Free Functional Inequalities on the Circle

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The Plan

- Classical Poincare inequality
- Passage to the free Poincare inequality via random matrices
- Free inequalities on the circle
- Back to the classical case

Poincaré and transportation inequality on manifolds

Assume that M is a Riemannian manifold and $\mu(dx) = e^{-V(x)}dx$.
The Poincaré inequality for the base measure μ states that there exists a constant $\rho > 0$ such that

$$\rho \mathrm{Var}_\mu(\phi) \leq \int |\nabla \phi|^2 \, d\mu \qquad (P(\rho))$$

for any smooth function $\phi : M \to \mathbb{R}$.
The constant ρ can also be interpreted as the spectral gap of the operator $L = -\Delta + \nabla V \cdot \nabla$.
If the Bakry-Emery condition $\mathrm{Ric} + \mathrm{Hess} V \geq \rho$, then $P(\rho)$ holds true.

The transportation inequality $T(\rho)$ states that

$$\rho W_2^2(\mu,\nu) \leq H(\mu|\nu).$$

$H(\mu|\nu) = \int \frac{d\mu}{d\nu} \log(\frac{d\mu}{d\nu}) d\nu$ and
$W_2^2(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \iint d(x,y)^2 \pi(dx,dy)$, with $\Pi(\mu,\nu)$: probability on $M \times M$ with marginals μ and ν
Again, if the Bakry-Emery holds true, then, transportation holds with the same constant ρ.
Otto and Villani, JFA 2001, show that transportation implies the Poincaré.

Random Matrix Models

On matrices consider the probability measure (on real or unitary matrices)

$$\mathcal{P}_Q^n(dA) = \frac{1}{C_n(Q)} e^{-n\mathrm{Tr_n Q(A)}} dA.$$

The distribution of the eigenvalues is given by:

$$\Lambda_n(dx) = \frac{1}{Z_n(Q)} e^{-n\sum_{i=1}^n Q(x_i)} \prod_{1\le i<j\le n} |x_i - x_j|^2 \prod_{i=1}^n dx_i$$

$$= \frac{1}{Z_n(Q)} \exp\left(-n^2\left(\frac{1}{n}\sum_{i=1}^n Q(x_i) - \frac{1}{n^2}\sum_{1\le i\neq j\le n} \log|x_i - x_j|\right)\right) \prod_{i=1}^n dx_i$$

$$= \frac{1}{Z_n(Q)} e^{-n^2\left(\int Q(t)\eta_n(x)(dt) - \iint_{t\neq s} \log|t-s|\eta_n(x)(dt)\eta_n(x)(ds)\right)} \prod_{i=1}^n dx_i$$

where $Z_n(Q) = d_n C_n(Q)$, d_n depending only on n.

$$\eta_n(x) = \frac{1}{n}\sum_{k=1}^n \delta_{x_i} \xrightarrow[n\to\infty]{a.s.} \mu_Q$$

Equilibrium Measures and Free Entropy

Q smooth on a closed subset K of $\mathbb{C}$ such that $\lim_{|x|\to\infty} \frac{Q(x)}{\log(1+|x|^2)} = \infty$ if K is unbounded.
Cases we consider $K = \mathbb{R}$, S the unit circle or $\mathbb{C}$.

$$E_Q(\mu) = \int Q(x)\mu(x) - \iint \log|x-y|\mu(dx)\mu(dy).$$

There is a unique probability measure μ_Q such that

$$E_Q := E_Q(\mu_Q) = \inf_{\mu\in\mathcal{P}(K)} E_Q(\mu).$$

In addition, μ_Q has compact support.

Variational Characterization

The variational characterization of μ_Q:

$$Q(x) \geq 2 \int \log|x-y| \mu_Q(dx) + C \quad \text{with equality for } x \in supp(\mu).$$

If $K = \mathbb{R}$ and the support of μ is a union of intervals, then for a.e. $x \in supp(\mu)$:

$$Q'(x) = \int \frac{2}{x-y} \mu_Q(dx).$$

For the circle case $S = [-\pi, \pi)$, a.s. on the support of μ_Q

$$Q'(x) = \int \cot\left(\frac{x-y}{2}\right) \mu_Q(dx).$$

The relative free entropy is defined as

$$E_Q(\mu|\mu_Q) = E_Q(\mu) - E_Q(\mu_Q).$$

It is always positive, unless $\mu = \mu_Q$.

If $Q(x) = x^2/2$ on $\mathbb{R}$, then the minimizer of the free entropy μ_Q is given by the semicircular law

$$\mu_Q(dx) = \frac{1}{2\pi} \mathbb{1}_{[-2,2]}(x) \sqrt{4 - x^2} dx.$$

with $E_Q = 3/4$.
If $Q = 0$ on S, then the equilibrium measure μ_Q is the Haar measure on S.

Free Talagrand on $\mathbb{R}$

Theorem (Biane & Voiculescu, GAFA 2003)

If $Q(x) = x^2/2$ *then*

$$\frac{1}{2} W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Theorem (Hiai & Ueda & Petz, PTRF 2004)

If $Q''(x) \geq 2\rho$, *then*

$$\rho W_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Still on $\mathbb{R}$

Without using any random matrices.
Ledoux&P. JFA 2009, TRAM, 2013,
P. JFA 2013
Houdre&P., IMRN, 2014,

Theorem

If $Q''(x) \geq 2\rho$ is convex for $\rho > 0$, then

$$E(\mu, \mu_Q) \leq \frac{1}{4\rho} I_Q(\mu|\mu_Q).$$

If μ_Q has support $[-2, 2]$, then the following Poincaré inequality:

$$\rho \iint_{[-2,2]^2} \left(\frac{\phi(x) - \phi(y)}{x - y}\right)^2 \frac{(4 - xy)dxdy}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} \leq \int_{-2}^{2} (\phi'(x))^2 \mu_Q(dx)$$

Random matrix limit

$\mathbb{P}_n(dM) = \frac{1}{Z_n^Q} e^{-nTrQ(M)} \mathcal{H}(dM)$ on $n \times n$ unitary matrices, where $\mathcal{H}$ is the Haar measure on the unitary group, $U(n)$.

$Ric = n(n-1)/2$ except in one direction where it is 0.

The fix is to consider the subgroup of matrices of determinant 1 and on this $Ricci = n(n-1)/2$.

Random matrix approximations gives

Theorem (Hia-Petz-Ueda, PTRF 2004)

If $Q'' \geq 2\rho - 1/2$, then the transportation inequality

$$\rho W_2^2(\mu, \mu_Q) \leq E(\mu|\mu_Q).$$

where W_2 is the standard Wasserstein distance.

$LSI(\rho)$ holds in the form

$$E(\mu|\mu_Q) \leq \int (H\mu - Q')^2 d\mu - \left(\int Q' d\mu \right)^2 := I(\mu|\mu_Q).$$

Using random matrices one can get the free transportation, *LS* but not *HWI*.

There is also a free Poincaré inequality which can be deduced from the matrix models in some particular cases of the potential Q. Apply the classical Poincare to $F(M) = Tr\, f(M)$ on $U(N)/S$

$$2\rho\, Var_{\mathbb{P}_n}(F) \leq \int |\nabla F|^2 d\mathbb{P}_n - \left| \int \langle \nabla F, e \rangle d\mathbb{P}_n \right|^2$$

which becomes for $n \to \infty$ (from fluctuations of random matrices)

$$2\rho \iint \left| \frac{f(z) - f(w)}{z - w} \right|^2 \alpha(dz)\, \alpha(dw) \leq \int |f'|^2 d\mu_Q - \left| \int f' d\mu_Q \right|^2 .$$

Free Poincaré on the Circle

Definition (P., Advances in Math, 2018)

μ on S satisfies a free Poincaré inequality, $P(\rho)$, $\rho > 0$ if

$$2\rho \iint \left| \frac{f(z) - f(w)}{z - w} \right|^2 \alpha(dz)\,\alpha(dw) \leq \int |f'|^2 d\mu - \left| \int f' d\mu \right|^2$$

for any smooth $f : S \to \mathbb{C}$ where α is the Haar measure.

If $\mu = \alpha$, and

$$(\mathcal{N}\phi)(z) = \int \frac{(\phi'(z) - \phi'(w))(z + w)}{i(z - w)} \alpha(dw)$$

$$(\mathcal{E}\phi)(z) = -2 \int \log|z - w| \phi(w) \alpha(dw)$$

then $\mathcal{N} z^{\pm n} = n z^{\pm n}$, $\mathcal{E} z^{\pm n} = \frac{1}{n} z^{\pm n}$ $(n \neq 0)$ and $\mathcal{N}^2 f = \mathcal{L} f = -f''$.

$$P(1/2) \iff \langle \mathcal{N} f, f \rangle \leq \langle \mathcal{L} f, f \rangle.$$

This is sharp!

If μ has $P(\rho)$, then necessarily its support is the whole S. Moreover, if μ has a density w w.r.t. α, then w must be positive on S.

Theorem

If $Q'' + 1/2 \geq \rho$, then

$$2\rho \iint \left|\frac{f(z)-f(w)}{z-w}\right|^2 \alpha(dz)\,\alpha(dw) \leq \int |f'|^2 d\mu_Q - \left|\int f' d\mu_Q\right|^2 .$$

$$\langle \mathcal{N}\phi, \phi\rangle = \langle \mathcal{E}(\phi' - \int \phi' d\mu_Q), \phi' - \int \phi' d\mu_Q\rangle.$$

$$\langle \mathcal{E}(\phi' - \int \phi' d\mu_Q), \phi' - \int \phi' d\mu_Q\rangle \leq \langle \phi' - \int \phi' d\mu_Q, \phi' - \int \phi' d\mu_Q\rangle$$

$$= \left\langle \frac{1}{1-\mathcal{N}Q}\left(\phi' - \int \phi' d\mu_Q\right), \phi' - \int \phi' d\mu_Q \right\rangle_{\mu_Q}$$

$$= \int \frac{\left|\phi' - \int \phi'\, d\mu_Q\right|^2}{1-\mathcal{N}Q} d\mu_Q \leq \frac{1}{2\rho}\int \left|\phi' - \int \phi'\, d\mu_Q\right|^2 d\mu_Q,$$

The modified Wasserstein

For a measure μ on S,

$$\bar{\mu}(A) = \sum_{n\in\mathbb{Z}} \mu(\exp(A \cap [2n\pi, 2(n+1)\pi)).$$

$\bar{\mu}_u$ is simply the restriction of $\bar{\mu}$ to the interval $[u, u + 2\pi)$.

Definition

For $\mu, \nu \in \mathcal{P}(S)$,

$$\mathcal{W}_2(\mu,\nu) = \sup_{u,v\in[0,2\pi]} \left\{ W_2(\bar{\mu}_u, \bar{\nu}_v) : \int x\, \bar{\mu}_u(dx) = \int x\, \bar{\nu}_v(dx), \right\},$$

where $W_2(\bar{\mu}_u, \bar{\nu}_v)$ is the standard Wasserstein distance on the real line.

Proposition

1. $W_2(\mu,\nu) \leq \mathcal{W}_2(\mu,\nu)$.
2. $\mathcal{W}_2(\delta_a,\delta_b) = \infty$ *unless* $a = b$.
3. $\mathcal{W}_2(\delta_a,\alpha) = \left(\frac{2\pi^3}{3}\right)^{1/3}$ *so* $\mathcal{W}_2(\delta_a,\delta_b) > \mathcal{W}_2(\delta_a,\alpha) + \mathcal{W}_2(\alpha,\delta_b)$.
4. *If* $\mu,\nu \in \mathcal{P}_{as}(S)$, *there exists* $t \in [0,2\pi]$ *s.t.* $\int x\,\bar{\mu}_t(dx) = \int x\,\bar{\nu}_t(dx)$. *Moreover, given* $u \in [0,2\pi]$, *there exists* $v \in [0,2\pi]$ *such that* $\int x\,\bar{\mu}_u(dx) = \int x\,\bar{\nu}_v(dx)$.
5. *If* $\mu,\zeta,\nu \in \mathcal{P}_{as}(S)$, *then* $\mathcal{W}_2(\mu,\nu) \leq \mathcal{W}_2(\mu,\zeta) + \mathcal{W}_2(\zeta,\nu)$.
6. *On* $\mathcal{P}_{as}(S)$, *the topology induced by* $\mathcal{W}_2$ *is the topology of weak convergence.*

Theorem

If $Q'' + 1/2 \geq \rho$, then

$$\frac{\rho}{2}\mathcal{W}_2^2(\mu, \mu_Q) \leq E_Q(\mu|\mu_Q).$$

Even for $Q \equiv 0$ ($\mu_Q = \alpha$) this is not sharp! I conjecture that in this case

$$\frac{1}{2}\mathcal{W}_2^2(\mu, \alpha) \leq E_Q(\mu|\alpha).$$

Back to the classical case

Conjecture

Assume M is a Riemannian manifold and G a Lie Group acting isometrically on M. If $\mu(dx) = e^{-V(x)}dx$ with $HessV + Ric \geq \rho$, then

$$\rho Var_{\mu}(f) \leq \int |\nabla f|^2 d\mu - \left| \int D_G f d\mu \right|^2$$

In the case μ is invariant with respect to the action, the last integral is 0.

An interesting challenge is to see a similar modification for the transportation.

A weak preliminary result on S

Proposition

If μ is a probability measure on S such that for some positive constant $C > 0$, $\frac{1}{C} \leq \frac{d\mu}{d\alpha} \leq C$,then for some $K > 0$

$$K\left(\int f^2 d\mu - \left(\int f d\mu\right)^2\right) \leq \int (f')^2 d\mu - \left(\int f' d\mu\right)^2 .$$

The proof

$$
\begin{aligned}
\int f^2 d\mu - \left(\int f d\mu\right)^2 &= \frac{1}{2} \iint (f(x) - f(y))^2 \mu(dx)\mu(dy) \\
&\leq C^2 \frac{1}{2} \iint (f(x) - f(y))^2 \alpha(dx)\alpha(dy) \\
&= C^2 \left(\int f^2 \alpha(dx) - \left(\int f d\alpha\right)^2 \right) \\
\text{(classical Poincare for } \alpha\text{)} &\leq C^2 \int (f')^2 d\alpha \\
\left(\int f' d\alpha = 0\right) \qquad &= C^2 \left(\int (f')^2 d\alpha - \left(\int f' d\alpha\right)^2 \right) \\
&= \frac{C^2}{2} \iint (f'(x) - f'(y))^2 \alpha(dx)\alpha(dy) \\
&\leq \frac{C^4}{2} \iint (f'(x) - f'(y))^2 \mu(dx)\mu(dy) \\
&= C^4 \left(\int (f')^2 d\mu - \left(\int f' d\mu\right)^2 \right)
\end{aligned}
$$

Log Sobolev: $H(\nu|\mu) \le K\left(\int (\phi')^2 d\mu - \left(\int \phi' d\mu\right)^2\right)$

$$\text{Proof: } \int \phi \log\left(\frac{\phi}{\int \phi d\mu}\right) d\mu = \inf_{t\ge 0} \int (\phi \log(\phi) - \phi \log(t) - \phi + t) d\mu$$

$$\le C \inf_{t\ge 0} \int (\phi \log(\phi) - \phi \log(t) - \phi + t) d\alpha$$

$$= C \int \phi \log\left(\frac{\phi}{\int \phi d\alpha}\right) d\alpha$$

(Log-Sobolev for α)

$$\le C \int (\phi')^2 d\alpha$$

$$= C\left(\int (\phi')^2 d\alpha - \left(\int \phi' d\alpha\right)^2\right)$$

$$= C \iint (\phi'(x) - \phi'(y))^2 \alpha(dx)\alpha(dy)$$

$$\le C^3 \iint (\phi'(x) - \phi'(y))^2 \mu(dx)\mu(dy)$$

THANK YOU!