Well-posedness of semilinear dissipative SPDEs with singular drift and semimartingale noise

Carlo Marinelli

Department of Mathematics
University College London

and

IZKS, Universität Bonn

goo.gl/4GKJP

Based on joint work with Luca Scarpa

September 2018

The problem

Consider the SPDE

$$du + Au\,dt + \beta(u)\,dt = B(u)\,dZ, \qquad u(0) = u_0,$$

with

- A linear maximal monotone on $H := L^2(D)$, $D \subset \mathbb{R}^n$ smooth bounded domain;
- $\beta : \mathbb{R} \to \mathbb{R}$ increasing;
- Z semimartingale with values in a (separable) Hilbert space K;
- $B : \Omega \times \mathbb{R}_+ \times H \to \mathscr{L}(K, H)$ Lipschitz continuous.

– Questions –

(a) Notion of solution
(b) Existence of solutions
(c) Uniqueness (conditional vs. unconditional)
(d) Continuity of the map $u_0 \mapsto u$
(e) Regularity properties

Motivation: random vs. non-random evolution equations

Let E be any Banach space, A nonlinear m-accretive on E, $\varphi \in L_1(0, T; E)$.

Theorem. (Crandall-Liggett) The Cauchy problem

$$\frac{du}{dt} + Au \ni \varphi, \qquad u(0) = u_0$$

is well-posed in the CL-mild sense.

- If E is Hilbert space, CL-mild solution is SOLA. (Bénilan-Brézis)
- If $A = A_0 + f$, A_0 linear m-accretive and f m.m. graph in $\mathbb{R} \times \mathbb{R}$, the problem is reduced to summing m-accretive operators.

✗ All proofs break down if φ is not a function.

⇝ No well-posedness theory can be deduced for

$$du + Au\,dt = B\,dW, \qquad u(0) = u_0.$$

✔ A stochastic well-posedness theory is available in some cases, e.g.

- $A : V \to V'$, $V \hookrightarrow H \hookrightarrow V'$ (Pardoux, Krylov-Rozovskiĭ, Győngy)
- special settings, e.g. $A = -\Delta\psi$, $E = H^{-1}$ (Barbu-Da Prato-Röckner)

Preliminaries: stochastic integration

Definition. A positive increasing adapted process C is a control process for Z if

$$\text{(D)} \qquad \mathbb{E}\big(Y \cdot Z\big)^{*2}_{T-} \leq \mathbb{E} C_{T-}\big(\|Y\|^2 \cdot C\big)_{T-}$$

for every elementary $\mathscr{L}(K, G)$-valued Y, separable G, stopping time T.

Theorem. Z adapted càdlàg K-valued process is a semimartingale if and only if it admits a control process.

Definition. A strongly predictable $\mathscr{L}(K, G)$-valued process Y is integrable with respect to Z if there exists a control process C for Z such that the process

$$\lambda^C(Y) := C\big(\|Y\|^2 \cdot C\big)$$

is finite.

The construction of $Y \cdot Z$ implies that the pre-stopped Doob inequality (D) remains true for every control process C and every Y with $\lambda^C(Y)$ finite.

Assumptions

- A is variational: $A \in \mathscr{L}(V, V')$ isomorphism, V Hilbert space densely, continuously, and compactly embedded in H, hence

$$\langle Av, v \rangle \geq c\|v\|_V^2;$$

- $A \cap (V \times H)$ admits m-accretive extension A_1 to $L^1(D)$;
- A_1 has sub-Markovian resolvent $(I + \lambda A_1)^{-1}$;
- $(I + \lambda A_1)^{-m} \in \mathscr{L}(L^1(D), L^\infty(D))$ for some $m \in \mathbb{N}$.

- β is an odd maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, $\mathsf{D}(\beta) = \mathbb{R}$, $0 \in \beta(0)$

$\rightsquigarrow$ $\exists$ l.s.c. convex even positive j such that $\beta = \partial j$, $j(0) = 0$,

$$\lim_{x \to \infty} \frac{j^*(x)}{|x|} = \infty.$$

- $B : \Omega \times \mathbb{R}_+ \times H \to \mathscr{L}(K, H)$ is such that $B(\cdot, \cdot, u)$ is strongly predictable for every $u \in H$, and $\forall C \in \mathscr{C}(Z)$ there exists $L \in \mathscr{A}_+^{\uparrow}$ r.c. such that

$$\left\| B(u_-) - B(v_-) \right\|_{\mathscr{L}(K,H)}^2 \cdot C \leq \left(u - v \right)_-^{*2} \cdot L,$$

$$\left\| B(u_-) \right\|_{\mathscr{L}(K,H)}^2 \cdot C \leq \left(1 + u_-^{*2} \right) \cdot L$$

for every adapted càdlàg processes u, v.

Strong solutions

Let $T_0 \in \mathbb{R}_+$ be a fixed final time.

Definition. Let $T \leq T_0$ be a stopping time. A strong solution on $[\![0, T]\!]$ is a pair (u, ξ), with

- u adapted càdlàg H-valued,
- ξ adapted $L^1(D)$-valued,

such that

(a) $\mathbb{1}_{[\![0,T]\!]} u \in L^1(0, T_0; V)$, $\mathbb{1}_{[\![0,T]\!]} \xi \in L^1([0, T_0] \times D)$, $\xi \in \beta(u)$ a.e. in $[\![0, T[\![\times D$;

(b) $\mathbb{1}_{[\![0,T]\!]} B(u_-)$ is integrable with respect to Z;

(c) one has, as an identity in $V' \cap L^1(D)$,

$$u^T + \int_0^{\cdot \wedge T} Au(s)\, ds + \int_0^{\cdot \wedge T} \xi(s)\, ds = u_0 + \big(\mathbb{1}_{[\![0,T]\!]} B(u_-)\big) \cdot Z.$$

Well-posedness

Theorem. Let $u_0 \in L^0(\Omega, \mathscr{F}_0; H)$ and $T_0 \in \mathbb{R}_+$. Then

(a) there exists a strong solution (u, ξ) on $[0, T_0]$;

(b) the solution is the only one such that

$$u^*_{T_0} + \|u\|_{L^2(0,T_0;V)} + \|u\xi\|_{L^1([0,T_0]\times D)} < \infty \qquad \text{a.s.}$$

(c) u and ξ are optional;

(d) The solution map

$$\begin{aligned} L^0(\Omega; H) &\longrightarrow L^0(\Omega; D([0, T_0]; H) \cap L^2(0, T_0; V)) \\ u_0 &\longmapsto u \end{aligned}$$

is continuous.

Remarks.

(i) Unconditional well-posedness is open.

(ii) More "precise" results if Z is a Wiener process.

Main steps of the proof

1. Consider the equation

$$du_\lambda + A_\lambda u_\lambda \, dt + \beta_\lambda(u_\lambda) \, dt = G \, dZ, \qquad u_\lambda(0) = u_0,$$

   with
   - A_λ, β_λ Yosida approximations;
   - G independent of u_λ and taking values in $V_0 \hookrightarrow V \cap L^\infty(D)$;
   - $u_0 \in L^2(\Omega, \mathscr{F}_0; H)$.

2. A priori estimates on u_λ and $\beta_\lambda(u_\lambda)$, both pathwise and in expectation.

$\rightsquigarrow$ Compactness and passage to the limit in the equation solving

$$du + Au \, dt + \beta(u) \, dt = G \, dZ, \qquad u(0) = u_0.$$

3. Removal of all restrictions on the data by further limiting procedures and localization.

4. General case $G = B(u_-)$ by fixed point, construction of a local solution, and extension thereof.

Existence for an auxiliary problem

We first consider the simpler problem

$$\text{(G)} \qquad du + Au\,dt + \beta(u)\,dt = G\,dZ, \qquad u(0) = u_0,$$

with $G : \Omega \times [0, T_0] \to \mathscr{L}(K, V_0)$, V_0 separable Hilbert space continuously embedded in V and $L^\infty(D)$, $\mathbb{E}\lambda^C_{T_0-}(G) < \infty$, $\mathbb{E}\|u_0\|^2 < \infty$.

Proposition. There exists a unique strong solution u to (G).

Let

$$A_\lambda := \frac{1}{\lambda}(I - (I + \lambda A)^{-1}), \qquad \beta_\lambda := \frac{1}{\lambda}(I - (I + \lambda\beta)^{-1}), \quad \lambda > 0$$

be the Yosida regularizations of A and β, and consider

$$du_\lambda + A_\lambda u_\lambda\,dt + \beta_\lambda(u_\lambda)\,dt = G\,dZ, \qquad u_\lambda(0) = u_0.$$

Since A_λ and β_λ are Lipschitz-continuous, a unique classical solution

$$u_\lambda \in \mathbb{S}^2(0, T_0)$$

exists.

A priori estimates

By the assumptions on G it follows that $G \cdot Z \in L^\infty(0,T;V_0)$. The regularized equation can be written as

$$(u_\lambda - G\cdot W)' + A_\lambda u_\lambda + \beta_\lambda(u_\lambda) = 0 \quad \text{in } V'.$$

The integration-by-parts formula yields

$$\frac{1}{2}\left\|u_\lambda - G\cdot Z\right\|^2 + \int_0^\cdot \langle A_\lambda u_\lambda, u_\lambda - G\cdot Z\rangle \\ + \int_0^\cdot\int_D \beta_\lambda(u_\lambda)(u_\lambda - G\cdot Z) = \frac{1}{2}\|u_0\|^2.$$

Lemma. For every ω in a set of probability one there exists $N = N(\omega)$ such that

$$\left\|u_\lambda(\omega)\right\|_{L^\infty(0,T_0;H)} < N(\omega),$$

$$\left\|J_\lambda u_\lambda(\omega)\right\|_{L^2(0,T_0;V)} < N(\omega),$$

$$\left\|\beta_\lambda(u_\lambda(\omega))u_\lambda(\omega)\right\|_{L^1([0,T_0]\times D)} < N(\omega).$$

Compactness properties of u_λ and $\beta_\lambda(u_\lambda)$

Recall that $xy \leq j(x) + j^*(y)$, with equality if and only if $y \in \beta(x)$. Since

$$j(J_\lambda u_\lambda) + j^*(\beta_\lambda(u_\lambda)) = \beta_\lambda(u_\lambda) J_\lambda u_\lambda \leq \beta_\lambda(u_\lambda) u_\lambda, \qquad J_\lambda := (I + \lambda\beta)^{-1},$$

it follows by the symmetry of j that

$$\big\| j^*(|\beta_\lambda(u_\lambda)(\omega)|) \big\|_{L^1([0,T_0]\times D)} < N(\omega).$$

The hypothesis $\mathrm{D}(\beta) = \mathbb{R}$ implies $\displaystyle\lim_{|r|\to\infty} \frac{j^*(r)}{|r|} = +\infty$, hence, thanks to the criterion by de la Vallé-Poussin, $(\beta_\lambda(u_\lambda))$ is uniformly integrable. The Dunford-Pettis theorem then yields the

Lemma. $(\beta_\lambda(u_\lambda)(\omega))$ is relatively weakly compact in $L^1([0,T_0] \times D)$.

One has $y'_\lambda + A_\lambda u_\lambda + \beta_\lambda(u_\lambda) = 0$, $y_\lambda := u_\lambda - G \cdot Z \;\rightsquigarrow\; y'_\lambda$ is bounded in $L^1(0,T_0;V'_0)$. Since y_λ is bounded in $L^2(0,T_0;V)$, Simon's compactness criterion implies the

Lemma. $(u_\lambda(\omega))$ is relatively compact in $L^2(0,T_0;H)$.

Passage to the limit

Keep $\omega \in \Omega$ fixed. By the pathwise estimates, we have (extracting subsequences)

$$\begin{aligned} u_\lambda &\longrightarrow u \quad \text{strongly in } L^2(0, T_0; H), \\ u_\lambda &\longrightarrow u \quad \text{weakly* in } L^\infty(0, T_0; H), \\ J_\lambda u_\lambda &\longrightarrow u \quad \text{weakly in } L^2(0, T_0; V), \\ \beta_\lambda(u_\lambda) &\longrightarrow \xi \quad \text{weakly in } L^1([0, T_0] \times D). \end{aligned}$$

This is enough to pass to the limit in the approximated equation. Moreover, by the strong-weak closure of β we deduce

$$\xi \in \beta(u) \quad \text{a.e. in } (0, T) \times D$$

and, by the lower semicontinuity of convex integrals,

$$\xi u = j(u) + j^*(\xi) \in L^1((0, T) \times D)$$

Measurability of the processes u and ξ

The construction of u and ξ does not imply their measurability in ω, as the subsequences depend on ω.

However, energy estimates and a uniqueness argument imply that there exists a sequence (λ_n) independent of ω along which the previous convergences hold.

Using the Mazur lemma and lower semicontinuity of the norms, this implies the

Proposition. u and ξ are measurable adapted processes (in fact optional), and

$$u \in L^2(\Omega; L^\infty(0, T_0; H)) \cap L^2(\Omega \times [0, T_0]; V),$$

$$\xi \in L^1(\Omega \times [0, T_0] \times D),$$

$$\xi \in \beta(u) \quad \text{a.e. in } \Omega \times [0, T_0] \times D,$$

$$j(u) + j^*(\xi) \in L^1(\Omega \times [0, T_0] \times D).$$

A generalized Itô formula for the square of the H-norm implies that u is càdlàg.

Additive noise

Consider the problem

$$du + Au\,dt + \beta(u)\,dt \ni G\,dZ, \quad u(0) = u_0,$$

with $G : \Omega \times [0, T_0] \to \mathscr{L}(K, H)$ predictable, $\mathbb{E}\lambda^C_{T_0-}(G) < \infty$, $\mathbb{E}\|u_0\|^2 < \infty$.

Set $G^\varepsilon := (I + \varepsilon A)^{-m} G$, with $m \in \mathbb{N}$ such that

$$G^\varepsilon \in L^0(\Omega; L^0(0, T_0; \mathscr{L}(K, V_0))), \qquad \mathbb{E}\lambda^C_{T_0-}(G^\varepsilon) < \infty$$

(this is possible because a power of the resolvent of A_1 is ultracontractive by assumption).
The regularized problem

$$du^\varepsilon + Au^\varepsilon\,dt + \beta(u^\varepsilon)\,dt \ni G^\varepsilon\,dZ, \qquad u^\varepsilon(0) = u_0$$

has a unique solution by the previous results.

Uniform estimates and passage to the limit as $\varepsilon \to 0$

By lower semicontinuity and estimates in expectation,

- (u^ε) is bounded in $L^2(\Omega; L^\infty(0, T_0; H)) \cap L^2(\Omega; L^2(0, T_0; V))$;
- (ξ^ε) is relatively weakly compact in $L^1(\Omega \times [0, T_0] \times D)$.

Moreover, since

$$\mathbb{E}\|u^\varepsilon - u^\delta\|^2_{L^\infty(0,T_0;H)\cap L^2(0,T_0;V)} \lesssim \mathbb{E}\lambda^C_{T_0-}(G^\varepsilon - G^\delta),$$

it follows that

- (u^ε) is a Cauchy sequence in $L^2(\Omega; L^\infty(0, T; H))$.

Further lower semicontinuity arguments imply that (u, ξ) is the unique solution to the equation with additive noise.

The càdlàg property of u follows by uniform convergence.

Localization

Let T be a finite stopping time. If C is a control process for Z, then C^{T-} is a control process for Z^{T-}. Considering the equation

$$dv + Av\,dt + \beta(v)\,dt = G\,dZ^{T-}, \qquad v(0) = u_0,$$

one obtains the

Lemma. If G is predictable with $\mathbb{E}\lambda^C_{T-}(G) < \infty$, and $\mathbb{E}\|u_0\|^2 < \infty$, then (G) admits a unique strong solution on $[\![0, T]\!]$.

The following continuity of $(u_0, G) \mapsto u$ holds.

Proposition. Let u_i be solutions on $[\![0, T_i]\!]$ with data u_{0i} and G_i. Setting $T := T_1 \wedge T_2$, one has

$$\mathbb{E}\big(u_1 - u_2\big)^{*2}_{T-} + \mathbb{E}\int_0^T \|u_1 - u_2\|^2_V \lesssim \mathbb{E}\|u_{01} - u_{02}\|^2 + \mathbb{E}\lambda^C_{T-}(G_1 - G_2).$$

General case (multiplicative noise)

We can now treat the original problem

$$du + Au\,dt + \beta(u)\,dt = B(u-)\,dZ, \quad u(0) = u_0.$$

Let $\alpha \in]0,1[$, $R > 0$, and

$$T^0 := \inf\{t \in [0, T_0] : C_t(L_t - L_0) \geq \alpha\} \wedge T_0, \qquad T := T^0 \mathbb{1}_{\{\|u_0\| \leq R\}}.$$

Fixed point argument: let $v \in \mathbb{S}^2(T_0-)$ and consider the problem

$$du + Au\,dt + \beta(u)\,dt \ni B(v-)\,dZ^{T-}, \quad u(0) = u_0,$$

which admits a unique strong solution, and

$$\|u_1 - u_2\|^2_{\mathbb{S}^2(T_0-)} + \mathbb{E}\|u_1 - u_2\|^2_{L^2(0,T_0;V)} \lesssim \alpha\|v_1 - v_2\|^2_{\mathbb{S}^2(T_0-)}.$$

Choosing α small enough, we get a solution $\tilde{u} \in \mathbb{S}^2(T_0-)$. Now set

$$u := u_0 \mathbb{1}_{\{T=0\}} + \tilde{u} \quad \text{on } [\![0, T[\![, \qquad u_T = u_{T-} + B(\tilde{u}_{T-})\Delta Z_T,$$

and $\xi = \tilde{\xi}\mathbb{1}_{[\![0,T]\!]}$. We have thus proved the

Proposition. There exists a stopping time $T \neq 0$ and a strong solution on $[\![0, T]\!]$.

General case (multiplicative noise)

Once local existence is established, one needs local uniqueness.

Proposition. If (u_i, ξ_i) is a local solution on $[\![0, T_i]\!]$, then $(u_i, \xi_i) = (u_j, \xi_j)$ on $[\![0, T_i \wedge T_j]\!]$.

This follows by the continuity of $(u_0, G) \mapsto u$, further pre-stopping arguments, and a stochastic Gronwall inequality by Métivier.

One can now iterate the procedure used for local existence defining

$$T_{n+1} := \begin{cases} T_n, & \text{if } \|u_{T_n}\| > n, \\ \inf\big\{t \geq T_n : \ C_t(L_t - L_{T_n}) > \alpha\big\} \wedge T_0, & \text{if } \|u_{T_n}\| \leq n, \end{cases}$$

and show that $\mathbb{P}(\lim_n T_n < T_0) = 0$ by linear growth of B.

Finally, the continuity of $u_0 \mapsto u$ is obtained using continuity of $(u_0, G) \mapsto u$ "prelocally" in $\mathbb{S}^2$ and the stochastic Gronwall inequality.