Workshop in Stochastic and PDE,
September 14-15, 2018, Bucharest

EDSR multivoques: L^p-formulation faibble variationelle $(1 < p < 2)$

[*Multivalued BSDE: L^p- variational weak formulation* $(1 < p < 2)$]

Aurel Răşcanu

Institut de mathématiques "Octav Mayer" de l'Académie Roumaine, Iaşi

e-mail: aurel.rascanu@uaic.ro

joint work with **Lucian Maticiuc**, "Al.I. Cuza" University, Iaşi, Romania

Object

In the present paper, we prove the existence and uniqueness of L^p solutions, in the case $p \in (1,2)$, of the backward stochastic variational inequality (BSVI for short):

$$\begin{cases} Y_t + \int_t^T dK_s = \eta + \int_t^T \left[F\left(s, Y_s, Z_s\right) ds + G\left(s, Y_s\right) dA_s\right] - \int_t^T Z_s dB_s \ , \\ \hfill t \in [0,T] \ , \\ dK_t \in \partial\varphi\left(Y_t\right) dt + \partial\psi\left(Y_t\right) dA_t \ , \quad \text{on } [0,T] \ , \end{cases}$$

where

- $\partial\varphi$ and $\partial\psi$ are the subdifferentials of two convex lower semicontinuous functions φ and ψ,
- $\{A_t : t \geq 0\}$ is a progressively measurable increasing continuous stochastic process

I. L^p-solution and reflected BSDE:

Briand, P., Delyon, B., Hu, Y., Pardoux, E. and Stoica, L. (2003) *L^p solutions...*

El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M.C. (1997) *Reflected solutions ...*

Lepeltier, J.P., Matoussi, A., Xu, M. (2005). *Reflected ....*

Hamadène, S., Popier, A. (2012). *L^p–solutions for reflected ...*

Klimsiak, T. (2013). *BSDEs ...reflecting barriers*

Rozkosz, A. and Słomiński, L. (2012). *L^p solutions of reflected ....*

Hanwu Li, Shige Peng (2017) *Reflected BSDE driven by G-Brownian ...*

II. BSVI :

starting paper: Pardoux, E. and Răşcanu, A. (1998, 1999) *BSDE with subdifferential ....*

Maticiuc, L. and Răşcanu, A. (2010, 2015)

Motivation :

- In several applications (finance, control, games, PDEs,...) the data are not square integrable and to assume them

so is somehow restrictive.

- **Example.**
  Let the parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann-Dirichlet boundary condition, or nonlinear Robin condition or impedance boundary conditions, from their application in electromagnetic problems, or convective boundary conditions, from their application in heat transfer problems (Hahn, 2012)):

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}_t u\,(t,x) + f\big(t,x,u(t,x),(\nabla u\sigma)(t,x)\big) \in \partial\varphi\big(u(t,x)\big), \\
\hfill t \in (0,T)\,,\ x \in D, \\
-\dfrac{\partial u(t,x)}{\partial n} + g\big(t,x,u(t,x)\big) \in \partial\psi\big(u(t,x)\big), \quad t \in (0,T)\,,\ x \in Bd\left(\overline{D}\right), \\
u(T,x) = h(x), \quad x \in \overline{D},
\end{cases}
\tag{1}
$$

where the operator $\mathcal{L}_t$ is given by

$$
\mathcal{L}_t v(x) = \frac{1}{2}\mathrm{Tr}\big[\sigma(t,x)\sigma^*(t,x)D^2 v(x)\big] + \big\langle b(t,x), \nabla v(x)\big\rangle,
$$

and D is an open connected bounded subset of $\mathbb{R}^m$ of the form

$$
D = \{x \in \mathbb{R}^m : \ell\,(x) < 0\}\,,\ \text{where } \ell \in C_b\,(\mathbb{R}^m)
$$

Let $(t,x) \in [0,T] \times \bar{\mathcal{D}}$ arbitrary fixed.

- Consider the system

$$
\begin{cases}
X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r - \int_t^s \nabla\ell(X_r^{t,x})dA_r^{t,x}, \\
A_s^{t,x} = \int_t^s \mathbf{1}_{\{X_r^{t,x} \in \mathrm{Bd}(\mathcal{D})\}} dA_r^{t,x}, \\
Y_s^{t,x} + \int_s^T U_r^{t,x} dr + \int_s^T V_r^{t,x}\, dA_r^{t,x} = h(X_T^{t,x}) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}\right) dr \\
\qquad\qquad + \int_s^T g\left(r, X_r^{t,x}, Y_r^{t,x}\right)\, dA_r^{t,x} - \int_s^T Z_r^{t,x} dB_r, \\
U_r^{t,x} dr \in \partial\varphi(Y_r^{t,x})dr, \qquad \text{and} \qquad V_r^{t,x}\, dA_r^{t,x} \in \partial\psi(Y_s^{t,x})dA_s^{t,x}, \quad \text{on } [t, T]\,.
\end{cases}
$$

The solution is an sextuple $(X_s^{t,x}, A_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, V_s^{t,x})_{s\in[0,T]}$ of $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$–valued $\mathcal{F}_s^t$–progressively measurable stochastic processes,where

$$
\mathcal{F}_s^t = \sigma(\mathbf{1}_N, B_r - B_t;\ t \le r \le s \vee t,\ N \in \mathcal{N}),
$$

where $\mathcal{N}$ is the set of $\mathbb{P}$–null events of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$

and $(B_s)_{s\geq 0}$ is a Brownian motion.

- $(X^{t,x}, Y^{t,x}) : \Omega \times [0,T] \to \bar{\mathcal{D}} \times \mathbb{R}$ are continuous stochastic processes,
- $A^{t,x} : \Omega \times [0,T] \to \mathbb{R}_+$ is an increasing continuous stochastic process,

We mention that $U_r^{t,x} dr \in \partial\varphi(Y_r^{t,x})dr$ on $[t,T]$ (similar for $V_r^{t,x}\, dA_r^{t,x} \in \partial\psi(Y_s^{t,x})dA_s^{t,x}$) means that

$$\int_{s_1}^{s_2} \left(v\left(r\right) - Y_r^{t,x}\right) U_r^{t,x} dr + \int_{s_1}^{s_2} \varphi\left(Y_r^{t,x}\right) dr \leq \int_{s_1}^{s_2} \varphi\left(v\left(r\right)\right) dr, \text{ a.s.,}$$

Remark that $Y_s^{t,x}$, $(t,x) \in [0,T] \times \overline{\mathcal{D}}$, is a determinist quantity since $Y_t^{t,x}$ is $\mathcal{F}_t^t = \sigma\left(\mathcal{N}\right)$–measurable.

Theorem *[Maticiuc & Răşcanu (2010)]*

$$u(t,x) = Y_t^{t,x}, \quad (t,x) \in [0,T] \times \overline{\mathcal{D}},$$

is the unique viscosity solution of the PVI with mixed nonlinear multivalued Neumann–Dirichlet boundary condition (1).

Notations and assumptions

- (Ω, F, P) is a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous and complete filtration generated by a standard k–dimensional Brownian motion $(B_t)_{t\geq 0}$;
- $S_m^p\,[0,T]$ is the space of continuous progressively measurable stochastic processes (p.m.s.p. for short) $X : \Omega \times [0,T] \to \mathbb{R}^m$ such that

$$\mathbb{E} \sup_{t\in[0,T]} |X_t|^p < +\infty, \quad \text{if } p > 0;$$

- $\Lambda_m^p\,(0,T)$ is the space of p.m.s.p. $X : \Omega \times (0,T) \to \mathbb{R}^m$ such that such that

$$\int_0^T |X_t|^2\, dt < +\infty, \quad \mathbb{P} - a.s., \qquad \text{if } p = 0, \text{ and}$$

$$\mathbb{E}\left(\int_0^T |X_t|^2\, dt\right)^{p/2} < +\infty, \quad \text{if } p > 0.$$

Recall the BSVI

$$\begin{cases} Y_t + \int_t^T dK_s = \eta + \int_t^T \left[F\left(s, Y_s, Z_s\right) ds + G\left(s, Y_s\right) dA_s\right] - \int_t^T Z_s dB_s \,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0, T] \,, \\ dK_t = U_t^{(1)} dt + U_t^{(2)} dA_t \\ U_t^{(1)} dt \in \partial\varphi\left(Y_t\right) dt \quad \text{and} \quad U_t^{(2)} dA_t \in \partial\psi\left(Y_t\right) dA_t \ , \quad \text{on } [0, T] \,, \end{cases} \tag{2}$$

Assumptions:

(A_1) $\quad p > 1$

(A_2) $\quad \eta : \Omega \to \mathbb{R}^m$ is random variable $\mathcal{F}_T$–measurable such that

$$\mathbb{E} \left|\eta\right|^p < \infty$$

(A_3) $\quad \varphi, \psi : \mathbb{R}^m \to [0, +\infty]$ are proper convex l.s.c. functions,

(a) $\partial\varphi$ and $\partial\psi$ denote their subdifferentials

(b) $0 = \varphi(0) \leq \varphi(y)$ and $0 = \psi(0) \leq \psi(y)$ for all $y \in \mathbb{R}^m$).

(A_4) $F : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ and $G : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$

(a) $F(\cdot,\cdot,y,z)$, $G(\cdot,\cdot,y)$ are p.m.s.p., for all $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{m\times k}$,

(b) $F(\omega,t,\cdot,\cdot)$, $G(\omega,t,\cdot)$ are continuous functions, $d\mathbb{P}\otimes dt$-a.e.

(c)

$$\mathbb{E}\left(\int_0^T |F(s,0,0)|\, ds\right)^p + \mathbb{E}\left(\int_0^T |G(s,0)|\, dA_s\right)^p < \infty, \tag{3}$$

(d) for all $t \geq 0$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times k}$, $\mathbb{P}$–a.s.

$$\begin{aligned} &\langle y' - y, F(t,y',z) - F(t,y,z)\rangle \leq \mu_t\, |y' - y|^2, \\ &\langle y' - y, G(t,y') - G(t,y)\rangle \leq \nu_t\, |y' - y|^2, \\ &|F(t,y,z') - F(t,y,z)| \leq \ell_t\, |z' - z|. \end{aligned} \tag{4}$$

where $\mu, \nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $\ell : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are p.m.s.p.

$$V_t \overset{def}{=} \int_0^t \left[\left(\mu_s + \frac{1}{2n_p\lambda} \ell_s^2 \right) ds + \nu_s dA_s \right] \geq 0, \text{ for all } t \in [0, T]$$

$$\mathbb{E} \exp\left(p \int_0^T \left(|\mu_s| + \frac{1}{2n_p\lambda} \ell_s^2 \right) ds + p \int_0^T |\nu_s| \, dA_s \right) < \infty$$

with $p > 1$, $n_p := 1 \wedge (p - 1)$, $0 < \lambda < 1$.

(A_5) Compatibility assumptions

$\forall\ \varepsilon>0,\ t\geq 0,\ y\in\mathbb{R}^m,\ z\in\mathbb{R}^{m\times k}:$

$$\begin{array}{rl} (i) & \left\langle \nabla\varphi_\varepsilon\left(y\right),\nabla\psi_\varepsilon\left(y\right)\right\rangle \geq 0, \\ (ii) & \left\langle \nabla\varphi_\varepsilon\left(y\right),G\left(t,y\right)\right\rangle \leq \left|\nabla\psi_\varepsilon\left(y\right)\right|\left|G\left(t,y\right)\right|,\ P\text{–a.s.}, \\ (iii) & \left\langle \nabla\psi_\varepsilon\left(y\right),F\left(t,y,z\right)\right\rangle \leq \left|\nabla\varphi_\varepsilon\left(y\right)\right|\left|F\left(t,y,z\right)\right|,\ P\text{–a.s..} \end{array} \tag{5}$$

where

$$\varphi_\varepsilon\left(y\right):=\inf\left\{\frac{1}{2\varepsilon}\left|y-v\right|^2+\varphi\left(v\right):v\in\mathbb{R}^m\right\}\quad\text{(Moreau-Yosida regularization)}$$

is a C^1–convex function with

$$\nabla\varphi_\varepsilon(x)=\partial\varphi_\varepsilon\left(x\right)\in\partial\varphi\left(J_\varepsilon\left(x\right)\right),\qquad \varphi\left(J_\varepsilon u\right)\leq\varphi_\varepsilon\left(u\right)$$

where $J_\varepsilon\left(x\right)=x-\varepsilon\nabla\varphi_\varepsilon(x)$ that satisfies

$$\left|J_\varepsilon\left(x\right)-J_\varepsilon\left(y\right)\right|\leq\left|x-y\right|\quad\text{and}\quad\left|\nabla\varphi_\varepsilon\left(x\right)-\nabla\varphi_\varepsilon\left(y\right)\right|\leq\frac{1}{\varepsilon}\left|x-y\right| \tag{6}$$

Example.

(a) If $\varphi = \psi$ then the compatibility assumptions (5) are satisfied.

(b) Let $m = 1$. Since $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are increasing monotone functions on R, we see that, if $y \cdot G(t, y) \leq 0$ and $y \cdot F(t, y, z) \leq 0$, for all t, y, z, then the compatibility assumptions (5) are satisfied.

(c) Let $a \leq 0 \leq b$. If $\varphi, \psi : R \to]-\infty, +\infty]$ are the convex indicator functions

$$\varphi(y) = I_{[a,+\infty[}(y) = \begin{cases} 0, & \text{if } y \geq a \\ +\infty, & \text{if } y < a \end{cases} \qquad \text{and } \nabla\varphi_\varepsilon(y) = -\frac{1}{\varepsilon}(y-a)^-$$

and

$$\psi(y) = I_{]-\infty,b]}(y) = \begin{cases} 0, & \text{if } y \leq b \\ +\infty, & \text{if } y > b \end{cases} \qquad \text{and } \nabla\psi_\varepsilon(y) = \frac{1}{\varepsilon}(y-b)^+$$

then the compatibility assumptions are reduced to

$$G(t, x, y) \geq 0 \textit{ for } y \leq a, \quad \textit{and} \quad F(t, x, y, z) \leq 0 \textit{ for } y \geq b$$

Notation.

We denote

$$Q_t(\omega) = t + A_t(\omega),$$

and let $\{\alpha_t : t \geq 0\}$ be the real positive p.m.s.p. such that $\alpha \in [0,1]$ and $dt = \alpha_t dQ_t$ and $dA_t = (1 - \alpha_t) dQ_t$.

Let us introduce the functions

$$\begin{aligned} H(t, y, z) &:= \alpha_t F(t, y, z) + (1 - \alpha_t) G(t, y), \\ \Psi(\omega, t, y) &:= \alpha_t(\omega) \varphi(y) + (1 - \alpha_t(\omega)) \psi(y), \end{aligned} \tag{7}$$

Intuitive introduction

Let $p \geq 0$, $n_p = (p-1)^+ \wedge 1$ and let be fixed an arbitrary $\lambda \in]\,0, 1[$. Define

$$V_t \overset{def}{=} \begin{cases} \displaystyle\int_0^t \left(\mu_r + \frac{1}{2n_p\lambda}\ell_r^2\right) dr + \int_0^t \nu_r dA_r, & \text{for } p > 1, \text{ and} \\ \displaystyle\int_0^t \mu_r dr + \int_0^t \nu_r dA_r, & \text{for } 0 \leq p \leq 1. \end{cases}$$

In the case $p \in [0,1]$ we shall put, in the sequel, $\ell = 0$ (i.e. H is independent of z) and $\ell_r^2/n_p = 0$.

Let us define the space $S_m^p(\gamma, N, R; V)$, $p \geq 0$, of the continuous stochastic process M of the form

$$M_t = \gamma - \int_0^t N_r dQ_r + \int_0^t R_r dB_r, \quad \text{or equivalent}$$

$$M_t = M_T + \int_t^T N_r dQ_r - \int_t^T R_r dB_r \ , \quad M_0 = \gamma$$

where $\gamma \in \mathbb{R}^m$ and $N : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$, $R : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m \times k}$ are progressively measurable stochastic processes such that for all $T > 0$:

$$\mathbb{E} \left(\int_0^T e^{V_r} N_r dr \right)^p + \mathbb{E} \left(\int_0^T e^{2V_r} |R_r|^2 dr \right)^{p/2} < \infty, \quad \text{if } p > 0$$

and

$$\int_0^T e^{V_r} N_r dr + \int_0^T e^{2V_r} |R_r|^2 dr < \infty, \ \mathbb{P} - a.s., \quad \text{if } p = 0.$$

Clearly M is a continuous progressively measurable stochastic process and

$$\mathbb{E}\left(\sup_{r\in[0,T]} e^{pV_r}\left|M_r\right|^p\right) < \infty, \quad \text{for } p > 0.$$

For a intuitive introduction let (Y, Z, U) be a strong a solution of (2) that is $Y, Z,$ and U are progressively measurable stochastic processes, Y has continuous trajectories, $\mathbb{P} - a.s.$:

$$\int_0^T e^{2V_r}\left|Z_r\right|^2 dr + \int_0^T e^{2V_r}\left|U_r\right|^2 dr < \infty$$

and

$$\begin{cases} Y_t + \int_t^T dK_s = \eta + \int_t^T H\left(s, Y_s, Z_s\right) dQ_s - \int_t^T Z_s dB_s, \quad \text{for all } t \in [0, T], \\ dK_s = U_t dQ_t \in \partial_y \Psi\left(s, Y_s\right) dQ_s; \end{cases} \tag{8}$$

For $0 < \delta \leq 1$ we define

$$\delta_p = \delta \mathbf{1}_{p<2} = \begin{cases} \delta, & \text{if } p < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $M \in S_m^0(\gamma, N, R; V)$. By Itô's formula for $(\Gamma_t)^q$, with $q \in \{2,\ p \wedge 2\}$,

$$\Gamma_t = \left(|M_t - Y_t|^2 + \delta_p \right)^{1/2}$$

and

$$M_t = M_T + \int_t^T N_r dQ_r - \int_t^T R_r dB_r,$$

we deduce for all $0 \leq t \leq s$ and for all $\delta \in]0,1]$,

$$\begin{aligned}
&(\Gamma_t)^q + \frac{q}{2}\int_t^s (\Gamma_r)^{q-4} \big[\, (q-1) \left|(R_r^* - Z_r^*)(M_r - Y_r)\right|^2 \\
&\quad + \left(|R_r - Z_r|^2 |M_r - Y_r|^2 - \left|(R_r^* - Z_r^*)(M_r - Y_r)\right|^2 \right) \\
&\quad + \delta_p |R_r - Z_r|^2 \,\big] dr + q\int_t^s (\Gamma_r)^{q-2} \langle M_r - Y_r, U_r dQ_r \rangle \\
&= (\Gamma_s)^q + q\int_t^s (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H(r, Y_r, Z_r) \rangle dQ_r \\
&\quad - q\int_t^s (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r)\, dB_r \rangle,
\end{aligned} \tag{9}$$

We have

$$\begin{aligned}
&(q-1) \left|(R_r^* - Z_r^*)(M_r - Y_r)\right|^2 + |R_r - Z_r|^2 |M_r - Y_r|^2 - \left|(R_r^* - Z_r^*)(M_r - Y_r)\right|^2 \\
&\qquad + \delta_p |R_r - Z_r|^2 \\
&\geq (q-1)\, \Gamma_r^2 \, |R_r - Z_r|^2
\end{aligned}$$

Since $U_t dQ_t \in \partial_y \Psi\left(t, Y_t\right) dQ_t$, then by the subdifferential inequality

$$\langle M_r - Y_r, U_t \rangle \, dQ_t + \Psi\left(r, Y_r\right) dQ_r \leq \Psi\left(r, M_r\right) dQ_r$$

it follows from (9):

$$\begin{aligned}
&\left(\Gamma_t\right)^q + \frac{q\left(q-1\right)}{2} \int_t^s \left(\Gamma_r\right)^{q-2} \left|R_r - Z_r\right|^2 dr + q \int_t^s \left(\Gamma_r\right)^{q-2} \Psi\left(r, Y_r\right) dQ_r \\
&= \left(\Gamma_s\right)^q + q \int_t^s \left(\Gamma_r\right)^{q-2} \Psi\left(r, M_r\right) dQ_r \\
&\qquad +q \int_t^s \left(\Gamma_r\right)^{q-2} \langle M_r - Y_r, N_r - H\left(r, Y_r, Z_r\right) \rangle dQ_r \\
&\qquad -q \int_t^s \left(\Gamma_r\right)^{q-2} \langle M_r - Y_r, \left(R_r - Z_r\right) dB_r \rangle,
\end{aligned} \tag{10}$$

Definition. *Let $p > 1$. We call $(Y_t, Z_t)_{t\geq 0}$ a L^p-weak variational solution of (8) if*

- $Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ *and* $Z : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m\times k}$ *are two progressively measurable stochastic processes, Y has continuous trajectories*

$$\mathbb{E}\left(\sup_{r\in[0,T]} e^{pV_r}\left|Y_r\right|^p\right) < \infty, \tag{11}$$

and

$$\mathbb{E}\ \left(\int_0^T e^{2V_r}\left|Z_r\right|^2 dr\right)^{p/2} + \mathbb{E}\ \left(\int_0^T e^{2V_r}\Psi\left(r, Y_r\right) dQ_r\right)^{p/2} < \infty; \tag{12}$$

- $(Y_t, Z_t) = (\eta, 0)$ *for* $t \geq T$

- *for* $q \in \{2, p \wedge 2\}$, $\delta_p = \delta \mathbf{1}_{p<2}$ *and* $\Gamma_t = \left(|M_t - Y_t|^2 + \delta_p \right)^{1/2}$ *it holds*

$$
\begin{aligned}
(\Gamma_t)^q + \frac{q(q-1)}{2} \int_t^s (\Gamma_r)^{q-2} \, |R_r - Z_r|^2 \, dr + q \int_t^s (\Gamma_r)^{q-2} \, \Psi(r, Y_r) \, dQ_r \\
\leq (\Gamma_s)^q + q \int_t^s (\Gamma_r)^{q-2} \, \Psi(r, M_r) \, dQ_r \\
+ q \int_t^s (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H(r, Y_r, Z_r) \rangle dQ_r \\
- q \int_t^s (\Gamma_r)^{q-2} \, \langle M_r - Y_r, (R_r - Z_r) \, dB_r \rangle;
\end{aligned}
\tag{13}
$$

for all $\delta \in]0, 1]$, *for all* $0 \leq t \leq s \leq T$, *for all* $M \in S_m^p(\gamma, N, R; V)$.

Remark. It is obviously that a strong solution for (8) is also a weak solution (see the intuitive introduction for inequality (13)).

Remark. For $q=2$ and the inequality (13) becomes

$$\begin{aligned} &\left|M_t - Y_t\right|^2 + \int_t^s \left|R_r - Z_r\right|^2 dr + 2\int_t^s \Psi\left(r, Y_r\right) dQ_r \\ &\leq \left|M_s - Y_s\right|^2 + 2\int_t^s \Psi\left(r, M_r\right) dQ_r \\ &\qquad + 2\int_t^s \langle M_r - Y_r, N_r - H\left(r, Y_r, Z_r\right)\rangle dQ_r \\ &\qquad - 2\int_t^s \langle M_r - Y_r, \left(R_r - Z_r\right) dB_r\rangle,\ \mathbb{P}-a.s.. \end{aligned} \tag{14}$$

and the stochastic integral $J_t = \int_0^t \langle M_r - Y_r, \left(R_r - Z_r\right) dB_r\rangle$ is a continuous martingale for $p \geq 2$, but generally is only a continuous progressively measurable stochastic process if $1 < p < 2$.

Proposition. *Let $M \in S_m^0\left(\gamma, N, R; V\right)$. Let $Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ and $Z : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m\times k}$ be two progressively measurable stochastic processes with*

Y having continuous trajectories and $\mathbb{P}-a.s.$

$$\int_0^T e^{2V_r}\left|Z_r\right|^2 dr+\int_0^T e^{2V_r}\Psi\left(r,Y_r\right)dQ_r<\infty,$$

1. *If the inequality (13) holds for* $q=2$, *then for all* $k>0$ *and for any stopping times* $0\leq\sigma\leq\theta\leq T$ *it follows*

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\theta e^{2V_r}\left|Z_r\right|^2 dr\right)^{k/2}+\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\theta e^{2V_r}\Psi\left(r,Y_r\right)dQ_r\right)^{k/2}\\&\leq C_{k,\lambda}\left[\mathbb{E}^{\mathcal{F}_\sigma}\sup_{r\in[\sigma,\theta]}e^{kV_r}\left|Y_r\right|^k+\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\theta e^{V_r}\left|Y_r\right|\left|H\left(r,0,0\right)\right|dQ_r\right)^{k/2}\right]\\&\leq 2C_{k,\lambda}\left[\mathbb{E}^{\mathcal{F}_\sigma}\sup_{r\in[\sigma,\theta]}e^{kV_r}\left|Y_r\right|^k+\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\theta e^{V_r}\left|H\left(r,0,0\right)\right|dQ_r\right)^{k}\right],\quad\mathbb{P}-a.s..\end{aligned}$$

2. *If the inequality (13) holds and for some fixed stopping times* $0\leq\sigma\leq\theta\leq$

T :

$$\mathbb{E}\left(\sup_{r\in[\sigma,\theta]} e^{qV_r}\left|Y_r\right|^q\right)<\infty,$$

then

$$\mathbb{E}^{\mathcal{F}_\sigma}\sup_{\tau\in[\sigma,\theta]} e^{qV_\tau}\left|Y_\tau\right|^q\leq C_{\lambda,q}\ \mathbb{E}^{\mathcal{F}_\sigma}\left[e^{qV_\theta}\left|Y_\theta\right|^q+\int_\sigma^\theta e^{qV_r}\left|Y_r\right|^{q-1}\left|H\left(r,0,0\right)\right|dQ_r\right]$$

and

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\left(\sup_{\tau\in[\sigma,\theta]} e^{qV_\tau}\left|Y_\tau\right|^q\right)+\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\theta e^{qV_r}\left|Y_r\right|^{q-2}\left|Z_r\right|^2dr\right)\\&\qquad\qquad+\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\theta e^{qV_r}\left|Y_r\right|^{q-2}\Psi\left(r,Y_r\right)dQ_r\right)\\&\leq C_{\lambda,q}\ \mathbb{E}^{\mathcal{F}_\sigma}\left[e^{qV_\theta}\left|Y_\theta\right|^q+\left(\int_\sigma^\theta e^{V_r}\left|H\left(r,0,0\right)\right|dQ_r\right)^q\right],\quad \mathbb{P}-a.s.\ .\end{aligned}$$

Theorem. *Let $p > 1$. The BSVI*

$$\begin{cases} Y_t + \int_t^T dK_s = \eta + \int_t^T H\left(s, Y_s, Z_s\right) dQ_s - \int_t^T Z_s dB_s, \quad \text{for all } t \in [0, T], \\ dK_s \in \partial_y \Psi\left(s, Y_s\right) dQ_s, \text{ on } [0, T]. \end{cases}$$

has a unique L^p-weak variational solution.

Proof. Existence.

Assume there exists $a > 1$ such that

$$\tilde{V}_t \overset{def}{=} \int_0^t \left[\left(\mu_r + \frac{a}{2}\ell_r^2\right) dt + \nu_r dA_r\right] \geq 0, \text{ a.s., for all } t \in [0, T].$$

and

$$\mathbb{E} \exp\left(4a \int_0^T \left(\ell_r\right)^2 dr\right) < \infty.$$

Step 1 : Strong solution in $S_m^2\left[0,T\right] \times \Lambda_{m\times k}^2\left[0,T\right]$

Basic assumptions for a strong solution are the followings

$$\mathbb{E}\int_0^T \left(F_\rho^{\#}\left(s\right)\right)^2 ds + \mathbb{E}\int_0^T \left(G_\rho^{\#}\left(s\right)\right)^2 dA_s < \infty, \quad \text{for all } \rho, T \geq 0, \tag{15}$$

where

$$F_\rho^{\#}\left(\omega,s\right) := \sup_{|y|\leq\rho} \left|F\left(\omega,s,y,0\right)\right|, \quad G_\rho^{\#}\left(\omega,s\right) := \sup_{|y|\leq\rho} \left|G\left(\omega,s,y\right)\right|\ ;$$

and

$$e^{\tilde{V}_T}\left|\eta\right| + \int_0^T e^{\tilde{V}_t}\left|H(r,0,0)\right| dQ_r \leq \rho_0, \text{ a.s.} \tag{16}$$

Let $0 < \varepsilon \leq 1$. We consider, for any $t \in \left[0,T\right]$, the approximating stochastic equation

$$Y_t^\varepsilon + \int_t^T \nabla_y \Psi^\varepsilon(r, Y_r^\varepsilon) dQ_r = \eta + \int_t^T H(r, Y_r^\varepsilon, Z_r^\varepsilon) dQ_r - \int_t^T Z_r^\varepsilon dW_r\,, \quad \mathbb{P}\text{–a.s.,} \tag{17}$$

where

$$\Psi^{\varepsilon}(\omega, r, y) := \alpha_r(\omega)\varphi_{\varepsilon}(y) + (1-\alpha_r(\omega))\psi_{\varepsilon}(y)$$
$$\nabla_y \Psi^{\varepsilon}(\omega, r, y) = \alpha_r(\omega)\nabla_y\varphi_{\varepsilon}(y) + (1-\alpha_r(\omega))\nabla_y\psi_{\varepsilon}(y).$$

Setting in Theorem 5.30 (Pardoux & Răşcanu 2014 *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations,* Springer, 2014) $p=4$, $a=2$, $\delta=2$, $q=4/3$ we infer that the approximating equation (17) has a unique solution such that

$$|Y_t^{\varepsilon}| \le C_1(\rho_0), \text{ for all } t\in[0,T], \text{ a.s.}$$
$$\mathbb{E}\Big(\int_0^T e^{2\tilde{V}_r}|Z_r^{\varepsilon}|^2 dr\Big) \le C_2(\rho_0)$$
$$\mathbb{E}\int_0^T e^{2\tilde{V}_r}\Big[|\nabla\varphi_{\varepsilon}(Y_r^{\varepsilon})|^2 dr + |\nabla\psi_{\varepsilon}(Y_r^{\varepsilon})|^2 dA_r\Big] \le C_3(\rho_0)$$

and

$$\mathbb{E}\sup_{r\in[0,T]} e^{2\tilde{V}_r}|Y_r^{\varepsilon}-Y_r^{\delta}|^2 + \mathbb{E}\int_0^T e^{2\tilde{V}_r}|Z_r^{\varepsilon}-Z_r^{\delta}|^2 dr \to 0, \text{ as } \varepsilon,\delta\to 0.$$

Hence there exists $\left(Y,Z,U^{(1)},U^{(2)}\right)\in S_m^2\left[0,T\right]\times\left(\Lambda_{m\times k}^2\left[0,T\right]\right)^3$ (the quadruple is also unique) such that

$$\left\{\begin{array}{l} Y_t+\displaystyle\int_t^T U_s^{(1)}dt+U_s^{(2)}dA_s=\eta+\int_t^T\left[F\left(s,Y_s,Z_s\right)ds+G\left(s,Y_s\right)dA_s\right] \\ \hfill -\displaystyle\int_t^T Z_s dB_s\,,\qquad t\in\left[0,T\right], \\ U_t^{(1)}dt\in\partial\varphi\left(Y_t\right)dt\quad\text{ and }\quad U_t^{(2)}dA_t\in\partial\psi\left(Y_t\right)dA_t\;,\quad\text{on }\left[0,T\right], \end{array}\right.$$

Step 2 : Weak variational solution.

We renounce at the assumption (16). In this case we consider the approximating equation

$$\begin{cases} Y_t^{(n)} + \displaystyle\int_t^T \left(U_s^{(1,n)} dt + U_s^{(2,n)} dA_s \right) = \eta^{(n)} \\ + \displaystyle\int_t^T \left[F^{(n)} \left(s, Y_s^{(n)}, Z_s^{(n)} \right) ds + G^{(n)} \left(s, Y_s^{(n)} \right) dA_s \right] - \int_t^T Z_s^{(n)} dB_s \,, \quad t \in [0,T] \,, \\ U_t^{(1,n)} dt \in \partial\varphi \left(Y_t^{(n)} \right) dt \quad \text{and} \quad U_t^{(2,n)} dA_t \in \partial\psi \left(Y_t^{(n)} \right) dA_t \,, \quad \text{on } [0,T] \,, \end{cases} \tag{18}$$

with

$$\begin{aligned} \Lambda_t &= \int_0^t \left[\left(|\mu_r| + \frac{a}{2} \ell_r^2 \right) dt + |\nu_r| \, dA_r \right] \\ \eta^{(n)} &= \eta \mathbf{1}_{[0,n]} \left(|\eta| + \Lambda_T \right) \\ F^{(n)} (s,y,z) &= F(s,y,z) - F(s,0,0) \, \mathbf{1}_{(n,\infty)} \left(|F(s,0,0)| + \Lambda_s \right) \\ G^{(n)} (s,y,z) &= G(s,y) - G(s,0) \, \mathbf{1}_{(n,\infty)} \left(|G(s,0)| + \Lambda_s \right) \end{aligned}$$

Clearly

$$e^{\tilde{V}_T}\left|\eta^{(n)}\right|+\int_0^T e^{\tilde{V}_t}\left|H^{(n)}(r,0,0)\right|dQ_r\leq\rho_n,\ \text{a.s.}$$

and consequently the BSDE (18) has unique strong solution $\left(Y^{(n)},Z^{(n)},U^{(1,n)},U^{(2,n)}\right)$. We obtain that

$$\mathbb{E}\sup_{r\in[0,T]}e^{pV_r}|Y_r^{(n+i)}-Y_r^{(n)}|^p+\mathbb{E}\left(\int_0^T e^{2V_r}|Z_r^{(n+i)}-Z_r^{(n)}|^2dr\right)^{p/2}\to 0,\ \text{as } n\to\infty.$$

but we can not obtain estimates on $\left(U^{(1,n)},U^{(2,n)}\right)$. Therefore we only can pass to limit in the weak variational form for (18).

Uniqueness. !!!

Let $q=p\wedge 2$. Let $\left(\hat{Y},\hat{Z}\right)$ and $\left(\tilde{Y},\tilde{Z}\right)$ be two L^p-variational weak solutions of (8) corresponding to $\left(\hat{\eta},\hat{H}\right)$ and $\left(\tilde{\eta},\tilde{H}\right)$ respectively, where $\hat{H}$ and $\tilde{H}$ have the same constants functions μ,ν,ℓ.

- Let $Y_r=\frac{1}{2}\left(\hat{Y}_r+\tilde{Y}_r\right)$.

- Let $\delta_p=\delta\mathbf{1}_{p<2}$,

$$\hat{\Gamma}_t^{\varepsilon}=\left(\left|M_t^{\varepsilon}-\hat{Y}_t\right|^2+\delta_p\right)^{1/2}\quad\text{and}\quad\tilde{\Gamma}_t^{\varepsilon}=\left(\left|M_t^{\varepsilon}-\tilde{Y}_t\right|^2+\delta_p\right)^{1/2}$$

with $(M^{\varepsilon},R^{\varepsilon})\in S_m^p\left[0,T\right]\times\Lambda_{m\times k}^p\left[0,T\right]$ the unique solution of the BSDE:

$$M_t^{\varepsilon}=\eta+\int_t^T 1_{[\varepsilon,\infty)}\left(r\right)\frac{1}{Q_{\varepsilon}}\left(Y_r-M_r^{\varepsilon}\right)dQ_r-\int_t^T R_r^{\varepsilon}dB_r\,,\;\;t\in\left[0,T\right],\quad\text{with}$$

In fact
$$M_t^\varepsilon = \mathbb{E}^{\mathcal{F}_t} \int_{t\vee\varepsilon}^{\infty} \frac{1}{Q_\varepsilon} e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}} Y_r dQ_r.$$
We shall denote
$$N_r^\varepsilon = 1_{[\varepsilon,\infty)}(r) \frac{1}{Q_\varepsilon} (Y_r - M_r^\varepsilon).$$

- $0 \le t \le r \le s \le T$ and the stopping times
$$s^* = Q_s^{-1}, \quad t^* = Q_t^{-1}, \quad r^* = Q_r^{-1},$$
where $Q_\cdot^{-1}(\omega)$ is the inverse mapping of $r \longmapsto Q_r(\omega) : [0,\infty) \to [0,\infty)$

and, for each $k, i \in \mathbb{N}^*$, the stopping time

$$\alpha_k = \inf \Big\{ s \geq 0 : \updownarrow V \updownarrow_s + \sup_{r \in [0,s]} \left| e^{V_r} \hat{Y}_r - \hat{Y}_0 \right| + \sup_{r \in [0,s]} \left| e^{V_r} \tilde{Y}_r - \tilde{Y}_0 \right|$$
$$+ \int_0^s e^{2V_r} \left| \hat{Z}_r \right|^2 dr + \int_0^s e^{2V_r} \left| \tilde{Z}_r \right|^2 dr$$
$$+ \int_0^s e^{V_r} \left| \hat{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) \right| dQ_r + \int_0^s e^{V_r} \left| \tilde{H}\left(r, \tilde{Y}_r, \tilde{Z}_r\right) \right| dQ_r$$
$$+ \int_0^s e^{2V_r} \Psi\left(r, \hat{Y}_r\right) dQ_r + \int_0^s e^{2V_r} \Psi\left(r, \tilde{Y}_r\right) dQ_r \geq k \Big\}.$$

and

$$t_k^* \stackrel{def}{=} T \wedge t^* \wedge \alpha_k \qquad \text{and} \qquad s_{k+i}^* \stackrel{def}{=} T \wedge s^* \wedge \alpha_{k+i} \ .$$

- With $\sigma = t_k^*$ and $\theta = s_{k+i}^*$ we have

$$
\begin{aligned}
&\left[e^{qV_\sigma}\left(\hat{\Gamma}_\sigma^\varepsilon\right)^q + e^{qV_\sigma}\left(\tilde{\Gamma}_\sigma^\varepsilon\right)^q\right] \\
&\qquad + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\left[\left(\hat{\Gamma}_r^\varepsilon\right)^q + \left(\tilde{\Gamma}_r^\varepsilon\right)^q\right] dV_r \\
&\qquad + \frac{q\,(q-1)}{2}\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\left[\left(\hat{\Gamma}_r\right)^{q-2}\left|R_r - \hat{Z}_r\right|^2 + \left(\tilde{\Gamma}_r\right)^{q-2}\left|R_r - \tilde{Z}_r\right|^2\right] dr \\
&\qquad + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\left[\left(\hat{\Gamma}_r^\varepsilon\right)^{q-2}\Psi\left(r,\hat{Y}_r\right) + \left(\tilde{\Gamma}_r^\varepsilon\right)^{q-2}\Psi\left(r,\tilde{Y}_r\right)\right] dQ_r \\
&\leq\ \mathbb{E}^{\mathcal{F}_\sigma}\left[e^{qV_\theta}\left(\hat{\Gamma}_\theta^\varepsilon\right)^q + e^{qV_\theta}\left(\tilde{\Gamma}_\theta^\varepsilon\right)^q\right] \\
&\qquad + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\left[\left(\hat{\Gamma}_r^\varepsilon\right)^{q-2} + \left(\tilde{\Gamma}_r^\varepsilon\right)^{q-2}\right]\Psi\left(r, M_r^\varepsilon\right) dQ_r \\
&\qquad + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\left(\hat{\Gamma}_r^\varepsilon\right)^{q-2}\langle M_r^\varepsilon - \hat{Y}_r, N_r^\varepsilon - \hat{H}\left(r,\hat{Y}_r,\hat{Z}_r\right)\rangle dQ_r \\
&\qquad + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\left(\tilde{\Gamma}_r^\varepsilon\right)^{q-2}\langle M_r^\varepsilon - \tilde{Y}_r, N_r^\varepsilon - \tilde{H}\left(r,\tilde{Y}_r,\tilde{Z}_r\right)\rangle dQ_r.
\end{aligned}
\tag{19}
$$

- For all $\beta > 0$ the following inequality holds

$$\frac{1}{2}\left[\frac{1+\beta}{\beta}\left|M^{\varepsilon}-Y\right|^{2}+\frac{1+\beta}{4}\left|\hat{Y}-\tilde{Y}\right|^{2}+\delta_{p}\right]^{(q-2)/2}\left|\hat{Z}-\tilde{Z}\right|^{2}$$
$$\leq\left(\hat{\Gamma}^{\varepsilon}\right)^{q-2}\left|R^{\varepsilon}-\hat{Z}\right|^{2}+\left(\tilde{\Gamma}^{\varepsilon}\right)^{q-2}\left|R^{\varepsilon}-\tilde{Z}\right|^{2}$$

- Also

$$\left[\left(\hat{\Gamma}_{r}^{\varepsilon}\right)^{q-2}\langle M_{r}^{\varepsilon}-\hat{Y}_{r},N_{r}^{\varepsilon}\rangle+\left(\tilde{\Gamma}_{r}^{\varepsilon}\right)^{q-2}\langle M_{r}^{\varepsilon}-\tilde{Y}_{r},N_{r}^{\varepsilon}\rangle\right]\leq 0$$

- We pass to limit in (19) as $\varepsilon\to 0_{+}$; then we pass to limit as $\beta\to 0_{+}$.
- We introduce the inequalities

$$2\Psi\left(r,Y_{r}\right)=2\Psi\left(r,\frac{1}{2}\hat{Y}_{r}+\frac{1}{2}\tilde{Y}_{r}\right)\leq\Psi\left(r,\hat{Y}_{r}\right)+\Psi\left(r,\tilde{Y}_{r}\right).$$

and

$$\langle \hat{Y}_r - \tilde{Y}_r, \hat{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) - \tilde{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) + \tilde{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) - \tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r)\rangle dQ_r$$

$$\leq \langle \hat{Y}_r - \tilde{Y}_r, \hat{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) - \tilde{H}\left(r, \hat{Y}_r, \hat{Z}_r\right)\rangle dQ_r + |\hat{Y}_r - \tilde{Y}_r|^2 dV_r$$

$$+ \frac{n_p \lambda}{2} |\hat{Z}_r - \tilde{Z}_r|^2 dr.$$

- It follows

$$
\begin{aligned}
&2\mathbb{E}\ e^{qV_{t_k^*}} \left(\frac{1}{4}\left|\hat{Y}_{t_k^*} - \tilde{Y}_{t_k^*}\right|^2 + \delta_p\right)^{q/2} \\
&\quad +2q\delta_q\ \mathbb{E}^{\mathcal{F}_{t_k^*}} \int_{t_k^*}^{s_{k+i}^*} e^{qV_r} \left(\frac{1}{4}\left|\hat{Y}_r - \tilde{Y}_r\right|^2 + \delta_p\right)^{(q-2)/2} dV_r \\
&\quad +\frac{q}{2}\left(q-1-\frac{n_p\lambda}{2}\right)\ \mathbb{E}^{\mathcal{F}_{t_k^*}} \int_{t_k^*}^{s_{k+i}^*} e^{qV_r} \left(\frac{1}{4}\left|\hat{Y} - \tilde{Y}\right|^2 + \delta_p\right)^{(q-2)/2} \left|\hat{Z}_r - \tilde{Z}_r\right|^2 dr \\
&\leq 2\ \mathbb{E}^{\mathcal{F}_{t_k^*}}\ e^{qV_{s_{k+i}^*}} \left(\frac{1}{4}\left|\hat{Y}_{s_{k+i}^*} - \tilde{Y}_{s_{k+i}^*}\right|^2 + \delta_p\right)^{q/2} \\
&\quad +\frac{q}{2}\ \mathbb{E}^{\mathcal{F}_{t_k^*}} \int_{t_k^*}^{s_{k+i}^*} e^{qV_r} \left(\frac{1}{4}\left|\hat{Y}_r - \tilde{Y}_r\right|^2 + \delta_p\right)^{\frac{q-2}{2}} \\
&\qquad\qquad \times\langle \hat{Y}_r - \tilde{Y}_r, \hat{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) - \tilde{H}\left(r, \hat{Y}_r, \hat{Z}_r\right)\rangle dQ_r,
\end{aligned} \tag{20}
$$

- We pass to limit successively as $\delta \to 0$, $i \to \infty$, $n \to \infty$, $s \to \infty$. We obtain

$$e^{qV_{t^*}} \left| \hat{Y}_{T\wedge t^*} - \tilde{Y}_{T\wedge t^*} \right|^q \leq \mathbb{E}^{\mathcal{F}_{t^*}} e^{qV_T} \left| \hat{\eta} - \tilde{\eta} \right|^q$$
$$+q\, \mathbb{E}^{\mathcal{F}_{T\wedge t^*}} \int_{T\wedge t^*}^{T} e^{qV_r} \left| \hat{Y}_r - \tilde{Y}_r \right|^{q-1} \left| \hat{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) - \tilde{H}\left(r, \hat{Y}_r, \hat{Z}_r\right) \right| dQ_r \ , \tag{21}$$

for all $t \in [0, T]$ and $t^* = Q_t^{-1}$.

- *Uniqueness.*
For $\hat{\eta} = \tilde{\eta}$ and $\hat{H} = \tilde{H}$ we obtain from (21):

$$\hat{Y}_{T\wedge t^*} = \tilde{Y}_{T\wedge t^*} \ , \quad \text{a.s.} \ , \forall\, t \geq 0 \text{ and } t^* = Q_t^{-1}.$$

Since the function $r \longmapsto Q_r(\omega) : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $Q_0(\omega) = 0$ and $\lim_{r\to\infty} Q_r(\omega) = \infty$, $\mathbb{P} - a.s.\ \omega \in \Omega$, we deduce that

$$\left[\hat{Y}_{T\wedge r} = \tilde{Y}_{T\wedge r} \ , \quad \text{a.s.} \ \right], \forall\, r \geq 0.$$

The continuity of the solutions yields then

$$\hat{Y}_r = \tilde{Y}_r \ , \quad \forall \ r \in [0, T] \, , \text{ a.s. } .$$

We put now in (20) $\hat{Y} = \tilde{Y}$, $\hat{H} = \tilde{H}$, $t = 0$, $i = 0$. We obtain

$$\mathbb{E} \int_0^{s_k^*} e^{qV_r} \left| \hat{Z}_r - \tilde{Z}_r \right|^2 dr = 0.$$

Passing to limit as $k \to \infty$ and $s \to \infty$

$$\mathbb{E} \int_0^{T} e^{2V_r} \left| \hat{Z}_r - \tilde{Z}_r \right|^2 dr = 0$$

that yields $\hat{Z} = \tilde{Z}$ in $\Lambda^0_{m \times k}$.

Merci bien pour votre attention !