \cdot \nabla v \, dx, \text{ for } u, v \in C_0^1(\mathbb{R}^d)$$

- If D is Caccioppoli then X is a semimartingale.

$$X(t) - X(0) = \underbrace{M_t}_{B(t)} + \underbrace{A_t}_{\int_0^t n(X_s) dL_s}, t \geq 0$$

- More general reflected diffusions in both finite and infinite dimensions!

3. BV functions.

Definition. $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a BV function if for any bounded open set $V \subset \mathbb{R}^d$

$$\int_V \frac{\partial v}{\partial x_i} \rho \, dx \leq ct \cdot |v|_\infty \quad \text{for all } v \in C^1_0(V), i = \overline{1:n}.$$

If $\pi_i(x) := x_i$, this rewrites as

$$\int \nabla \pi_i \cdot \nabla v \, \rho \, dx =: \mathcal{E}_\rho(\pi_i, v) \leq ct \cdot |v|_\infty$$

which means that $\pi_i(X^\rho)$ is a semimartingale.
Actually

$$\pi_i(X^\rho(t)) - \pi_i(X^\rho(0)) = B_i(t) + A_i(t)$$

$A_i(t)$ is in Revuz correspondence with the measure $\frac{\partial \rho}{\partial x_i}$.

- Works in infinite dimensions!

Dirichlet forms

- [Z. M. Ma and M. Röckner, Springer, 1992]

$$\mathcal{E}(u,v) = \sum_{i,j=1}^{d} \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i}^{d} \int u \frac{\partial v}{\partial x_i} d_i dx + \sum_{i}^{d} \int v \frac{\partial u}{\partial x_i} b_i dx +$$

$$+ \int uvc dx + \int \int_{x \neq y} (u(x) - u(y))(v(x) - v(y)) k(x,y) dxdy,$$

Formal definition of a Dirichlet form

Let $\mathcal{E}$ be a bilinear form on $L^2(E,\mu)$ with dense domain $\mathcal{F}$;
$\mathcal{E}_\alpha(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \alpha(\cdot,\cdot)_{L^2}$, $\alpha > 0$.

$(\mathcal{E},\mathcal{F})$ is a coercive closed form if:

- $\mathcal{E}(u,u) \geq 0$.
- $\mathcal{F}$ is a Hilbert space w.r.t. $\mathcal{E}_1(u,u)^{\frac{1}{2}}$.
- $|\mathcal{E}_1(u,v)| \leq const \cdot \mathcal{E}_1(u,u)^{\frac{1}{2}}\mathcal{E}_1(v,v)^{\frac{1}{2}},\ u,v \in \mathcal{F}$.

$\mathcal{E}(u,v) = (-Lu,v)_{L^2},\ u \in D(A), \quad P_t := e^{tL}$.

Dual structure: $\widehat{L}$, $\widehat{P}_t$, $t \geq 0$.

- Semi-Dirichlet form if $0 \leq P_t f \leq 1$ for all $0 \leq f \leq 1$.
- Dirichlet form if $0 \leq P_t f,\ \widehat{P}_t f \leq 1$ for all $0 \leq f \leq 1$.
- Lower-bounded (semi) Dirichlet form if there exists $\alpha > 0$ s.t. $(\mathcal{E}_\alpha,\mathcal{F})$ is a (semi) Dirichlet form.

Markov process associated to semi-Dirichlet forms

Assume that the (lower bounded) semi-Dirichlet form is quasi-regular.

- $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ is a right Markov process on E.

- $P_t f(x) = \mathbb{E}^x f(X_t)$, $t \geq 0$ its transition function;
$P_t^\alpha := e^{-\alpha t} P_t$.

- $u : E \to [0, \infty]$ is called α-excessive if $P_t^\alpha u \leq u$ and
$P_t^\alpha u \to u, t \to 0$.

Returning to semimartingales...

Q1: For which $u \in \mathcal{F}$ it follows that $u(X)$ is a real valued semimartingale?

If $u(X)$ is a semimartingale then $u(X) = M + A$.

Q2: Can we identify M and A merely in terms of $\mathcal{E}$ and u, and maybe compute

$$\mathbb{E}^x \int_0^t f(X_s) dA_s \quad \text{or} \quad \mathbb{E}^x \int_0^t f(X_s) d[M]_s ?$$

Symmetric case: many contributors

- **R. F. Bass, P. Hsu** (1990), The semimartingale structure of reflecting Brownian motion, PAMS.
- **R.J. Williams and W.A. Zheng** (1990), On reflecting Brownian motion—a weak convergence approach, AIHP.
- **E. Pardoux, R. J. Williams** (1994), Symmetric Reflected Diffusions, AIHP.
- **Z. Q. Chen**, (1993) On reflecting diffusion processes and Skorokhod decompositions, PTRF.
- **Z. Q. Chen, P. J. Fitzsimmons, R. J. Williams** (1993), Reflecting Brownian motions: quasimartingales and strong Caccioppoli sets, Pot. Anal.

- **M. Fukushima**
  - (1996) Distorted Brownian motions and BV functions, Trends in probability and related analysis,
  - (1999) On semi-martingale characerizations of functionals of symmetric Markov processes, EJP,
  - (2000) BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space, JFA,
  - (2001) with **M. Hino**: On the space of BV functions and a related stochastic calculus in infinite dimensions JFA.

- (2012) **M. Röckner, R.-C. Zhu and X.-C. Zhu**, The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple, The Anals of Prob.

Common tool: Fukushima decomposition!

Q1

- An increasing sequence of closed sets $F_n \subset E$, $n \geq 1$ is called a *nest* if $\bigcup_n \{v \in \mathcal{F} : v = 0 \text{ on } F_n^c\}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$.

Theorem

Let $\mathcal{E}$ be a lower-bounded semi-Dirichlet, $u \in \mathcal{F}$ and assume that there exists a "nest" $(F_n)_{n\geq 1}$ and constants c_n such that

$$(*) \quad |\mathcal{E}(u, v)| \leq c_n \|v\|_\infty \text{ for all bounded } v \in \mathcal{F}, \quad v = 0 \text{ on } F_n^c.$$

Then $u(X)$ is a semimartingale.

Proof: No Fukushima decomposition!

- $u(X)$ is a (local) quasimartingale (difference of supermartingales) by showing that u has finite variation w.r.t. (P_t) (new analytical object!)

Beznea, L., Cîmpean, I.: *Quasimartingales associated to Markov processes*, **TAMS (2018)**

Q2

Assume (*) so that $u(X) = M + A$.

Theorem

There exist two smooth measures μ and ν (signed) such that for all $v \in \bigcup_n \{v \in \mathcal{F} : v = 0 \text{ on } F_n^c\}$

$$\nu(v) = \mathcal{E}(u, v)$$

$$\mu(v) = 2\mathcal{E}(u, uv) - \mathcal{E}(u^2, v)$$

and for all bounded f and v α-co-excessive, $\alpha > 0$

$$\mathbb{E}_{v \cdot m}\{\int_0^t f(X_s) dA_s\} = \int_0^t \nu(f \widehat{P}_s v) ds$$

$$\mathbb{E}_{v \cdot m}\{\int_0^t f(X_s) d[M]_s\} = \int_0^t \mu(f \widehat{P}_s v) ds$$

Proof: Doob-Meyer decomp. + Revuz correspondence

Example

Let $b : (-1, 1) \to \mathbb{R}, b(x) = \sqrt{x+1}$ and set

$$\mathcal{E}(u, v) = \int_{-1}^{1} u'v'dx + \int_{-1}^{1} bu'vdx, \quad u, v \in \mathcal{F} = H_0^1(-1, 1)$$

$$\mathrm{L}u = u'' - bu'$$

Then $(\mathcal{E}, \mathcal{F})$ is a quasi-regular lower-bounded semi-Dirichlet form on $L^2(-1, 1)$, which is not Dirichlet: $\widehat{P}_t 1 \leq 1$ does NOT hold.
Take $u(x) = |x|(x^2 - 1),\ x \in (-1, 1)$. Then $u \in \mathcal{F}$

$$\nu(\cdot) = \mathcal{E}(u, \cdot) = 2\delta_0 + fdx$$

$$f(x) = \sqrt{x+1}(x^2 - 1)sgn(x) + (2x - 6)|x| \text{ and}$$

$$\mu(v) = 2\mathcal{E}(u, uv) - \mathcal{E}(u^2, v) = g(x)dx$$

$$g(x) = 4x^2 + (x^2 - 1)[x^2 + 2x^2 - 2x\sqrt{x+1}(5x^2 - 1) - 1]$$

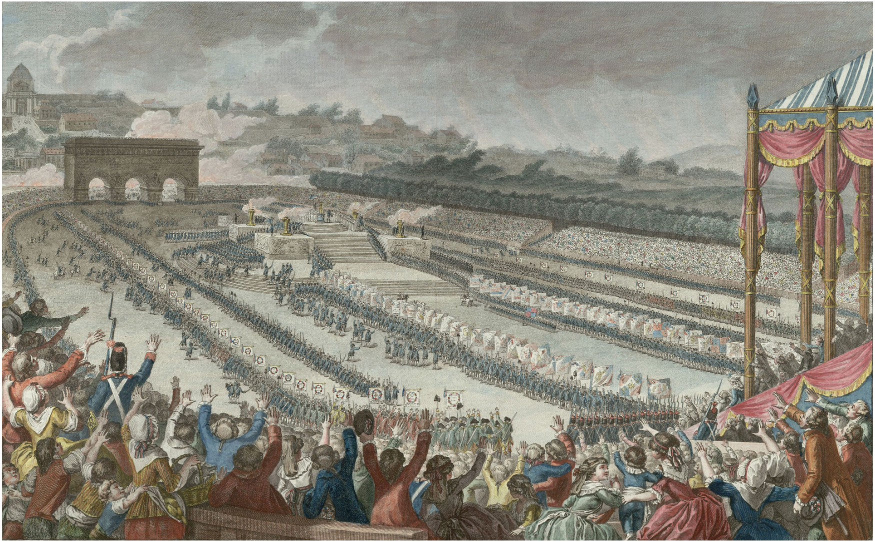
Work in progress

Partially, the previous results remain valid for generalized Dirichlet forms (no sector condition)!

100 years ago **1 December 1918**

Photo: Samoila Sturza

228 years ago **14 July 1790**



Painting: C. Monet

Thank You!