Rosenblatt Laplace motion

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Motivation: **A stochastic model for the hydraulic conductivity fields** in geophysics (e.g. a model for the groundwater flow or transport models).

Aim:

- we introduce the **Rosenblatt Laplace motion** (as the subordinated Rosenblatt process to an independent Gamma process);
- we derive the basic properties of this new stochastic process;
- we make a numerical analysis of this process;
- how our model fits with the hydraulic conductivity data.

Why to model the hydraulic conductivity by Rosenblatt Laplace motion?

- Logarithms of the hydraulic conductivity data exhibit a correlation structure similar to that of fractional Brownian motion, but the increment distributions are more peaked with heavier tails.

A stochastic model for the hydraulic conductivity: fractional Brownian motion subordinated to a Gamma process (it provides one alternative to reproduce the higher peaks and wider tails), called fractional Laplace motion.

[T. J. Kozubowski et all. *Fractional Laplace motion.* Adv. Appl. Prob., 2006].

- **Fractional Laplace motion,** $(U_t)_{t\geq 0}$, i.e. for every $t \geq 0$:

$$U_t := B^H_{G_t},$$

where $(B^H_t)_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $(G_t)_{t\geq 0}$ is an independent Gamma process.

- **Fractional Brownian motion with index** $H \in (0,1)$ is a centered Gaussian process $(B_t^H)_{t\geq 0}$ with $B_0^H = 0$ and covariance function

$$E(B_t^H B_s^H) = \frac{\sigma^2}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}), \quad t,s \geq 0, \text{ where } \sigma^2 = var(B_1^H).$$

- **Gamma process** $(G_t)_{t\geq 0}$ is a Lévy process such that for every $0 \leq s \leq t$,

$$G_t - G_s =^{(d)} \Gamma(t-s, 1).$$

The lack in the FLm model: the data do not necessary have a conditional Gaussian behavior, but they preserve the other properties.

Our aim is to enlarge the modelling tool kit by replacing the fractional Brownian by another stochastic process which has the same correlation structure, but it is non-Gaussian. This is the so-called **Rosenblatt process**.

[C.A. Tudor, Analysis of variations for self-similar processes. Springer, 2013].

Subordinated processes in the second Wiener chaos

- Let $(B(y), y \in \mathbb{R}_+)$ be **a Wiener process** on $(\Omega_1, \mathcal{F}_1, P_1)$.

Define the process $(X_t)_{t \geq 0}$ by

$$X_t = \int_{\mathbb{R}} \int_{\mathbb{R}} L_t(y_1, y_2) dB(y_1) dB(y_2) \tag{0.1}$$

where $L_t \in L_S^2(\mathbb{R}^2)$ for every $t \geq 0$ and $L_S^2(\mathbb{R}^2)$ is the set of square integrable symmetric functions defined on $\mathbb{R}^2$.

- The process X belongs to the second Wiener chaos generated by B.

- Let G be **a random variable** on $(\Omega_2, \mathcal{F}_2, P_2)$ **independent by** B:

$$G(\omega_2) > 0 \text{ for almost all } \omega_2 \in \Omega_2 \text{ and } E_2|G|^p < \infty \text{ for every } p > 0. \tag{0.2}$$

- Define X_G on $\Omega := \Omega_1 \times \Omega_2$ by

$$X_G(w_1, \omega_2) = X_{G(\omega_2)}(\omega_1) = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} L_t(y_1, y_2) dB(y_1) dB(y_2)(\omega_1) \right)_{t=G(\omega_2)}.$$

Aim: Analyse the probability distribution of the random variable X_G.

Proposition

Assume

$$E_2 \int_{\mathbb{R}} \int_{\mathbb{R}} L_G^2(y_1, y_2) dy_1 dy_2 < \infty. \tag{0.3}$$

Then we have P-almost surely

$$X_G = \int_{\mathbb{R}} \int_{\mathbb{R}} L_G(y_1, y_2) dB(y_1) dB(y_2). \tag{0.4}$$

Proof:

Let

$$Y_G = \int_{\mathbb{R}} \int_{\mathbb{R}} L_G(y_1, y_2) dB(y_1) dB(y_2).$$

(i.e. for every $\omega_2 \in \Omega_2$ we consider the multiple integral of order two with respect to B).

- $Y_G \in L^2(\Omega)$ and taking the conditional expectation given G

$$EY_G^2 = 2E_2 \int_{\mathbb{R}} \int_{\mathbb{R}} L_G^2(y_1, y_2) dy_1 dy_2 < \infty.$$

- The equality (0.4) follows by conditionning on G, since

$$E\left(X_G - Y_G\right)^2 = E_2 g(G) = 0$$

because $g(x) = E_1 \left(X_x - Y_x\right)^2 = 0$ for every $x \in \mathbb{R}$. □

Moments of the random variable X_G

Proposition

Let us assume that

$$E_2 \left(\int_{\mathbb{R}} \int_{\mathbb{R}} L_G^2(y_1, y_2) dy_1 dy_2 \right)^{\frac{q}{2}} < \infty. \tag{0.5}$$

Then $X_G \in L^q(\Omega)$ *for every* $q \geq 2$ *and*

$$E\left|X_G\right|^q \leq 2(q-1)^q E_2 \left(\int_{\mathbb{R}} \int_{\mathbb{R}} L_G^2(y_1, y_2) dy_1 dy_2 \right)^{\frac{q}{2}}.$$

Proof: We use the hypercontractivity property of the double Wiener-Itô integrals, $E\left|Y\right|^q \leq (q-1)^q E|Y^2|^{\frac{q}{2}}$ for every $q > 2$, see [5].

Cumulants of the random variable X_G

- For random variables in the second Wiener chaos **their probability law is determined by the cumulants.**
- Define the mth cumulant of $F \in L^m(\Omega)$ as $k_m(F)$, $m \geq 1$,

$$k_m(F) = (-i)^n \frac{\partial^n}{\partial t^n} \ln E(e^{itF})|_{t=0},$$

- The link between the moments and the cumulants of F,

$$k_m(F) = \sum_{\sigma=(a_1,..,a_r)\in\mathcal{P}(\{1,..,n\})} (-1)^{r-1}(r-1)! EX^{|a_1|} \dots EX^{|a_r|}, \qquad (0.6)$$

where $\mathcal{P}(b)$ is the set of all partitions of b.

- If $F = I_2(f)$, then its cumulants can be computed as

$$k_m(F) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} du_1 \dots du_m f(u_1,u_2) f(u_2,u_3) \dots f(u_{m-1},u_m) f(u_m,u_1). \qquad (0.7)$$

[I. Nourdin, 2012; C. Tudor, 2013].

Question: when the probability distribution of the random variable X_G is determined by its cumulants (or equivalently, by its moments)?

Answer:

Proposition

Consider the random variable X_G with G satisfying (0.2). Assume that there exists a constant $C > 0$ (no depending on k) such that for every $k \geq 2$,

$$E_2 \left(\int_{\mathbb{R}} \int_{\mathbb{R}} L_G^2(y_1, y_2) dy_1 dy_2 \right)^k < C. \tag{0.8}$$

Then the law of the random variable X_G is determined by its cumulants (or equivalently by its moments). Moreover, for every $m \geq 2$,

$$k_m(X_G) = 2^{m-1}(m-1)! E_2 \int_{\mathbb{R}^m} L_G(u_1, u_2) L_G(u_2, u_3) \dots L_G(u_m, u_1) du_1..du_m. \tag{0.9}$$

[O.Lupascu, C-A. Tudor, Rosenblatt Laplace motion, preprint, 2018].

Proof:

- we obtain that

$$E|X_G|^{2k} \leq 2C(2k-1)^{2k}$$

for every $k \geq 1$. This is enough to conclude that the law of X_G is determined by the cumulants (see e.g. Gwo Dong Lin, 2017).

- by conditionning on G, we find

$$\begin{aligned} k_m(X_G) &= E_2\left(k_m(I_2(L_t))\big|_{t=G}\right) \\ &= 2^{m-1}(m-1)!E_2\left[\left(\int_{\mathbb{R}^m} du_1..du_m L_t(u_1,u_2)...L_t(u_m,u_1)\right)\Big|_{t=G}\right] \\ &= 2^{m-1}(m-1)!E_2\int_{\mathbb{R}^m} L_G(u_1,u_2)L_G(u_2,u_3)\dots L_G(u_m,u_1)du_1..du_m. \end{aligned}$$

- Condition (0.8) ensures that the moments of X_G are finite, this implies that the cumulants of X_G are finite. □

Proposition

For every $\omega = (\omega_1, \omega_2) \in \Omega$ *we have*

$$X_G(\omega_1, \omega_2) = \sum_{j \geq 1} \lambda_{j,G}(\omega_2)(N_j^2(\omega_1) - 1) \tag{0.10}$$

where $(N_j)_{j \geq 1}$ *are i.i.d. standard normal random variables and for every* $\omega_2 \in \Omega_2$, $(\lambda_{j,G(\omega_2)})_{j \geq 1}$ *are the eigenvalues of the operator on* $L^2(\mathbb{R})$,

$$A_{G(\omega_2)} f = L_{G(\omega_2)} \otimes_1 f, \quad f \in L^2(\mathbb{R}). \tag{0.11}$$

The series in (0.10) is convergent in $L^2(\Omega_1, P_1)$ *and* P_1*-almost surely for every* $\omega_2 \in \Omega_2$.

Rosenblatt process

- If we take the kernel L_t from X given by

$$L_t^H(y_1, y_2) = d(H,2)\int_0^t ds(s-y_1)_+^{\frac{H}{2}-1}(s-y_2)_+^{\frac{H}{2}-1}, \quad t \geq 0 \qquad (0.12)$$

with $H \in (\frac{1}{2}, 1)$, we obtain a **Rosenblatt process** X^H.
We denote $x_+ = \max(x, 0)$.

- $(X_t^H)_{t \geq 0}$ is a H-self-similar stochastic process (i.e., $X_{ct}^H =^{(d)} c^H X_t^H$), with stationary increments. It has the same covariance as the fractional Brownian motion,

$$EX_t^H X_s^H = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \text{ for every } s, t \geq 0.$$

- The sample paths of X^H are Hölder continuous of order δ for every $\delta \in (0, H)$, i.e., $|X_t^H - X_s^H| \leq C|t-s|^{\delta}$, and it exhibits long-range dependence.

Rosenblatt process stopped at a random time

Let be the random variable

$$X_G^H(w_1, w_2) = X_{G(\omega_2)}(\omega_1) = \int_{\mathbb{R}} \int_{\mathbb{R}} L_G^H(y_1, y_2) dB(y_1) dB(y_2). \qquad (0.13)$$

Proposition

If X_G^H is given by (0.13), for every $m \geq 2$

$$E(X_G^H)^m = E_2(G^{mH}) E_1(X_1^H)^m \qquad (0.14)$$

and

$$\begin{aligned} k_m(X_G^H) &= 2^{\frac{m}{2}-1}(m-1)!(H(2H-1))^{\frac{m}{2}} E_2(G^{mH}) \int_0^1 ... \int_0^1 du_1..du_m \\ &\times |u_1 - u_2|^{H-1} |u_2 - u_3|^{H-1} \ldots |u_m - u_1|^{H-1}. \qquad (0.15) \end{aligned}$$

" $=^{(d)}$ " denotes the equality in distribution and " $\to^{(d)}$ " the convergence in distribution.

Proposition

If X_G^H is given by (0.13) and G satisfies (0.2), then

$$X_G^H =^{(d)} \sqrt{G^{2H}} X_1^H.$$

Proof: By self-similarity $X_t^H =^{(d)} t^H X_1^H$, for every $a \in \mathbb{R}$ and $t \geq 0$

$$E_1 e^{ia \sum_{j\geq 1} \lambda_{j,t}(N_j^2 - 1)} = E_1 e^{iat^H \sum_{j\geq 1} \lambda_{j,1}(N_j^2 - 1)}. \tag{0.16}$$

Replace t by $G(\omega_2)$ and take the expectation with respect to P_2, we have

$$Ee^{ia \sum_{j\geq 1} \lambda_{j,t}(N_j^2 - 1)} = Ee^{ia\sqrt{G^{2H}} \sum_{j\geq 1} \lambda_{j,1}(N_j^2 - 1)}$$

for every $a \in \mathbb{R}$ and the conclusion follows. □

The behavior with respect to the parameter H of X_G^H

Remark

i) *Recall the behavior of* X_1^H *with respect to* H*:*

- *if* $H \to \frac{1}{2}$ *then* X_1^H *converges in distribution to a standard normal random variable* Z*.*
- *if* $H \to 1$, *then* X_1^H *converges in distribution to the centered chi-square random variable* $\frac{1}{\sqrt{2}}(Z^2 - 1)$*.*

ii) *We deduce that*

$$X_G^H \to^{(d)}_{H \to \frac{1}{2}} \sqrt{G} Z \text{ and } X_G^H \to^{(d)}_{H \to 1} G \frac{1}{\sqrt{2}}(Z^2 - 1).$$

[M. S. Veillette and M. S. Taqqu, 2013];

[H. Araya and C. A. Tudor, 2017].

The Rosenblatt process subordinated to a Gamma process

- Let $(X_t^H)_{t\geq 0}$ be a Rosenblatt process on $(\Omega_1, \mathcal{F}_1, P_1)$ and $(G_t)_{t\geq 0}$ a Gamma process on $(\Omega_2, \mathcal{F}_2, P_2)$ independent of X^H.
- $(G_t)_{t\geq 0}$ is a Lévy process such that for every $0 \leq s \leq t$,

$$G_t - G_s =^{(d)} \Gamma(t-s, 1),$$

where $\Gamma(a, \lambda)$ is the Gamma distribution whose density is given by

$$f(x) = \frac{\lambda^a}{\Gamma(a)} e^{-\lambda x} x^{a-1} 1_{(0,\infty)}(x), \quad \text{with } a, \lambda > 0.$$

On $\Omega = \Omega_1 \times \Omega_2$ we define the Rosenblatt process subordinated to a Gamma process $(U_t)_{t>0}$, called **the Rosenblatt Laplace motion**, as

$$U_t := X^H_{G_t}, \text{ for every } t \geq 0. \tag{0.17}$$

Corollary

From every $t \geq 0$ and for every $m \geq 2$ we have

$$EU_t^m = \frac{\Gamma(mH+t)}{\Gamma(t)} E_1(X_1^H)^m.$$

Moreover, the covariance function of (0.17) is given by, for every $0 \leq s \leq t$,

$$EU_tU_s = \frac{1}{2}\left(\frac{\Gamma(2H+t)}{\Gamma(t)} + \frac{\Gamma(2H+s)}{\Gamma(s)} - \frac{\Gamma(2H+t-s)}{\Gamma(t-s)}\right).$$

Remark

Define the Rosenblatt Laplace noise $(W_j)_{j\geq 0}$ *defined by* $W_j = U_{j+1} - U_j$ *for every* $j \geq 0$. *Let* $r(n) := EW_jW_{j+n}$. *Then we have* $r(0) = \Gamma(2H+1)$ *and for* $n \geq 1$,

$$r(n) = \frac{1}{2}\left(\frac{\Gamma(2H+n+1)}{\Gamma(n+1)} + \frac{\Gamma(2H+n-1)}{\Gamma(n-1)} - 2\frac{\Gamma(2H+n)}{\Gamma(n)}\right).$$

Consequently (see [1]), for $H > \frac{1}{2}$, $r(n)$ *behaves, as* $n \to \infty$, *as*

$$H(2H-1)n^{2H-2}.$$

This means that the noise generated by U *has the same behavior as the fractional Brownian noise with Hurst index bigger than one-half.*
The series $\sum_{n\geq 1} r(n)$ *diverges, so the process* $(U_t)_{t\geq 0}$ *exhibits long-range dependence.*

Numerical approximations of cumulants and moments

Let $U_{t_0} := X^H_{G_{t_0}}$ be the subordinated procees at the fixed time t_0.

- U_{t_0} and $\sqrt{G^{2H}_{t_0}} X^H_1$ have the same moments and cumulants. So, we can obtain the m-cumulants at t_0, to multiply by $(G_{t_0})^{Hm}$ the m-cumulants of X^H_1, for every $m \geq 2$.
- To compute the m-cumulants of X^H_1 we improve the numerical schema introduced in [M. S. Veillette, M. S. Taqqu, 2013], see $https://github.com/markveillette/rosenblatt.git$.
- Then, we compute the moments from the cumulants by using relation (0.6).

Numerical results-cumulants

H	k_1	k_2	k_3	k_4	k_5
1	0.0524	0.0078	0.0017	$5.1178e-04$	$1.8963e-04$
0.95	0.0607	0.0104	0.0027	$9.1554e-04$	$3.9248e-04$
0.80	0.0945	0.0228	0.0087	0.0045	0.0029
0.70	0.1269	0.0333	0.0156	0.0099	0.0079
0.60	0.1705	0.0344	0.0168	0.0115	0.0100
0.55	0.1975	0.0219	0.0090	0.0054	0.0042
0.50	0.2289	0	0	0	0

Table : First 5 cumulants for the Rosenblatt Laplace process for same values of H at the fixed time $t_0 = 1$.

Numerical results-moments

H	m_1	m_2	m_3	m_4	m_5
1	0.0524	0.0078	0.0022	$7.2502e-04$	$2.9827e-04$
0.95	0.0607	0.0104	0.0033	0.0013	$6.1795e-04$
0.80	0.0945	0.0228	0.0113	0.0065	0.0047
0.70	0.1269	0.0333	0.0217	0.0153	0.0136
0.60	0.1705	0.0344	0.0317	0.0215	0.0215
0.55	0.1975	0.0219	0.0322	0.0140	0.0149
0.50	0.2289	0	0.0360	0	0.0094

Table : First 5 moments for the Rosenblatt Lapace process for same values of H at the fixed time $t_0 = 1$.

Remark: the approximations for the first two cumulants coincide with the approximations for the corresponding moments.

Paths simulations of the Rosenblatt Laplace process

We fix a time grid $t_1, \cdots, t_n$ on a fixed time interval $[0, T]$.

Step 1. We sample values of the subordinator G_{t_k} for points t_k. The discretized trajectory is $G_{t_i} = \sum_{k=1}^{i} S_k$, where S_k are independent Gamma variables with parameters $t_k - t_{k-1}$ and 1.

Step 2. We simulate a trajectory of $U_{t_k} = X^H_{G_{t_k}}$, which, conditionally on the values of G_{t_k}, it is a Rosenblatt process evaluated at the values of the Gamma process, adapting the software developed by J.-M. Bardet in 2010.

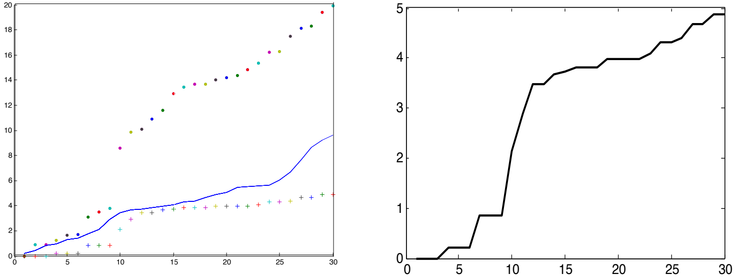

Figure : *Left:*

The "·" Data represents the pure-jump increasing Gamma process, the blue line is a continuous trajectory of the Ronsebaltt process $X^{0.8}$, and "$+$" Data is the path of the corresponding Rosenblatt Laplace motion. *Right:* Zoom of the discontinuous path of the subordinated Rosenblatt process, obtained by the interpolation of values from the left side.

The subordinated process has the same jump moments as the Gamma subordinator.

Application to modelling

- The experimental data for hydraulic conductivity is usually referred as the K data.
- $\ln(K)$ was modeled by a stochastic process with **long-range dependence** and with a **correlation structure** similar to those of the fBm.
- The Rosenblatt process is a possible alternative to replace fBm in this model, since our **Rosenblatt Laplace model** has
  - the **same correlation structure**;
  - **long-range dependence**;
  - **conditional non-gaussian behavior**;
  - **heavy-tailed distribution of U_t**:

$U_t \rightarrow^{(d)}_{H\rightarrow 1} G\frac{1}{\sqrt{2}}(Z^2-1)$ (which is a heavy-tail distribution), multiplied by an independent Gamma distribution.

Other possible applications: in some financial time series.

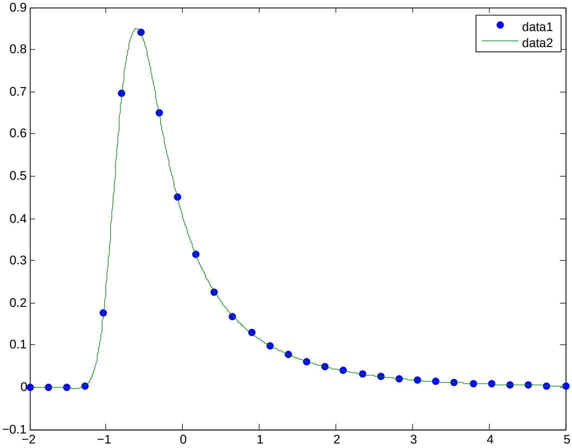

Figure : The plot of the Rosenblatt PDF for a with Hurst index $H = 0.8$. Data 1 represents the approximations of $P[X_1^{0.8} \leq x]$, $x \in (-2, 5)$, and data 2 is a linear interpolation in between.

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