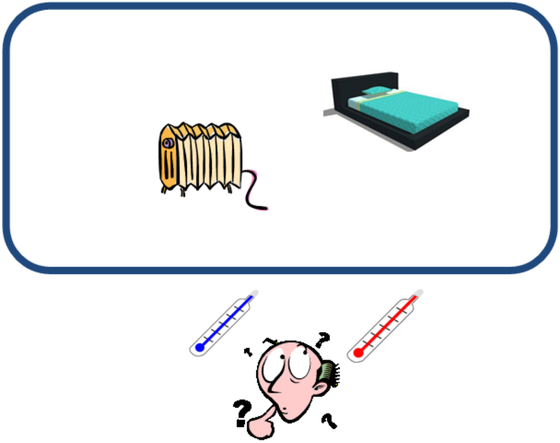
A new attempt to tackle the Hot Spots conjecture: Fixed-distance coupling of RBM and Iterated harmonic measure

Mihai N. Pascu

Transilvania University of Braşov, Romania

Atelier de travail en stochastique et EDP

Bucarest, Roumanie
September 14-15, 2018



– *Where should I put the bed, to keep warm* ***in the long run***?

Heuristics

Consider $u\left(t,x\right)$ the solution of the Neumann heat equation in a smooth bounded domain $D \subset \mathbb{R}^d$ with generic initial condition u_0.

Let x_t^+ be the hot spot at time t and x_t^- be the cold spot, i.e.

$$u\left(t,x_t^+\right) = \max_{x\in\overline{D}} u\left(t,x\right) \qquad \text{and} \qquad u\left(t,x_t^-\right) = \min_{x\in\overline{D}} u\left(t,x\right)$$

If the second Neumann eigenvalue λ_2 is simple, and φ_2 is a corresponding second Neumann eigenfunction, for large t we have

$$u\left(t,x\right) = \int_D u_0 + e^{-\lambda_2 t}\varphi_2\left(x\right)\int_D u_0\varphi_2 + R_2(t,x) \approx c_0 + c_1 e^{-\lambda_2 t}\varphi_2\left(x\right),$$

so x_t^+ and x_t^- are close to the maximum/minimum points of φ_2.
Hot spots (x_t^+) and cold spots (x_t^-) repel each other, so the distance between them tends to increase wrt t. In convex domains, the maximum distance is attained for points on the boundary.
This suggests the following.

Heuristics

Consider $u(t,x)$ the solution of the Neumann heat equation in a smooth bounded domain $D \subset \mathbb{R}^d$ with generic initial condition u_0.
Let x_t^+ be the hot spot at time t and x_t^- be the cold spot, i.e.

$$u\left(t, x_t^+\right) = \max_{x \in \overline{D}} u(t,x) \qquad \text{and} \qquad u\left(t, x_t^-\right) = \min_{x \in \overline{D}} u(t,x)$$

If the second Neumann eigenvalue λ_2 is simple, and φ_2 is a corresponding second Neumann eigenfunction, for large t we have

$$u(t,x) = \int_D u_0 + e^{-\lambda_2 t}\varphi_2(x) \int_D u_0 \varphi_2 + R_2(t,x) \approx c_0 + c_1 e^{-\lambda_2 t}\varphi_2(x),$$

so x_t^+ and x_t^- are close to the maximum/minimum points of φ_2.
Hot spots (x_t^+) and cold spots (x_t^-) repel each other, so the distance between them tends to increase wrt t. In convex domains, the maximum distance is attained for points on the boundary.
This suggests the following.

Heuristics

Consider $u\left(t,x\right)$ the solution of the Neumann heat equation in a smooth bounded domain $D\subset\mathbb{R}^d$ with generic initial condition u_0.
Let x_t^+ be the hot spot at time t and x_t^- be the cold spot, i.e.

$$u\left(t,x_t^+\right)=\max_{x\in\overline{D}} u\left(t,x\right) \qquad \text{and} \qquad u\left(t,x_t^-\right)=\min_{x\in\overline{D}} u\left(t,x\right)$$

If the second Neumann eigenvalue λ_2 is simple, and φ_2 is a corresponding second Neumann eigenfunction, for large t we have

$$u\left(t,x\right)=\int_D u_0+e^{-\lambda_2 t}\varphi_2\left(x\right)\int_D u_0\varphi_2+R_2(t,x)\approx c_0+c_1e^{-\lambda_2 t}\varphi_2\left(x\right),$$

so x_t^+ and x_t^- are close to the maximum/minimum points of φ_2.
Hot spots (x_t^+) and cold spots (x_t^-) repel each other, so the distance between them tends to increase wrt t. In convex domains, the maximum distance is attained for points on the boundary.
This suggests the following.

Heuristics

Consider $u(t,x)$ the solution of the Neumann heat equation in a smooth bounded domain $D \subset \mathbb{R}^d$ with generic initial condition u_0.
Let x_t^+ be the hot spot at time t and x_t^- be the cold spot, i.e.

$$u\left(t, x_t^+\right) = \max_{x \in \overline{D}} u(t,x) \qquad \text{and} \qquad u\left(t, x_t^-\right) = \min_{x \in \overline{D}} u(t,x)$$

If the second Neumann eigenvalue λ_2 is simple, and φ_2 is a corresponding second Neumann eigenfunction, for large t we have

$$u(t,x) = \int_D u_0 + e^{-\lambda_2 t} \varphi_2(x) \int_D u_0 \varphi_2 + R_2(t,x) \approx c_0 + c_1 e^{-\lambda_2 t} \varphi_2(x),$$

so x_t^+ and x_t^- are close to the maximum/minimum points of φ_2.
Hot spots (x_t^+) and cold spots (x_t^-) repel each other, so the distance between them tends to increase wrt t. In convex domains, the maximum distance is attained for points on the boundary.
This suggests the following.

Heuristics

Consider $u(t,x)$ the solution of the Neumann heat equation in a smooth bounded domain $D \subset \mathbb{R}^d$ with generic initial condition u_0.
Let x_t^+ be the hot spot at time t and x_t^- be the cold spot, i.e.

$$u\left(t, x_t^+\right) = \max_{x \in \overline{D}} u(t,x) \qquad \text{and} \qquad u\left(t, x_t^-\right) = \min_{x \in \overline{D}} u(t,x)$$

If the second Neumann eigenvalue λ_2 is simple, and φ_2 is a corresponding second Neumann eigenfunction, for large t we have

$$u(t,x) = \int_D u_0 + e^{-\lambda_2 t} \varphi_2(x) \int_D u_0 \varphi_2 + R_2(t,x) \approx c_0 + c_1 e^{-\lambda_2 t} \varphi_2(x),$$

so x_t^+ and x_t^- are close to the maximum/minimum points of φ_2.
Hot spots (x_t^+) and cold spots (x_t^-) repel each other, so the distance between them tends to increase wrt t. In convex domains, the maximum distance is attained for points on the boundary.
This suggests the following.

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y \in D} \varphi_2(y) < \varphi_2(x) < \max_{y \in D} \varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y\in D}\varphi_2(y) < \varphi_2(x) < \max_{y\in D}\varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y\in D}\varphi_2(y) < \varphi_2(x) < \max_{y\in D}\varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y \in D} \varphi_2(y) < \varphi_2(x) < \max_{y \in D} \varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y\in D}\varphi_2(y) < \varphi_2(x) < \max_{y\in D}\varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y\in D}\varphi_2(y) < \varphi_2(x) < \max_{y\in D}\varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y \in D} \varphi_2(y) < \varphi_2(x) < \max_{y \in D} \varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y \in D} \varphi_2(y) < \varphi_2(x) < \max_{y \in D} \varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y \in D} \varphi_2(y) < \varphi_2(x) < \max_{y \in D} \varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

Hot Spots conjecture (Jeffrey Rauch, 1974)

Conjecture 1 (Hot Spots conjecture)

For any smooth bounded convex domain $D \subset \mathbb{R}^d$

$$\min_{y \in D} \varphi_2(y) < \varphi_2(x) < \max_{y \in D} \varphi_2(y), \qquad x \in D,$$

where φ_2 is any second Neumann eigenfunction of the Laplacian on D.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex D): min$\in D$, max$\in \partial D$
- R. Bass and K. Burdzy: stronger counterexample (non-convex D): min, max$\in D$
- D. Jerisson and N. Nadirashvili: true if D has two orthogonal axis of symmetry
- R. Bañuelos and K. Burdzy: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 symmetric wrt it
- P: true if D has two orthogonal axes of symmetry, or just one axis of symmetry and φ_2 antisymmetric wrt it, or ... (some condition on the nodal set of φ_2)
- Other results: true for obtuse triangles, for some some doubly connected domains (Burdzy), for nearly circular domains (Miyamoto), for certain acute triangles (Siudeja).

HS still open in its full generality! (e.g., proof for **acute triangles**?...)

A possible new approach to tackle HS

Using a *mirror coupling* of RBMs, we showed [JFA'11] that the Neumann heat kernel in the ball satisfies

$$p_{\mathbb{U}}(t,y,z) \le p_{\mathbb{U}}(t,x,z),$$

whenever $\|y\| \le \|x\|$ and $\|x-z\| \le \|y-z\|$.

Using this we obtained

$$\int_{\partial\mathbb{U}} p_{\mathbb{U}}(t,x+ru,x)\,d\sigma(u) \le p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x\right) \le p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x+r\|x\|\right), \qquad (1)$$

which implies that the radial derivative of $p_{\mathbb{U}}(t,x,x)$ is non-negative, hence $p_{\mathbb{U}}(t,x,x)$ is increasing in $\|x\|$ (**Laugesen-Morpurgo conjecture**).

In turn, LM conjecture implies the HS conjecture.

A possible new approach to tackle HS

Using a *mirror coupling* of RBMs, we showed [JFA'11] that the Neumann heat kernel in the ball satisfies

$$p_{\mathbb{U}}(t,y,z) \leq p_{\mathbb{U}}(t,x,z),$$

whenever $\|y\| \leq \|x\|$ and $\|x-z\| \leq \|y-z\|$.

Using this we obtained

$$\int_{\partial\mathbb{U}} p_{\mathbb{U}}(t,x+ru,x)\,d\sigma(u) \leq p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x\right) \leq p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x+r\|x\|\right), \qquad (1)$$

which implies that the radial derivative of $p_{\mathbb{U}}(t,x,x)$ is non-negative, hence $p_{\mathbb{U}}(t,x,x)$ is increasing in $\|x\|$ (**Laugesen-Morpurgo conjecture**).

In turn, LM conjecture implies the HS conjecture.

A possible new approach to tackle HS

Using a *mirror coupling* of RBMs, we showed [JFA'11] that the Neumann heat kernel in the ball satisfies

$$p_{\mathbb{U}}\left(t,y,z\right) \leq p_{\mathbb{U}}\left(t,x,z\right),$$

whenever $\|y\| \leq \|x\|$ and $\|x-z\| \leq \|y-z\|$.

Using this we obtained

$$\int_{\partial\mathbb{U}} p_{\mathbb{U}}\left(t,x+ru,x\right) d\sigma(u) \leq p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x\right) \leq p_{\mathbb{U}}\left(t,x+r\frac{x}{\|x\|},x+r\|x\|\right), \qquad (1)$$

which implies that the radial derivative of $p_{\mathbb{U}}\left(t,x,x\right)$ is non-negative, hence $p_{\mathbb{U}}\left(t,x,x\right)$ is increasing in $\|x\|$ (**Laugesen-Morpurgo conjecture**).

In turn, LM conjecture implies the HS conjecture.

A possible new approach to tackle HS

Using a *mirror coupling* of RBMs, we showed [JFA'11] that the Neumann heat kernel in the ball satisfies

$$p_{\mathbb{U}}(t, y, z) \le p_{\mathbb{U}}(t, x, z),$$

whenever $\|y\| \le \|x\|$ and $\|x - z\| \le \|y - z\|$.

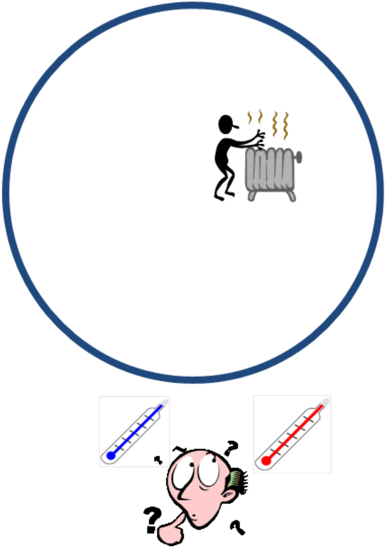
Using this we obtained

$$\int_{\partial \mathbb{U}} p_{\mathbb{U}}(t, x + ru, x)\, d\sigma(u) \le p_{\mathbb{U}}\left(t, x + r\frac{x}{\|x\|}, x\right) \le p_{\mathbb{U}}\left(t, x + r\frac{x}{\|x\|}, x + r\|x\|\right), \qquad (1)$$

which implies that the radial derivative of $p_{\mathbb{U}}(t, x, x)$ is non-negative, hence $p_{\mathbb{U}}(t, x, x)$ is increasing in $\|x\|$ (**Laugesen-Morpurgo conjecture**).

In turn, LM conjecture implies the HS conjecture.

Laugesen-Morpurgo conjecture: a furniture arrangement problem



– *Where should I put the radiator, to feel warmest* ***at all times****?*

A possible new approach to tackle HS

Raw idea: borrowing from this, try to prove the following inequality for a general convex domain D

$$p_D\left(t,x,x\right)\leq\int_{\partial\mathbb{U}}p_D\left(t,x^*,x+ru\right)d\mu(u)\leq p_D\left(t,x^*,x^*\right), \tag{2}$$

for a certain probability measure μ on $\partial\mathbb{U}$ and x^* with $\|x-x^*\|=r$ (depending on D and x).

Remark: the above would prove a general LM conjecture, thus also proving HS conjecture.

A possible new approach to tackle HS

Raw idea: borrowing from this, try to prove the following inequality for a general convex domain D

$$p_D\left(t,x,x\right) \leq \int_{\partial\mathbb{U}} p_D\left(t,x^*,x+ru\right) d\mu(u) \leq p_D\left(t,x^*,x^*\right), \tag{2}$$

for a certain probability measure μ on $\partial\mathbb{U}$ and x^* with $\|x-x^*\| = r$ (depending on D and x).

Remark: the above would prove a general LM conjecture, thus also proving HS conjecture.

Fixed-distance couplings of RBMs

The ideea for proving the first inequality is to construct a *fixed-distance* coupling of RBMs.

Remark: there are no *shy couplings* in sufficiently regular domains (Burdzy, Kendall et.al.)!

Ideea is still feasible, if we require fixed-distance when processes are **away** from the boundary.

Example: in the case of the upper half=plane $\mathcal{H}$, if $\tilde{B}_t = (X_t, Y_t)$ is a free 2-dimensional BM, then

$$B_t = (X_t, |Y_t|), \qquad W_t = (X_t + a, |Y_t + b|) \tag{3}$$

defines a fixed-distance coupling of RBMs in $\mathcal{H}$:

$$|W_t - B_t| = |v_t| = |v_0|,$$

at all times t when processes are away from the boundary, where $v_0 = W_0 - B_0 = (a, b)$.

Fixed-distance couplings of RBMs

The ideea for proving the first inequality is to construct a *fixed-distance* coupling of RBMs.

Remark: there are no *shy couplings* in sufficiently regular domains (Burdzy, Kendall et.al.)!

Ideea is still feasible, if we require fixed-distance when processes are **away** from the boundary.

Example: in the case of the upper half=plane $\mathcal{H}$, if $\tilde{B}_t = (X_t, Y_t)$ is a free 2-dimensional BM, then

$$B_t = (X_t, |Y_t|), \qquad W_t = (X_t + a, |Y_t + b|) \tag{3}$$

defines a fixed-distance coupling of RBMs in $\mathcal{H}$:

$$|W_t - B_t| = |v_t| = |v_0|,$$

at all times t when processes are away from the boundary, where $v_0 = W_0 - B_0 = (a, b)$.

Fixed-distance couplings of RBMs

The ideea for proving the first inequality is to construct a *fixed-distance* coupling of RBMs.

Remark: there are no *shy couplings* in sufficiently regular domains (Burdzy, Kendall et.al.)!

Ideea is still feasible, if we require fixed-distance when processes are **away** from the boundary.

Example: in the case of the upper half=plane $\mathcal{H}$, if $\tilde{B}_t = (X_t, Y_t)$ is a free 2-dimensional BM, then

$$B_t = (X_t, |Y_t|), \qquad W_t = (X_t + a, |Y_t + b|) \tag{3}$$

defines a fixed-distance coupling of RBMs in $\mathcal{H}$:

$$|W_t - B_t| = |v_t| = |v_0|,$$

at all times t when processes are away from the boundary, where $v_0 = W_0 - B_0 = (a, b)$.

Fixed-distance couplings of RBMs

The ideea for proving the first inequality is to construct a *fixed-distance* coupling of RBMs.

Remark: there are no *shy couplings* in sufficiently regular domains (Burdzy, Kendall et.al.)!

Ideea is still feasible, if we require fixed-distance when processes are **away** from the boundary.

Example: in the case of the upper half=plane $\mathcal{H}$, if $\tilde{B}_t = (X_t, Y_t)$ is a free 2-dimensional BM, then

$$B_t = (X_t, |Y_t|), \qquad W_t = (X_t + a, |Y_t + b|) \tag{3}$$

defines a fixed-distance coupling of RBMs in $\mathcal{H}$:

$$|W_t - B_t| = |v_t| = |v_0|,$$

at all times t when processes are away from the boundary, where $v_0 = W_0 - B_0 = (a, b)$.

Fixed-distance couplings of RBMs

The ideea for proving the first inequality is to construct a *fixed-distance* coupling of RBMs.

Remark: there are no *shy couplings* in sufficiently regular domains (Burdzy, Kendall et.al.)!

Ideea is still feasible, if we require fixed-distance when processes are **away** from the boundary.

Example: in the case of the upper half=plane $\mathcal{H}$, if $\tilde{B}_t = (X_t, Y_t)$ is a free 2-dimensional BM, then

$$B_t = (X_t, |Y_t|), \qquad W_t = (X_t + a, |Y_t + b|) \tag{3}$$

defines a fixed-distance coupling of RBMs in $\mathcal{H}$:

$$|W_t - B_t| = |v_t| = |v_0|,$$

at all times t when processes are away from the boundary, where $v_0 = W_0 - B_0 = (a, b)$.

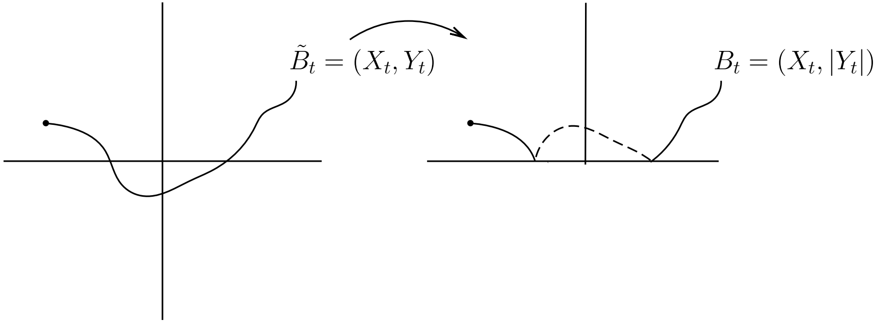


Figure : From BM to RBM in a half-plane (i.e. Tanaka's formula)

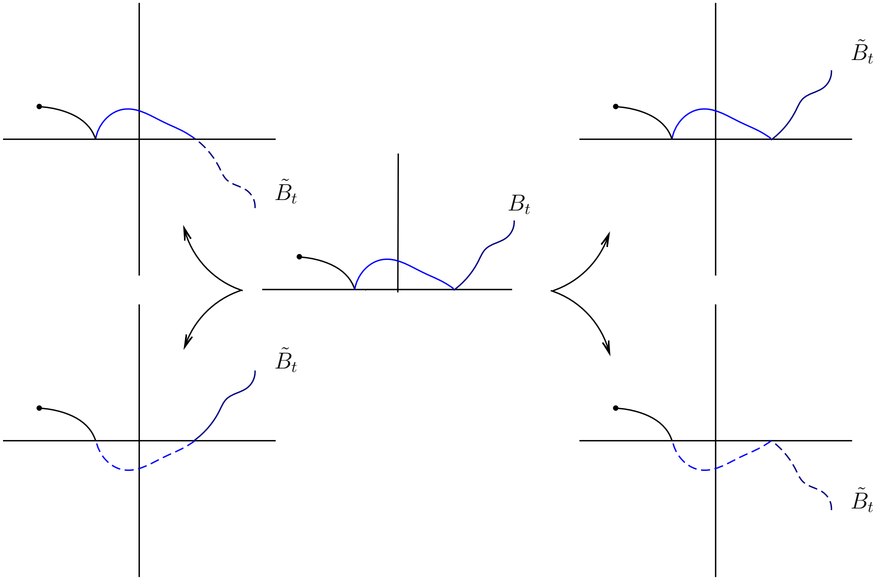


Figure : From RBM in a half-plane to BM

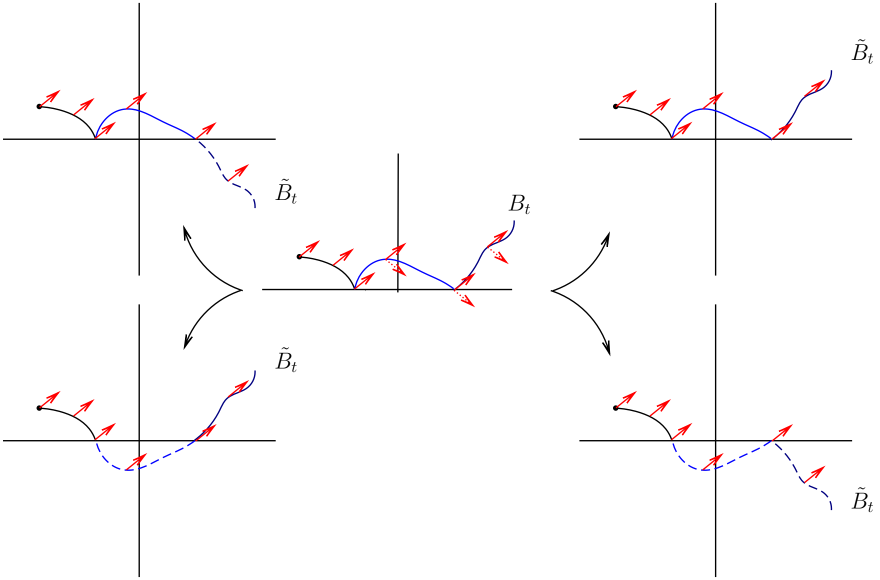


Figure : From RBM in a half-plane to BM

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = \left(B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), \left|\mathrm{Im}(\tilde{B}_t + v_t)\right|\right)\right)$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = \left(B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), \left|\mathrm{Im}(\tilde{B}_t + v_t)\right|\right)\right)$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = \left(B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), \left|\mathrm{Im}(\tilde{B}_t + v_t)\right|\right)\right)$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = \left(B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), |\mathrm{Im}(\tilde{B}_t + v_t)|\right)\right)$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = \left(B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), |\mathrm{Im}(\tilde{B}_t + v_t)|\right)\right)$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = \left(B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), |\mathrm{Im}(\tilde{B}_t + v_t)|\right)\right)$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = (B_t, (\mathrm{Re}(\tilde{B}_t + v_t), |\mathrm{Im}(\tilde{B}_t + v_t)|))$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Dynamics of the spin ("translation vector") $v_t = W_t - B_t = v_0 e^{i\theta_t}$:

- constant magnitude
- constant direction during excursions
- direction can change (by reflection in the boundary) with probability $1/2$ at beginning of excursions

$(B_t, W_t) = (B_t, \left(\mathrm{Re}(\tilde{B}_t + v_t), |\mathrm{Im}(\tilde{B}_t + v_t)|\right))$ defines a "fixed-distance coupling" (shy coupling) of RBMs, save the times when near the boundary.

Same can be done in

- wedges of angles $\frac{\pi}{n}$ (perhaps any angle)
- equilateral triangles (perhaps any triangles)
- convex polygonal domains (perhaps)
- smooth convex domains (perhaps)

Gain: $P^x(B_t \in dy) = \int_0^{2\pi} P^{x+v_0}(W_t \in v_0 e_t^{i\theta} + dy, \theta_t \in d\theta)$

Control (monotonicity) of IHM $\Longrightarrow$ control (monotonicity) of θ_t

$\Longrightarrow p_D(t,x,y) \lesseqgtr p_D(t,x+v_0,y+v)$ for some $v = v(x,y,t)$
(monotonicity of transition density of RBM in D)

$\Longrightarrow$ monotonicity of second Neumann eigenfunction(s) for D
(a.k.a. *Hot Spots conjecture*)

Gain: $P^x(B_t \in dy) = \int_0^{2\pi} P^{x+v_0}(W_t \in v_0 e_t^{i\theta} + dy, \theta_t \in d\theta)$

Control (monotonicity) of IHM $\Longrightarrow$ control (monotonicity) of θ_t

$\Longrightarrow p_D(t,x,y) \lesseqgtr p_D(t,x+v_0,y+v)$ for some $v = v(x,y,t)$
(monotonicity of transition density of RBM in D)

$\Longrightarrow$ monotonicity of second Neumann eigenfunction(s) for D
(a.k.a. *Hot Spots conjecture*)

Gain: $P^x(B_t \in dy) = \int_0^{2\pi} P^{x+v_0}(W_t \in v_0 e_t^{i\theta} + dy, \theta_t \in d\theta)$

Control (monotonicity) of IHM $\Longrightarrow$ control (monotonicity) of θ_t

$\Longrightarrow p_D(t,x,y) \lesseqgtr p_D(t,x+v_0,y+v)$ for some $v = v(x,y,t)$
(monotonicity of transition density of RBM in D)

$\Longrightarrow$ monotonicity of second Neumann eigenfunction(s) for D
(a.k.a. *Hot Spots conjecture*)

Gain: $P^x(B_t \in dy) = \int_0^{2\pi} P^{x+v_0}(W_t \in v_0 e_t^{i\theta} + dy, \theta_t \in d\theta)$

Control (monotonicity) of IHM $\Longrightarrow$ control (monotonicity) of θ_t

$\Longrightarrow p_D(t,x,y) \lesseqgtr p_D(t,x+v_0,y+v)$ for some $v = v(x,y,t)$
(monotonicity of transition density of RBM in D)

$\Longrightarrow$ monotonicity of second Neumann eigenfunction(s) for D
(a.k.a. *Hot Spots conjecture*)

Gain: $P^x(B_t \in dy) = \displaystyle\int_0^{2\pi} P^{x+v_0}(W_t \in v_0 e_t^{i\theta} + dy, \theta_t \in d\theta)$

Control (monotonicity) of IHM $\Longrightarrow$ control (monotonicity) of θ_t

$\Longrightarrow p_D(t,x,y) \lesseqgtr p_D(t,x+v_0,y+v)$ for some $v = v(x,y,t)$
(monotonicity of transition density of RBM in D)

$\Longrightarrow$ monotonicity of second Neumann eigenfunction(s) for D
(a.k.a. *Hot Spots conjecture*)

Harmonic measure

Harmonic measure (Rolf Nevanlinna, 1928-1929, [9]): the measure $\omega_D(z,\alpha)$ on the boundary of a domain $D \subset \mathbb{C}$, which is harmonic and bounded with respect to $z \in D$, and assumes the value 1 on $\alpha \subset \partial D$, and 0 on $\partial D - \alpha$.

Remarks:

- extends naturally to higher dimensional Euclidean spaces
- important tool in the study of harmonic and analytic functions: maximum principles of analytic functions, solution of the first boundary problem, connection to Poisson kernel and Green's function, aso.

Probabilistic interpretation (S. Kakutani, 1944, [7]): *harmonic measure* is just the exit distribution of the Brownian motion from the domain, that is:

$$\omega(z,C) = P^z\left(B_{\tau_D} \in C\right), \qquad C \in \partial D. \tag{4}$$

Connection with differential equations: the solution of the Dirichlet problem for D with continuous boundary data $f : \partial D \to \mathbb{R}$ has the representation

$$u(z) = E^z f(B_\tau) = \int_{\partial D} f(w)\,\omega(z,dw), \tag{5}$$

i.e. the kernel of the harmonic measure $\omega(z,\cdot)$ is just the Poisson kernel for the Dirichlet problem for D with respect to $z \in D$.

Harmonic measure

Harmonic measure (Rolf Nevanlinna, 1928-1929, [9]): the measure $\omega_D(z,\alpha)$ on the boundary of a domain $D \subset \mathbb{C}$, which is harmonic and bounded with respect to $z \in D$, and assumes the value 1 on $\alpha \subset \partial D$, and 0 on $\partial D - \alpha$.

Remarks:

- extends naturally to higher dimensional Euclidean spaces
- important tool in the study of harmonic and analytic functions: maximum principles of analytic functions, solution of the first boundary problem, connection to Poisson kernel and Green's function, aso.

Probabilistic interpretation (S. Kakutani, 1944, [7]): *harmonic measure* is just the exit distribution of the Brownian motion from the domain, that is:

$$\omega(z,C) = P^z\left(B_{\tau_D} \in C\right), \qquad C \in \partial D. \tag{4}$$

Connection with differential equations: the solution of the Dirichlet problem for D with continuous boundary data $f : \partial D \to \mathbb{R}$ has the representation

$$u(z) = E^z f(B_\tau) = \int_{\partial D} f(w)\,\omega(z,dw), \tag{5}$$

i.e. the kernel of the harmonic measure $\omega(z,\cdot)$ is just the Poisson kernel for the Dirichlet problem for D with respect to $z \in D$.

Harmonic measure

Harmonic measure (Rolf Nevanlinna, 1928-1929, [9]): the measure $\omega_D(z, \alpha)$ on the boundary of a domain $D \subset \mathbb{C}$, which is harmonic and bounded with respect to $z \in D$, and assumes the value 1 on $\alpha \subset \partial D$, and 0 on $\partial D - \alpha$.

Remarks:

- extends naturally to higher dimensional Euclidean spaces
- important tool in the study of harmonic and analytic functions: maximum principles of analytic functions, solution of the first boundary problem, connection to Poisson kernel and Green's function, aso.

Probabilistic interpretation (S. Kakutani, 1944, [7]): *harmonic measure* is just the exit distribution of the Brownian motion from the domain, that is:

$$\omega(z, C) = P^z(B_{\tau_D} \in C), \qquad C \in \partial D. \tag{4}$$

Connection with differential equations: the solution of the Dirichlet problem for D with continuous boundary data $f : \partial D \to \mathbb{R}$ has the representation

$$u(z) = E^z f(B_\tau) = \int_{\partial D} f(w)\, \omega(z, dw), \tag{5}$$

i.e. the kernel of the harmonic measure $\omega(z, \cdot)$ is just the Poisson kernel for the Dirichlet problem for D with respect to $z \in D$.

Harmonic measure

Harmonic measure (Rolf Nevanlinna, 1928-1929, [9]): the measure $\omega_D(z,\alpha)$ on the boundary of a domain $D \subset \mathbb{C}$, which is harmonic and bounded with respect to $z \in D$, and assumes the value 1 on $\alpha \subset \partial D$, and 0 on $\partial D - \alpha$.

Remarks:

- extends naturally to higher dimensional Euclidean spaces
- important tool in the study of harmonic and analytic functions: maximum principles of analytic functions, solution of the first boundary problem, connection to Poisson kernel and Green's function, aso.

Probabilistic interpretation (S. Kakutani, 1944, [7]): *harmonic measure* is just the exit distribution of the Brownian motion from the domain, that is:

$$\omega(z, C) = P^z\left(B_{\tau_D} \in C\right), \qquad C \in \partial D. \tag{4}$$

Connection with differential equations: the solution of the Dirichlet problem for D with continuous boundary data $f : \partial D \to \mathbb{R}$ has the representation

$$u(z) = E^z f(B_\tau) = \int_{\partial D} f(w)\,\omega(z, dw), \tag{5}$$

i.e. the kernel of the harmonic measure $\omega(z,\cdot)$ is just the Poisson kernel for the Dirichlet problem for D with respect to $z \in D$.

Harmonic measure

Harmonic measure (Rolf Nevanlinna, 1928-1929, [9]): the measure $\omega_D(z, \alpha)$ on the boundary of a domain $D \subset \mathbb{C}$, which is harmonic and bounded with respect to $z \in D$, and assumes the value 1 on $\alpha \subset \partial D$, and 0 on $\partial D - \alpha$.

Remarks:

- extends naturally to higher dimensional Euclidean spaces
- important tool in the study of harmonic and analytic functions: maximum principles of analytic functions, solution of the first boundary problem, connection to Poisson kernel and Green's function, aso.

Probabilistic interpretation (S. Kakutani, 1944, [7]): *harmonic measure* is just the exit distribution of the Brownian motion from the domain, that is:

$$\omega(z, C) = P^z\left(B_{\tau_D} \in C\right), \qquad C \in \partial D. \tag{4}$$

Connection with differential equations: the solution of the Dirichlet problem for D with continuous boundary data $f : \partial D \to \mathbb{R}$ has the representation

$$u(z) = E^z f(B_\tau) = \int_{\partial D} f(w)\, \omega(z, dw), \tag{5}$$

i.e. the kernel of the harmonic measure $\omega(z, \cdot)$ is just the Poisson kernel for the Dirichlet problem for D with respect to $z \in D$.

Extension: *iterated harmonic measure*

Idea: replace the Brownian motion by a reflecting Brownian motion in D (the same up to the first exit time from the domain).

Consider a partition $A_1, \ldots, A_m \subset \partial D$ of ∂D into measurable sets, and define the succesive hitting times $T_1 < T_2 < \ldots$ of the boundary ∂D when the reflected Brownian motion hits a set A_i different from the previous hit.

We define the *iterated harmonic measure* by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_m) = P^z\left(B_{T_1} \in C_1, \ldots, B_{T_m} \in C_m\right). \tag{6}$$

The name *iterated harmonic measure was suggested to us by Theorem 3 below, which gives an "iterative" construction of the above generalized harmonic measure.*

Extension: *iterated harmonic measure*

Idea: replace the Brownian motion by a reflecting Brownian motion in D (the same up to the first exit time from the domain).
Consider a partition $A_1, \ldots, A_m \subset \partial D$ of ∂D into measurable sets, and define the succesive hitting times $T_1 < T_2 < \ldots$ of the boundary ∂D when the reflected Brownian motion hits a set A_i different from the previous hit.

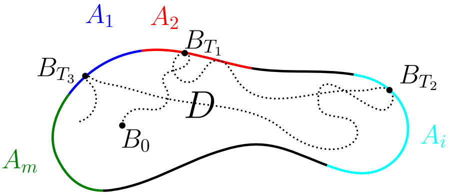
We define the *iterated harmonic measure* by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_m) = P^z\left(B_{T_1} \in C_1, \ldots, B_{T_m} \in C_m\right). \tag{6}$$

The name *iterated harmonic measure was suggested to us by Theorem 3 below, which gives an "iterative" construction of the above generalized harmonic measure.*

Extension: *iterated harmonic measure*

Idea: replace the Brownian motion by a reflecting Brownian motion in D (the same up to the first exit time from the domain).
Consider a partition $A_1, \dots, A_m \subset \partial D$ of ∂D into measurable sets, and define the succesive hitting times $T_1 < T_2 < \dots$ of the boundary ∂D when the reflected Brownian motion hits a set A_i different from the previous hit.



We define the *iterated harmonic measure* by

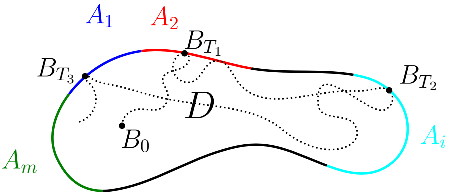
$$\omega_{D,n}^{A_1,\dots,A_m}(z, C_1, \dots, C_m) = P^z\left(B_{T_1} \in C_1, \dots, B_{T_m} \in C_m\right). \qquad (6)$$

The name *iterated harmonic measure was suggested to us by Theorem 3 below, which gives an "iterative" construction of the above generalized harmonic measure.*

Extension: *iterated harmonic measure*

Idea: replace the Brownian motion by a reflecting Brownian motion in D (the same up to the first exit time from the domain).

Consider a partition $A_1, \ldots, A_m \subset \partial D$ of ∂D into measurable sets, and define the succesive hitting times $T_1 < T_2 < \ldots$ of the boundary ∂D when the reflected Brownian motion hits a set A_i different from the previous hit.



We define the *iterated harmonic measure* by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_m) = P^z(B_{T_1} \in C_1, \ldots, B_{T_m} \in C_m). \tag{6}$$

The name *iterated harmonic measure was suggested to us by Theorem 3 below, which gives an "iterative" construction of the above generalized harmonic measure.*

IHM (setup)

Consider $D \subset \mathbb{R}^2$ a smooth planar domain (Lipschitz or $C^{1,\alpha}$ domain $(0 < \alpha < 1)$).

Let $A_1, \ldots, A_m \subset \partial D$ be unions of disjoint open arcs, $A = \cup_{i=1}^{m} A_i$, and $A_0 = \partial D - A$.

Denote by T_C the first hit of C by RBM $(B_t)_{t\geq 0}$ in D.

Define the sequence $(T_n)_{n\geq 1}$ of random times defined by $T_1 = T_A$, and for $i \geq 1$ by

$$T_{i+1} = \inf\left\{t \geq T_i : B_t \in A_i'\right\}, \tag{7}$$

where $A_i' = A - A_j$ if $B_{T_i} \in A_j$ for some $j \in \{1, \ldots, n\}$.

Remark. $(T_n)_{n\geq 1}$ is a strictly increasing sequence of a.s. finite stopping times.

IHM (setup)

Consider $D \subset \mathbb{R}^2$ a smooth planar domain (Lipschitz or $C^{1,\alpha}$ domain $(0 < \alpha < 1)$).

Let $A_1, \ldots, A_m \subset \partial D$ be unions of disjoint open arcs, $A = \cup_{i=1}^m A_i$, and $A_0 = \partial D - A$.

Denote by T_C the first hit of C by RBM $(B_t)_{t \geq 0}$ in D.

Define the sequence $(T_n)_{n \geq 1}$ of random times defined by $T_1 = T_A$, and for $i \geq 1$ by

$$T_{i+1} = \inf \left\{ t \geq T_i : B_t \in A_i' \right\}, \tag{7}$$

where $A_i' = A - A_j$ if $B_{T_i} \in A_j$ for some $j \in \{1, \ldots, n\}$.

Remark. $(T_n)_{n \geq 1}$ is a strictly increasing sequence of a.s. finite stopping times.

IHM (setup)

Consider $D \subset \mathbb{R}^2$ a smooth planar domain (Lipschitz or $C^{1,\alpha}$ domain $(0 < \alpha < 1)$).

Let $A_1, \ldots, A_m \subset \partial D$ be unions of disjoint open arcs, $A = \cup_{i=1}^m A_i$, and $A_0 = \partial D - A$.

Denote by T_C the first hit of C by RBM $(B_t)_{t \geq 0}$ in D.

Define the sequence $(T_n)_{n \geq 1}$ of random times defined by $T_1 = T_A$, and for $i \geq 1$ by

$$T_{i+1} = \inf \left\{ t \geq T_i : B_t \in A_i' \right\}, \tag{7}$$

where $A_i' = A - A_j$ if $B_{T_i} \in A_j$ for some $j \in \{1, \ldots, n\}$.

Remark. $(T_n)_{n \geq 1}$ is a strictly increasing sequence of a.s. finite stopping times.

IHM (setup)

Consider $D \subset \mathbb{R}^2$ a smooth planar domain (Lipschitz or $C^{1,\alpha}$ domain $(0 < \alpha < 1)$).

Let $A_1, \ldots, A_m \subset \partial D$ be unions of disjoint open arcs, $A = \cup_{i=1}^{m} A_i$, and $A_0 = \partial D - A$.

Denote by T_C the first hit of C by RBM $(B_t)_{t \geq 0}$ in D.

Define the sequence $(T_n)_{n \geq 1}$ of random times defined by $T_1 = T_A$, and for $i \geq 1$ by

$$T_{i+1} = \inf \left\{ t \geq T_i : B_t \in A_i' \right\}, \tag{7}$$

where $A_i' = A - A_j$ if $B_{T_i} \in A_j$ for some $j \in \{1, \ldots, n\}$.

Remark. $(T_n)_{n \geq 1}$ is a strictly increasing sequence of a.s. finite stopping times.

IHM (definition)

Definition 2

For $n \geq 1$ fixed, the *iterated harmonic measure* with respect D, $z \in \overline{D}$, and the partition $A_0, A_1, \ldots, A_m$ is defined by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = P^z(B_{T_1} \in C_1, \ldots, B_{T_n} \in C_n), \tag{8}$$

for measurable subsets $C_1, \ldots, C_n \subset \partial D$, where P^z denotes the probability distribution of the RBM $(B_t)_{t\geq 0}$ in D starting at $B_0 = z \in \overline{D}$.

Remarks

- fixing $(n-1)$ sets C_i in the definition (8), IHM becomes a measure on $\mathcal{B}(\partial D)$
- in particular, $\omega_{D,n}^{A_1,\ldots,A_m}(z, \partial D, \ldots, \partial D, C) = P^z(B_{T_n} \in C)$ is a measure on ∂D. The case $n = 1$ gives the usual harmonic measure.
- using a version of Carathéodory's extension theorem, one can extend uniquely $\omega_{D,n}^{A_1,\ldots,A_m}(z, \cdot)$ to a measure on $\mathcal{B}(\partial D^n)$.

IHM (definition)

Definition 2

For $n \geq 1$ fixed, the *iterated harmonic measure* with respect D, $z \in \overline{D}$, and the partition $A_0, A_1, \ldots, A_m$ is defined by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = P^z\left(B_{T_1} \in C_1, \ldots, B_{T_n} \in C_n\right), \tag{8}$$

for measurable subsets $C_1, \ldots, C_n \subset \partial D$, where P^z denotes the probability distribution of the RBM $(B_t)_{t\geq 0}$ in D starting at $B_0 = z \in \overline{D}$.

Remarks

- fixing $(n-1)$ sets C_i in the definition (8), IHM becomes a measure on $\mathcal{B}(\partial D)$
- in particular, $\omega_{D,n}^{A_1,\ldots,A_m}(z, \partial D, \ldots, \partial D, C) = P^z(B_{T_n} \in C)$ is a measure on ∂D. The case $n = 1$ gives the usual harmonic measure.
- using a version of Carathéodory's extension theorem, one can extend uniquely $\omega_{D,n}^{A_1,\ldots,A_m}(z, \cdot)$ to a measure on $\mathcal{B}(\partial D^n)$.

IHM (definition)

Definition 2

For $n \geq 1$ fixed, the *iterated harmonic measure* with respect D, $z \in \overline{D}$, and the partition $A_0, A_1, \ldots, A_m$ is defined by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = P^z(B_{T_1} \in C_1, \ldots, B_{T_n} \in C_n), \quad (8)$$

for measurable subsets $C_1, \ldots, C_n \subset \partial D$, where P^z denotes the probability distribution of the RBM $(B_t)_{t\geq 0}$ in D starting at $B_0 = z \in \overline{D}$.

Remarks

- fixing $(n-1)$ sets C_i in the definition (8), IHM becomes a measure on $\mathcal{B}(\partial D)$
- in particular, $\omega_{D,n}^{A_1,\ldots,A_m}(z, \partial D, \ldots, \partial D, C) = P^z(B_{T_n} \in C)$ is a measure on ∂D. The case $n = 1$ gives the usual harmonic measure.
- using a version of Carathéodory's extension theorem, one can extend uniquely $\omega_{D,n}^{A_1,\ldots,A_m}(z, \cdot)$ to a measure on $\mathcal{B}(\partial D^n)$.

IHM (definition)

Definition 2

For $n \geq 1$ fixed, the *iterated harmonic measure* with respect D, $z \in \overline{D}$, and the partition $A_0, A_1, \ldots, A_m$ is defined by

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = P^z\left(B_{T_1} \in C_1, \ldots, B_{T_n} \in C_n\right), \tag{8}$$

for measurable subsets $C_1, \ldots, C_n \subset \partial D$, where P^z denotes the probability distribution of the RBM $(B_t)_{t \geq 0}$ in D starting at $B_0 = z \in \overline{D}$.

Remarks

- fixing $(n-1)$ sets C_i in the definition (8), IHM becomes a measure on $\mathcal{B}(\partial D)$
- in particular, $\omega_{D,n}^{A_1,\ldots,A_m}(z, \partial D, \ldots, \partial D, C) = P^z(B_{T_n} \in C)$ is a measure on ∂D. The case $n = 1$ gives the usual harmonic measure.
- using a version of Carathéodory's extension theorem, one can extend uniquely $\omega_{D,n}^{A_1,\ldots,A_m}(z, \cdot)$ to a measure on $\mathcal{B}(\partial D^n)$.

Main results on IHM

Theorem 3 (Properties of iterated harmonic measure)

a) $\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n)$ *is a bounded harmonic function of* $z \in D$*, continuous on* $\overline{D}$*, and it is a measure in each of the variables* C_i*,* $i = 1, \ldots, n$*.*

b) *(Conformal invariance) If* $f : D \to D'$ *is a conformal map, then*

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = \omega_{D',n}^{f(A_1),\ldots,f(A_m)}(f(z), f(C_1), \ldots, f(C_n)), \quad z \in \overline{D}, \tag{9}$$

for any $n \geq 1$ *and any measurable subsets* $C_1, \ldots, C_n \subset \partial D$*.*

c) *(Recursive/iterative construction) If* A_0 *is polar for Brownian motion, then*

$$\omega_{D,1}^{A_1,\ldots,A_m}(z, \cdot) = \omega_D(z, \cdot), \tag{10}$$

and for any $n > 1$ *and measurable sets* $C_1, \ldots, C_n \subset \partial D$ *we have*

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = \int_{C_1} \omega_{D,1}^{A_1,\ldots,A_m}(z, dy_1) \int_{C_2} \omega_{D,2}^{A_1,\ldots,A_m}(y_1, dy_2) \cdots \int_{C_n} \omega_{D,2}^{A_1,\ldots,A_m}(y_{n-1}, dy_n). \tag{11}$$

Main results on IHM

Theorem 3 (Properties of iterated harmonic measure)

a) $\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n)$ *is a bounded harmonic function of* $z \in D$*, continuous on* $\overline{D}$*, and it is a measure in each of the variables* C_i*,* $i = 1, \ldots, n$*.*

b) *(Conformal invariance) If* $f : D \to D'$ *is a conformal map, then*

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = \omega_{D',n}^{f(A_1),\ldots,f(A_m)}(f(z), f(C_1), \ldots, f(C_n)), \quad z \in \overline{D}, \tag{9}$$

for any $n \geq 1$ *and any measurable subsets* $C_1, \ldots, C_n \subset \partial D$*.*

c) *(Recursive/iterative construction) If* A_0 *is polar for Brownian motion, then*

$$\omega_{D,1}^{A_1,\ldots,A_m}(z, \cdot) = \omega_D(z, \cdot), \tag{10}$$

and for any $n > 1$ *and measurable sets* $C_1, \ldots, C_n \subset \partial D$ *we have*

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = \int_{C_1} \omega_{D,1}^{A_1,\ldots,A_m}(z, dy_1) \int_{C_2} \omega_{D,2}^{A_1,\ldots,A_m}(y_1, dy_2) \cdots \int_{C_n} \omega_{D,2}^{A_1,\ldots,A_m}(y_{n-1}, dy_n). \tag{11}$$

Main results on IHM

Theorem 3 (Properties of iterated harmonic measure)

a) $\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n)$ *is a bounded harmonic function of* $z \in D$*, continuous on* $\overline{D}$*, and it is a measure in each of the variables* C_i*,* $i = 1, \ldots, n$*.*

b) *(Conformal invariance) If* $f : D \to D'$ *is a conformal map, then*

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = \omega_{D',n}^{f(A_1),\ldots,f(A_m)}(f(z), f(C_1), \ldots, f(C_n)), \quad z \in \overline{D}, \tag{9}$$

for any $n \geq 1$ *and any measurable subsets* $C_1, \ldots, C_n \subset \partial D$*.*

c) *(Recursive/iterative construction) If* A_0 *is polar for Brownian motion, then*

$$\omega_{D,1}^{A_1,\ldots,A_m}(z, \cdot) = \omega_D(z, \cdot), \tag{10}$$

and for any $n > 1$ *and measurable sets* $C_1, \ldots, C_n \subset \partial D$ *we have*

$$\omega_{D,n}^{A_1,\ldots,A_m}(z, C_1, \ldots, C_n) = \int_{C_1} \omega_{D,1}^{A_1,\ldots,A_m}(z, dy_1) \int_{C_2} \omega_{D,2}^{A_1,\ldots,A_m}(y_1, dy_2) \cdots \int_{C_n} \omega_{D,2}^{A_1,\ldots,A_m}(y_{n-1}, dy_n). \tag{11}$$

The case of the half-plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

Consider $A_1 = (-\infty, 0)$, $A_2\,(0,1)$, $A_3 = (1, \infty)$, and $A_0 = \{0, 1\}$.

Theorem 4

The distribution of the RBM $(B_t)_{t\geq 0}$ in $\mathcal{H}$ starting at $B_0 = u \in \partial\mathcal{H} - \{0,1\}$ at time T_2 is given by

$$P^u\left(\operatorname{Re} B_{T_2} \leq v\right) = \begin{cases} 1 - \frac{2}{\pi}\arctan\sqrt{\frac{1-v}{u-1}}, & v < 1 < u, \\ 1 - \frac{2}{\pi}\arctan\sqrt{-\frac{u}{v}}, & u < 0 < v, \\ 1 - \frac{2}{\pi}\arctan\sqrt{\frac{v(1-u)}{u(v-1)}}, & u \in (0,1),\ v \in (-\infty, 0) \cup (1, \infty), \end{cases} \tag{12}$$

In particular, the Poisson kernel for the BM in $\mathcal{H}$ reflected on the part $A_i \subset \partial\mathcal{H}$ where the Brownian motion starts ($i = 1, 2,$ or 3), and killed on the remaining part of $\partial\mathcal{H}$, is given by

$$k_{\mathcal{H}}(u,v) = \frac{\omega_{\mathcal{H},2}^{A_1,A_2,A_3}(u, A_i, dv)}{dv} = \frac{P^u\left(\operatorname{Re} B_{T_2} \in dv\right)}{dv} = \begin{cases} \frac{1}{\pi}\frac{1}{u-v}\sqrt{\frac{u-1}{1-v}}, & v < 1 < u, \\ \frac{1}{\pi}\frac{1}{v-u}\sqrt{\frac{-u}{v}}, & u < 0 < v, \\ \frac{1}{\pi}\frac{1}{v-u}\sqrt{\frac{(1-u)u}{(v-1)v}}, & u \in (0,1),\ v \notin [0,1], \end{cases} \tag{13}$$

The case of the half-plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

Consider $A_1 = (-\infty, 0)$, $A_2\,(0,1)$, $A_3 = (1, \infty)$, and $A_0 = \{0, 1\}$.

Theorem 4

The distribution of the RBM $(B_t)_{t\geq 0}$ in $\mathcal{H}$ starting at $B_0 = u \in \partial\mathcal{H} - \{0,1\}$ at time T_2 is given by

$$P^u\left(\operatorname{Re} B_{T_2} \leq v\right) = \begin{cases} 1 - \frac{2}{\pi}\arctan\sqrt{\frac{1-v}{u-1}}, & v < 1 < u, \\ 1 - \frac{2}{\pi}\arctan\sqrt{-\frac{u}{v}}, & u < 0 < v, \\ 1 - \frac{2}{\pi}\arctan\sqrt{\frac{v(1-u)}{u(v-1)}}, & u \in (0,1),\ v \in (-\infty, 0) \cup (1, \infty), \end{cases} \tag{12}$$

In particular, the Poisson kernel for the BM in $\mathcal{H}$ reflected on the part $A_i \subset \partial\mathcal{H}$ where the Brownian motion starts ($i = 1, 2,$ or 3), and killed on the remaining part of $\partial\mathcal{H}$, is given by

$$k_{\mathcal{H}}(u,v) = \frac{\omega_{\mathcal{H},2}^{A_1,A_2,A_3}(u, A_i, dv)}{dv} = \frac{P^u\left(\operatorname{Re} B_{T_2} \in dv\right)}{dv} = \begin{cases} \frac{1}{\pi}\frac{1}{u-v}\sqrt{\frac{u-1}{1-v}}, & v < 1 < u, \\ \frac{1}{\pi}\frac{1}{v-u}\sqrt{\frac{-u}{v}}, & u < 0 < v, \\ \frac{1}{\pi}\frac{1}{v-u}\sqrt{\frac{(1-u)u}{(v-1)v}}, & u \in (0,1),\ v \notin [0,1], \end{cases} \tag{13}$$

Proof.

Consider the case $B_0 = u > 1$ (reflection on A_3), and let $v < 1$ be arbitrarily fixed.

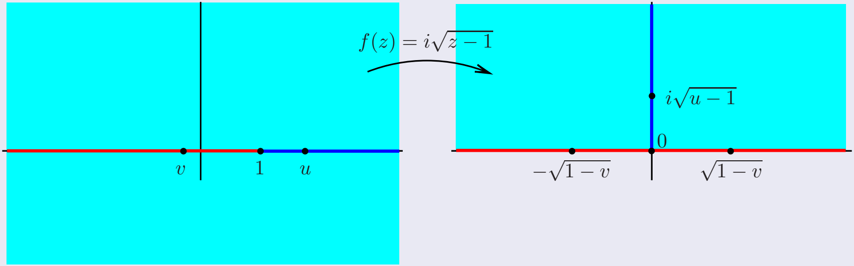


Figure : The image of the slit plane $D = \mathbb{C} - (-\infty, 1]$ under the conformal map $f(z) = i\sqrt{z-1}$.

$$P^u\left(\operatorname{Re} B_{T_2} \leq v\right) = P^u\left(\operatorname{Re} Z_{\tau_D} \leq v\right) = P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \leq -\sqrt{1-v}\right) + P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \geq \sqrt{1-v}\right).$$

The last two probabilities are equal by symmetry, and using the Poisson kernel for the half-plane we conclude

$$P^u\left(\operatorname{Re} B_{T_1} \leq v\right) = 2P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \leq -\sqrt{1-v}\right) = \frac{2}{\pi}\int_{-\infty}^{-\sqrt{1-v}} \frac{\sqrt{u-1}}{x^2+u-1}dx = 1 - \frac{2}{\pi}\arctan\sqrt{\frac{1-v}{u-1}}.$$

Proof.

Consider the case $B_0 = u > 1$ (reflection on A_3), and let $v < 1$ be arbitrarily fixed.

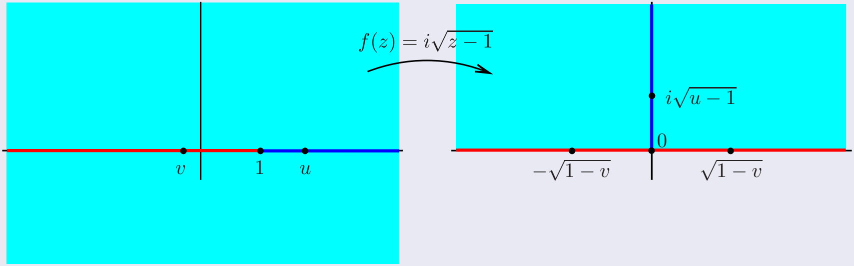


Figure : The image of the slit plane $D = \mathbb{C} - (-\infty, 1]$ under the conformal map $f(z) = i\sqrt{z-1}$.

$$P^u\left(\mathrm{Re}\, B_{T_2} \leq v\right) = P^u\left(\mathrm{Re}\, Z_{\tau_D} \leq v\right) = P^{i\sqrt{u-1}}\left(\mathrm{Re}\, W_{\tau_{\mathcal{H}}} \leq -\sqrt{1-v}\right) + P^{i\sqrt{u-1}}\left(\mathrm{Re}\, W_{\tau_{\mathcal{H}}} \geq \sqrt{1-v}\right).$$

The last two probabilities are equal by symmetry, and using the Poisson kernel for the half-plane we conclude

$$P^u\left(\mathrm{Re}\, B_{T_1} \leq v\right) = 2P^{i\sqrt{u-1}}\left(\mathrm{Re}\, W_{\tau_{\mathcal{H}}} \leq -\sqrt{1-v}\right) = \frac{2}{\pi}\int_{-\infty}^{-\sqrt{1-v}} \frac{\sqrt{u-1}}{x^2+u-1}dx = 1 - \frac{2}{\pi}\arctan\sqrt{\frac{1-v}{u-1}}.$$

Proof.

Consider the case $B_0 = u > 1$ (reflection on A_3), and let $v < 1$ be arbitrarily fixed.

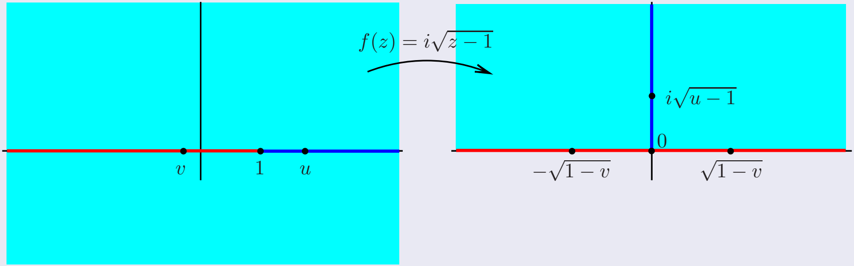


Figure : The image of the slit plane $D = \mathbb{C} - (-\infty, 1]$ under the conformal map $f(z) = i\sqrt{z-1}$.

$$P^u\left(\operatorname{Re} B_{T_2} \le v\right) = P^u\left(\operatorname{Re} Z_{\tau_D} \le v\right) = P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \le -\sqrt{1-v}\right) + P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \ge \sqrt{1-v}\right).$$

The last two probabilities are equal by symmetry, and using the Poisson kernel for the half-plane we conclude

$$P^u\left(\operatorname{Re} B_{T_1} \le v\right) = 2P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \le -\sqrt{1-v}\right) = \frac{2}{\pi}\int_{-\infty}^{-\sqrt{1-v}} \frac{\sqrt{u-1}}{x^2+u-1}dx = 1 - \frac{2}{\pi}\arctan\sqrt{\frac{1-v}{u-1}}.$$

Proof.

Consider the case $B_0 = u > 1$ (reflection on A_3), and let $v < 1$ be arbitrarily fixed.

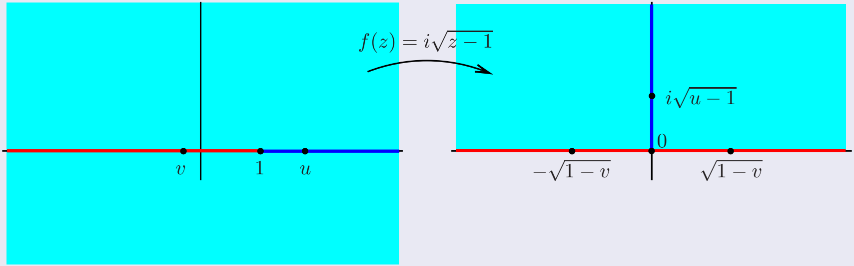


Figure : The image of the slit plane $D = \mathbb{C} - (-\infty, 1]$ under the conformal map $f(z) = i\sqrt{z-1}$.

$$P^u\left(\operatorname{Re} B_{T_2} \le v\right) = P^u\left(\operatorname{Re} Z_{\tau_D} \le v\right) = P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \le -\sqrt{1-v}\right) + P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \ge \sqrt{1-v}\right).$$

The last two probabilities are equal by symmetry, and using the Poisson kernel for the half-plane we conclude

$$P^u\left(\operatorname{Re} B_{T_1} \le v\right) = 2P^{i\sqrt{u-1}}\left(\operatorname{Re} W_{\tau_{\mathcal{H}}} \le -\sqrt{1-v}\right) = \frac{2}{\pi}\int_{-\infty}^{-\sqrt{1-v}} \frac{\sqrt{u-1}}{x^2+u-1}dx = 1 - \frac{2}{\pi}\arctan\sqrt{\frac{1-v}{u-1}}.$$

Corollary 5

For arbitrarily fixed $z \in \mathcal{H}$, $n \geq 1$ and measurable subsets $C_1, \ldots, C_n \subset \partial\mathcal{H}$, we have

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, C_1, \ldots, C_n) = \int_{C_1 \times \ldots \times C_n} k(z, y_1)\, k_{\mathcal{H}}(y_1, y_2) \cdot \ldots \cdot k_{\mathcal{H}}(y_{n-1}, y_n)\, dy_1 \ldots dy_n, \tag{14}$$

where $k(z,y) = \frac{1}{\pi}\frac{\operatorname{Im} z}{|z-y|^2}$, $z \in \mathcal{H}$, $y \in \partial\mathcal{H}$, is the Poisson kernel for $\mathcal{H}$, and $k_{\mathcal{H}}$ is the Poisson kernel for the Brownian motion in $\mathcal{H}$ with reflection on a part of the boundary, given by (13).

Monotonicity properties of IHM in $\mathcal{H}$

Theorem 6

Consider $A_1 = (-\infty, 0)$, $A_2\,(0,1)$, $A_3 = (1,\infty)$, $C_i \subset \{A_1, A_2, A_3\}$, *and* $z, \tilde{z} \in \mathcal{H}$. *We have*

a) *If* $z, \tilde{z}$ *are symmetric wrt the circle* $C(0,1)$ *and* $|z| < 1$, *we have:*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_2, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_2, C_2, \ldots, C_n).$$

b) *If* $z, \tilde{z}$ *are symmetric wrt the line* $y = mx$, $\operatorname{Re} z < \operatorname{Re} \tilde{z}$, *then*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_1, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_1, C_2, \ldots, C_n).$$

c) *If* $z, \tilde{z}$ *are symmetric wrt the line* $y = m(x-1)$, $\operatorname{Re} z > \operatorname{Re} \tilde{z}$, *then*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_3, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_3, C_2, \ldots, C_n).$$

Monotonicity properties of IHM in $\mathcal{H}$

Theorem 6

Consider $A_1 = (-\infty, 0)\,, A_2\,(0,1)\,, A_3 = (1, \infty)$, $C_i \subset \{A_1, A_2, A_3\}$, *and* $z, \tilde{z} \in \mathcal{H}$. *We have*

a) *If* $z, \tilde{z}$ *are symmetric wrt the circle* $C(0,1)$ *and* $|z| < 1$, *we have:*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_2, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_2, C_2, \ldots, C_n).$$

b) *If* $z, \tilde{z}$ *are symmetric wrt the line* $y = mx$, $\mathrm{Re}z < \mathrm{Re}\tilde{z}$, *then*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_1, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_1, C_2, \ldots, C_n).$$

c) *If* $z, \tilde{z}$ *are symmetric wrt the line* $y = m(x-1)$, $\mathrm{Re}z > \mathrm{Re}\tilde{z}$, *then*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_3, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_3, C_2, \ldots, C_n).$$

Monotonicity properties of IHM in $\mathcal{H}$

Theorem 6

Consider $A_1 = (-\infty, 0)$, $A_2\,(0,1)$, $A_3 = (1, \infty)$, $C_i \subset \{A_1, A_2, A_3\}$, *and* $z, \tilde{z} \in \mathcal{H}$. *We have*

a) *If* $z, \tilde{z}$ *are symmetric wrt the circle* $C(0,1)$ *and* $|z| < 1$, *we have:*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_2, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_2, C_2, \ldots, C_n).$$

b) *If* $z, \tilde{z}$ *are symmetric wrt the line* $y = mx$, $\mathrm{Re}\, z < \mathrm{Re}\, \tilde{z}$, *then*

$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_1, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_1, C_2, \ldots, C_n).$$

c) *If* $z, \tilde{z}$ *are symmetric wrt the line* $y = m(x-1)$, $\mathrm{Re}\, z > \mathrm{Re}\, \tilde{z}$, *then*

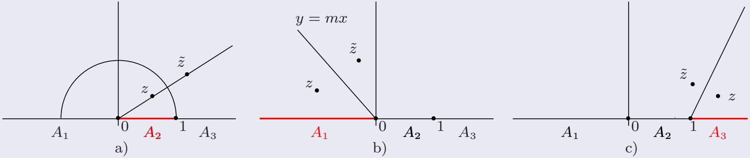
$$\omega_{\mathcal{H},n}^{A_1,A_2,A_3}(z, A_3, C_2, \ldots, C_n) \geq \omega_{\mathcal{H},n}^{A_1,A_2,A_3}(\tilde{z}, A_3, C_2, \ldots, C_n).$$

Proof.

a) Consider $(B_t, W_t) = \left(B_t, \frac{B_t}{|B_t|^2}\right)$ a coupling of (time-changed) RBM in $\mathcal{H}$ starting at $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_2 before hitting $C(0,1)$, hence before coupling with B_t.

b) Consider (B_t, W_t) a *mirror coupling of RBM wrt $y = mx$, starting ar $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_1 before coupling first with B_t.*

c) Similar to b).

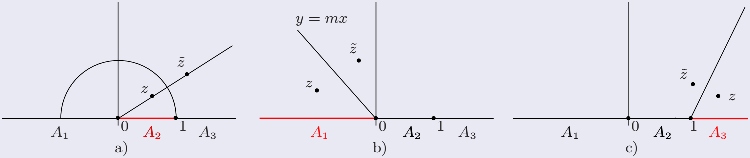


Proof.

a) Consider $(B_t, W_t) = \left(B_t, \frac{B_t}{|B_t|^2}\right)$ a coupling of (time-changed) RBM in $\mathcal{H}$ starting at $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_2 before hitting $C(0, 1)$, hence before coupling with B_t.

b) Consider (B_t, W_t) a *mirror coupling of RBM wrt* $y = mx$*, starting ar* $(z, \tilde{z})$*. Geometric considerations show that* W_t *cannot hit* A_1 *before coupling first with* B_t*.*

c) Similar to b).

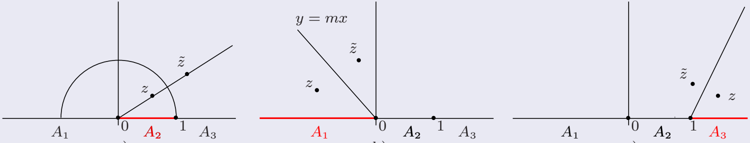


Proof.

a) Consider $(B_t, W_t) = \left(B_t, \frac{B_t}{|B_t|^2}\right)$ a coupling of (time-changed) RBM in $\mathcal{H}$ starting at $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_2 before hitting $C(0,1)$, hence before coupling with B_t.

b) Consider (B_t, W_t) a *mirror coupling of RBM wrt $y = mx$, starting ar $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_1 before coupling first with B_t.*

c) Similar to b).



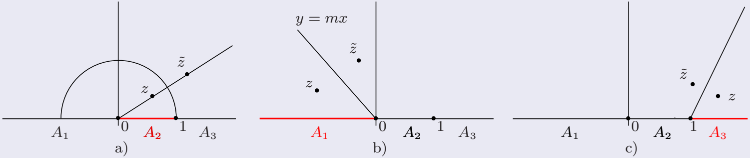
a) b) c)

Proof.

a) Consider $(B_t, W_t) = \left(B_t, \frac{B_t}{|B_t|^2}\right)$ a coupling of (time-changed) RBM in $\mathcal{H}$ starting at $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_2 before hitting $C(0, 1)$, hence before coupling with B_t.

b) Consider (B_t, W_t) a *mirror coupling of RBM wrt $y = mx$, starting ar $(z, \tilde{z})$. Geometric considerations show that W_t cannot hit A_1 before coupling first with B_t.*

c) Similar to b).



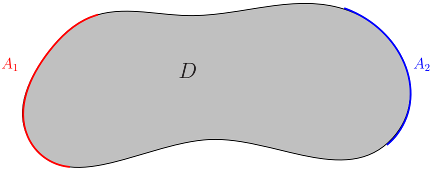
Application: connection of IHM to extremal distance

The *extremal distance* $d_D(A_1, A_2)$ between the disjoint closed arcs $A_1, A_2 \subset \partial D$ is given by

$$d_D(A_1, A_2)^{-1} = \int_D |\nabla u|^2 dxdy,$$

where u is the solution of the mixed Dirichlet-Neumann problem for D

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = 1 & \text{on } A_1 \\ u = 0 & \text{on } A_2 \\ \frac{\partial u}{\partial n} = 0 & \text{on } A_0 = \partial D - (A_1 \cup A_2) \end{cases},$$



Remark. The solution of the above problem is given by $u(z) = \omega_{D,2}^{A_1,A_2}(z, A_1, A_2)$.

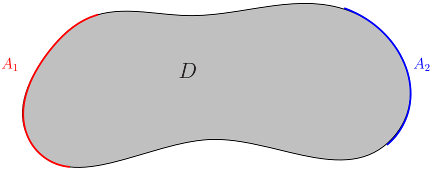
Application: connection of IHM to extremal distance

The *extremal distance* $d_D(A_1, A_2)$ between the disjoint closed arcs $A_1, A_2 \subset \partial D$ is given by

$$d_D(A_1, A_2)^{-1} = \int_D |\nabla u|^2 dxdy,$$

where u is the solution of the mixed Dirichlet-Neumann problem for D

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = 1 & \text{on } A_1 \\ u = 0 & \text{on } A_2 \\ \frac{\partial u}{\partial n} = 0 & \text{on } A_0 = \partial D - (A_1 \cup A_2) \end{cases},$$



Remark. The solution of the above problem is given by $u(z) = \omega_{D,2}^{A_1,A_2}(z, A_1, A_2)$.

References

[1] R. F. Bass, Probabilistic techniques in analysis, Probability and its Applications, Springer-Verlag, New York, 1995. MR1329542 (p.213)

[2] K. Burdzy, Z. Q. Chen, D. E. Marshall, Traps for reflected Brownian motion, Math. Z. 252 (2006), no. 1, 103–132. MR2209154 (sau COMPARISON OF POTENTIAL THEORETIC PROPERTIES)

[3] R. F. Bass, P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, Ann. Probab. 19 (1991), No. 2, 486–508. MR1106272

[4] R. F. Bass, P. Hsu, The semimartingale structure of reflecting Brownian motion, Proc. Amer. Math. Soc. 108 (1990), No. 4, 1007–1010. MR1007487

[5] M. Fukushima, Y. Oshima, and M. Takeda. Dirichlet Forms and Symmetric Markov Processes, volume 19 of Studies In Mathematics. de Gruyter, 1994.

[6] J. B. Garnett, D. E. Marshall, Harmonic measure, Cambridge University Press, Cambridge, 2005. MR2150803 (pp. 484).

[7] S. Kakutani, Two.dimensional Brownian Motion and Harmonic Functions, Proc. Imp. Acad. Tokyo 20, (1944). 706–714. MR0014647

[8] A. Klenke, Probability theory, Springer, London, 2014. MR3112259

[9] R. Nevanlinna, Analytic functions Das harmonische MaB von Punktrnengen und seine Anwendung in der Funktionentheorie. 8. Congr. Math. scand. Stockholm 1934.

[10] M. N. Pascu, Scaling coupling of reflecting Brownian motions and the hot spots problem, Trans. Amer. Math. Soc. 354 (2002), No. 11, pp. 4681 – 4702. MR1926894

[11] R. G. Pinsky, Transcience/recurrence for normally reflected Brownian motion in unbounded domains, Ann. Probab. 37 (2009), No. 2, pp. 676 – 686. MR2510020

[12] Ch. Pommerenke, Boundary behaviour of conformal maps, Grundlehren der Mathematischen Wissenschaften 299, Springer-Verlag, Berlin, 1992. MR1217706