Basins of attraction for the Granular media equation

I) Introduction

* V, F potentials on $\mathbb{R}^d$ (convexes at infinity).
$\sigma > 0$.

$$\frac{\partial}{\partial t}\mu_t = \frac{\sigma^2}{2}\Delta\mu_t + \mathrm{div}\left\{\mu_t\left(\nabla V + \nabla F * \mu_t\right)\right\} \qquad (GME)$$

* Probabilistic interpretation:

$$X_t = X_0 + \sigma W_t - \int_0^t \nabla V(X_s)\,ds - \int_0^t \left[\nabla F * \mathcal{L}(X_s)\right](X_s)\,ds \qquad (SSD)$$

$$X_0 \sim \mu_0,\ (W_t)_{t \geq 0} \text{ B.M.} \Rightarrow \mathcal{L}(X_t) = \mu_t.$$

* Particles interpretation:

$$X_t^i = X_0^i + \sigma W_t^i - \int_0^t \nabla V(X_s^i)\,ds - \int_0^t \frac{1}{N}\sum_{j=1}^N \nabla F(X_s^i - X_s^j)\,ds \qquad (MFE)$$

$N \longrightarrow \infty$ then $\sup_{[0;T]} \mathbb{E}\left[\|X_t^i - X_t\|^2\right] \xrightarrow[N\infty]{} 0,\ \forall T > 0.$

* Some applications:

- Economic point of view: a lot of interacting agents.
- Contraction of the muscolar cells.
- Neuronal conduction.
- Plasmas.

II) Invariant probabilities and convergence

Theorem (Phase transition) Tug14

$V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2} x^2$, $\alpha > 0$.

$\forall \alpha > 0$, $\exists\, \sigma_c(\alpha) > 0$ s.t.

* $\forall \sigma \geq \sigma_c(\alpha)$: $\#\{\text{inv. } \mathbb{P}\} = 1$.
* $\forall \sigma < \sigma_c(\alpha)$: $\#\{\text{inv. } \mathbb{P}\} = 3$.

Theorems (convergence) BRV98, BCCP98, CMV03, CGM08.

Here, V is convex. F also.

Then, $\mu_t \xrightarrow[t \to \infty]{} \mu^\sigma$ where μ^σ is the unique invariant probability.

Theorem (convergence) Tug13a, Tug13b

$V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) = \frac{\alpha}{2} x^2$, $\alpha > 0$.

If :
* $\int x^{32} \mu_0(dx) < \infty$
* μ_0 is absolutely continuous w.r.t. Lebesgue measure.
* $\int \mu_0 \log \mu_0 < \infty$

Then: $\mu_t \xrightarrow[t \to \infty]{} \mu^\sigma$ where μ^σ is an invariant probability.

III) Basins of attraction

$V(x) = \frac{x^4}{4} - \frac{x^2}{2}$, $F(x) = \frac{\alpha}{2} x^2$, $\underline{\alpha > 1}$, $\sigma < \sigma_c(\alpha)$.

$\xi(t) := \mathbb{E}\left[|X_t - 1|^2 \right] = \int (x-1)^2 \mu_t(dx)$.

Main theorem (basin of attraction) Tug 18

If μ_0 has a compact support $K \subset \mathbb{R}_+^*$

$\Rightarrow \forall \lambda > 0$, $\exists T_\lambda \geq 0$ s.t. $\xi(t) \leq \lambda^2$ $\forall t \geq T_\lambda$, if $\sigma < \sigma_\lambda$.

Corollary Tug 18

If moreover $\int \mu_0 \log(\mu_0) < \infty$ and $\int x^{32} \mu_0(dx) < \infty$

$\Rightarrow \mu_t \xrightarrow[t\to\infty]{} \mu_1^\sigma$ where μ_1^σ is the unique invariant probability with positive expectation

IV) Proof of Main theorem

$\mathcal{S}_\rho := \left\{ x \in \mathbb{R} : (x-1)V'(x) > \rho |x-1|^2 \right\}$.

S_ρ is the connex part of $\mathcal{S}_\rho$ which contains 1.

If ρ is small enough, $S_\rho \neq \emptyset$.

We assume $K \subset S_\rho$ (ρ small enough and depends on compact K)

We put $\tau_\rho(\sigma) := \inf\{ t \geq 0 : X_t \notin S_\rho \}$.

We know, see Tug 17: $\exists h(\rho) > 0$ s.t.

$$\mathbb{P}\left\{ e^{\frac{2}{\sigma^2} h(\rho)} < \tau_\rho(\sigma) \right\} \xrightarrow[\sigma \to 0]{} 1 \quad \text{(Kramers' law)}.$$

Lemma

$\exists\, \beta > 0$ s.t. $\forall t \geqslant 0$:

$$\xi'(t) \leqslant -2\rho\, \xi(t) + \sigma^2 + \beta \sqrt{\mathbb{P}(X_t \notin S_\rho)}$$

Proof: We use Itô formula, we take the expectation then we derive:

$$\xi'(t) = -2\,\mathbb{E}\left[(X_t - 1)\, V'(X_t)\right] + \sigma^2 - 2\alpha \underbrace{\mathbb{E}\left[(X_t - 1)(X_t - \mathbb{E}[X_t])\right]}_{= \operatorname{Var}(X_t - 1) \geqslant 0}$$

$$\leqslant \sigma^2 - 2\,\mathbb{E}\left[(X_t - 1)\, V'(X_t)\right].$$

But,
$$\begin{aligned}
\mathbb{E}\left[(X_t - 1)\, V'(X_t)\right] &= \mathbb{E}\left[(X_t - 1) V'(X_t) \mathbb{1}_{X_t \in S_\rho}\right] + \mathbb{E}\left[(X_t - 1) V'(X_t) \mathbb{1}_{X_t \notin S_\rho}\right] \\
&\geqslant \rho\, \mathbb{E}\left[(X_t - 1)^2 \mathbb{1}_{X_t \in S_\rho}\right] + \mathbb{E}\left[(X_t - 1) V'(X_t) \mathbb{1}_{X_t \notin S_\rho}\right] \\
&\geqslant \rho\, \mathbb{E}\left[|X_t - 1|^2\right] + \mathbb{E}\left\{\left((X_t - 1) V'(X_t) - \rho |X_t - 1|^2\right) \mathbb{1}_{X_t \notin S_\rho}\right\} \\
&\geqslant \rho\, \xi(t) - \mathbb{E}\left[\Psi(X_t) \mathbb{1}_{X_t \notin S_\rho}\right] \\
&\geqslant \rho\, \xi(t) - \underbrace{\sqrt{\mathbb{E}\left[\Psi(X_t)^2\right]}}_{\leqslant \frac{\beta}{2}} \sqrt{\mathbb{P}(X_t \notin S_\rho)}
\end{aligned}$$

$\square$

Corollary:

$\forall\, \lambda > 0$, $\exists\, \sigma_0 > 0$ and $T_\lambda \geqslant 0$ s.t.

$\forall\, \sigma < \sigma_0$, $\xi(T_\lambda) \leq \left(\frac{\lambda}{2}\right)^2$.

Remark : If σ is small enough,

$$\forall t \in \left[T_\lambda ; e^{\frac{2h(\rho)}{\sigma^2}}\right] : \xi(t) \leq \left(\frac{3\lambda}{4}\right)^2.$$

We put $T_\lambda(\sigma) := \inf\{t \geq T_\lambda : \xi(t) \geq \lambda^2\}$, $\inf \emptyset = +\infty$.

We consider

$$Y_{T_\lambda+t} = X_{T_\lambda} + \sigma\left(W_{T_\lambda+t} - W_{T_\lambda}\right) - \int_{T_\lambda}^{T_\lambda+t} V'(Y_s)\,ds - \alpha \int_{T_\lambda}^{T_\lambda+t} (Y_s - 1)\,ds.$$

Lemma : $\forall \theta > 0$, we have

$$\mathbb{P}\left\{\sup_{t\in[T_\lambda;T_\lambda(\sigma)]} |X_t - Y_t| \geq \theta\right\} = 0$$

if λ is small enough.

Proof :
$$d|X_t - Y_t|^2 = -2(X_t - Y_t)\left(W'_{\mu_t}(X_t) - W'_{\mu_t}(Y_t)\right)dt$$
$$-2(X_t - Y_t)\left(W'_{\mu_t}(Y_t) - W'_{\delta_1}(Y_t)\right)dt$$

where $W_\mu(x) := V(x) + \frac{\alpha}{2}\left(x - \mathbb{E}[\mu]\right)^2$ and $\mu_t = \mathcal{L}(X_t)$.

Since $\alpha > 1$, $(X_t - Y_t)\left(W'_{\mu_t}(X_t) - W'_{\mu_t}(Y_t)\right) \geq \underbrace{(\alpha - 1)}_{>0} |X_t - Y_t|^2$.

Also, $-(X_t - Y_t)\left(W'_{\mu_t}(Y_t) - W'_{\delta_1}(Y_t)\right)$

$$\leq |X_t - Y_t|\,\left|\mathbb{E}[X_t] - 1\right| \leq |X_t - Y_t| \underbrace{\sqrt{\mathbb{E}\left[|X_t - 1|^2\right]}}_{\leq \lambda \text{ if } t \in [T_\lambda; T_\lambda(\sigma)]}$$

However, $X_{T_\lambda} = Y_{T_\lambda} \Rightarrow \forall t \in [T_\lambda; T_\lambda(\sigma)]$, we have

$$|X_t - Y_t| \leq \frac{\alpha}{\alpha - 1}\lambda.$$

Taking $\lambda < \frac{\alpha-1}{\alpha}\theta$ yields the result.

$\square$

We take $\gamma > 0$ small s.t.

$\mathbb{B}(1;\gamma) \subset S_{\rho+\delta} \subset S_\rho$ for some $\delta > 0$.

By taking $\theta := d\left(\mathbb{B}(1;\gamma); S_\rho^c\right)$, we obtain

$$\mathbb{P}\left(X_t \notin S_\rho\right) \leq \mathbb{P}\left(Y_t \notin \mathbb{B}(1;\gamma)\right).$$

But, $\mathbb{P}\left(Y_t \notin \mathbb{B}(1;\gamma)\right) \leq \dfrac{\mathbb{E}\left[|Y_t - 1|^2\right]}{\gamma^2}$.

By putting $\hat{\xi}(t) := \mathbb{E}\left[|Y_t - 1|^2\right]$, we get

$\hat{\xi}'(t) \leq \sigma^2 - 2(\alpha-1)\hat{\xi}(t)$.

$$\Longrightarrow \hat{\xi}(t) \leq \frac{\sigma^2}{2(\alpha-1)} + \left(\frac{3\lambda}{4}\right)^2 e^{-2(\alpha-1)(t-T_\lambda)} \quad \forall t \geq T_\lambda.$$

In particular, $\mathbb{P}\left(X_t \notin S_\rho\right) \xrightarrow[\sigma \to 0]{} 0$ if $t \geq e^{\frac{2h(\rho)}{\sigma^2}}$.

However, if $t < e^{\frac{2h(\rho)}{\sigma^2}}$, $\mathbb{P}\left(X_t \notin S_\rho\right) \xrightarrow[\sigma \to 0]{} 0$.

$\Rightarrow \forall t \geq 0 : \quad \xi'(t) \leq -2\rho\, \xi(t) + f(\sigma)$, with $f(\sigma) \xrightarrow[\sigma \to 0]{} 0$.

We deduce $\xi\left(T_\lambda(\sigma)\right) \leq \left(\frac{3\lambda}{4}\right)^2$. By definition, $\xi\left(T_\lambda(\sigma)\right) = \lambda^2$.

$\Longrightarrow T_\lambda(\sigma) = +\infty$.