Optimal approximation of internal controls for a wave-type problem with fractional Laplacian

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The control for the continuous wave equation

Given $T \geq 2$ and $(u^0, u^1) \in L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$ there exists a control function $v \in C^0([0,T],\mathbb{C})$ such that the solution of the wave equation

$$\begin{cases} u''(t,x) - u_{xx}(t,x) = 0 & t \in (0,T),\ x \in (0,1), \\ u(t,0) = 0 & t \in (0,T), \\ u(t,1) = v(t) & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1}$$

satisfies

$$u(T,x) = u'(T,x) = 0 \qquad (x \in (0,1)).$$

The discrete model of the wave equation

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. For $T > 0$, we consider the following semi-discrete space approximation of the wave equation by the explicit finite-differences method:

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} = 0 & 1 \le j \le N,\ t > 0, \\ u_0(t) = 0 & t \in (0, T), \\ u_{N+1}(t) = v_h(t) & t \in (0, T), \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \le j \le N. \end{cases} \tag{2}$$

The discrete controllability problem

Given $T \ge 2$, $h > 0$ *and* $((u_j^0, u_j^1))_{1 \le j \le N} \in \mathbb{C}^{2N}$, *there exists a control function* $v_h \in C^0([0, T])$ *such that the solution of the equation* (31) *verifies*

$$u_j(T) = u_j'(T) = 0 \qquad (j = 1, 2, ..., N).$$

Bad news

The main problem

The sequence of discrete controls $(v_h)_{h>0}$ converges to a control v of the continuous wave equation?

In general, there exist high-frequency spurious solutions generated by the discretization process that make the discrete controls diverge when the mesh-size h goes to zero. (Glowinski - Li - Lions ('90), Infante - Zuazua ('99), Micu ('03), Zuazua ('05), etc...)

Basically, this difficulty can be overcame by using an appropriate filtering technique to eliminate the short wave length components of the solutions of the discrete system, i.e. the large frequencies (of order $|n| = N$) of the discretized problem.

Our aim

- To filtering the initial data in an optimal range in order to restore the uniform controllability property.

- To obtain a relation between the range of filtration and the minimal time of control, recovering in many cases the usual minimal time to control for the (continuous) wave equation.

How we can control such a special "wave"?

Now, we can control everything!

Back to the math reality: The adjoint problem

Let us consider the corresponding homogeneous adjoint problem:

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t) - 2w_j(t) + w_{j-1}(t)}{h^2} = 0 & 1 \leq j \leq N, \ t > 0, \\ w_0(t) = 0 & t \in (0, T), \\ w_{N+1}(t) = 0 & t \in (0, T), \\ w_j(0) = w_j^0, \quad w_j' = w_j^1 & 1 \leq j \leq N. \end{cases} \tag{3}$$

The discretisation matrix

We define the matrix $A_h \in \mathcal{M}_{N\times N}(\mathbb{R})$ as follows:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The adjoint problem (3) can be rewritten in a matricial form as follows:

$$\begin{cases} W''(t) + A_h W(t) = 0 & t > 0, \\ W(0) = W^0, \quad W'(0) = W^1, & \end{cases} \tag{4}$$

where $W(t) = (w_1(t), ..., w_N(t))^T \in \mathbb{C}^N$ and the initial data is $\begin{pmatrix} W^0 \\ W^1 \end{pmatrix} = \begin{pmatrix} (w_j^0)_{1\leq j\leq N} \\ (w_j^1)_{1\leq j\leq N} \end{pmatrix} \in \mathbb{C}^{2N}$.

The operator $\mathcal{A}_h$

Now, if we set $Z(t) = \begin{pmatrix} W(t) \\ W'(t) \end{pmatrix}$ and $Z^0 = \begin{pmatrix} W^0 \\ W^1 \end{pmatrix}$, then (4) has the following equivalent vectorial form

$$\begin{cases} Z'(t) + \mathcal{A}_h Z(t) = 0 \\ Z(0) = Z^0, \end{cases}$$

where the operator $\mathcal{A}_h$ is given by $\mathcal{A}_h = \begin{pmatrix} 0 & -I_N \\ A_h & 0 \end{pmatrix}$ and I_N is the identity matrix of size N.

The eigenvalues and eigenvectors of the operator $\mathcal{A}_h$

The eigenvalues of $\mathcal{A}_h$ are given by the family $(i\,\lambda_n)_{1\leq|n|\leq N}$, where

$$\lambda_n = \frac{2}{h}\sin\left(\frac{n\pi h}{2}\right), \qquad 1 \leq |n| \leq N, \tag{5}$$

and the corresponding eigenvectors are

$$\Phi_h^n = \begin{pmatrix} \frac{1}{i\lambda_n}\varphi_h^n \\ -\varphi_h^n \end{pmatrix} \qquad (1 \leq |n| \leq N),$$

where

$$(\varphi_h^n)_{1\leq|n|\leq N} = \begin{pmatrix} \sin(n\pi h) \\ \sin(2n\pi h) \\ \dots \\ \sin(n\pi h N) \end{pmatrix} \in \mathbb{C}^{2N}.$$

Note that $(\Phi_h^n)_{1\leq|n|\leq N}$ forms an orthonormal basis in $\mathbb{C}^{2N}$.

Previous result

S. Micu [Numer. Math. ('02)] proved that if the initial data are given by

$$\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} = \sum_{1\leq |n|\leq M} a_{hn}^0 \Phi_h^n,$$

with $M = \sqrt{N}$, then there exists a sequence of bounded controls $(v_h)_{h>0}$ for (31) provided that the initial condition verifies some conditions on its Fourier coefficients and that the time is large enough (but no quantitative estimate of this minimal time is given).

Moreover, S. Micu [Numer. Math. ('02)] proved that there exists regular initial data (with $M = N$) for which there exists no sequence of discrete controls uniformly bounded in $L^2(0,T)$.

The main results

Open problem

What about the range between $\sqrt{N}$ and N?

Optimal filtration for the approximation of boundary controls for the wave equation

- By filtering the initial data in an optimal range, we restore the uniform controllability property.
- We obtain a relation between the range of filtration and the minimal time of control, recovering in many cases the usual minimal time to control for the (continuous) wave equation.

A sufficient and necessary condition for the null-controllability

Lemma

Given $T > 0$, system (31) is null-controllable at time T if, and only if, for any initial data $U^0 = (u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{C}^{2N}$, there exists $v_h \in C^0([0,T],\mathbb{C})$ which verifies

$$\int_0^T v_h(t)\,\frac{\overline{w_N(t)}}{h}\,dt = h\sum_{1\leq j\leq N}(u_j^0\overline{w}_j^1 - u_j^1\overline{w}_j^0)\ \left(\left((w_j^0, w_j^1)_{1\leq j\leq N}\right) \in \mathbb{C}^{2N}\right),$$

where $W = (w_1(t), ..., w_N(t))^T$ is the solution of (4).

The moment problem

Lemma

Given $T > 0$, system (31) is null-controllable at time T if, and only if, for any initial data $(u_j^0, u_j^1)_{1\leq j\leq N} = \sum_{1\leq|n|\leq N} a_n \Phi_h^n$ there exists $v_h \in C^0([0,T],\mathbb{C})$ which verifies

$$\int_0^T v_h(t)\, e^{-i\lambda_n t}\, dt = \frac{(-1)^n h}{\sin(n\pi h)} a_n \qquad (1 \leq |n| \leq N)\,. \tag{6}$$

The moment problems have been, from the very beginning, one of the most successful method for controllability problems (see the books of Avdonin and Ivanov ('95), Coron ('07), Komornic and Loretti ('05), Russel ('78), Tucsnak and Weiss ('09)).

The biorthogonals

A sequence $(\theta_m)_{1\leq|m|\leq N} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is *biorthogonal to the family of exponential functions* $\left(e^{i\lambda_n t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{i\overline{\lambda}_n t} dt = \delta_{mn} \qquad (1 \leq |m|, |n| \leq N)\,. \tag{7}$$

An explicit formula for the discrete control

Once we have a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ we can construct a control thanks to the following formula :

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_n \frac{T}{2}} a_n(h)\theta_n\left(t-\frac{T}{2}\right),$$

where $a_n(h)$ is related to a_n by the following relations:

$$a_n(h) := \begin{cases} 0, & |n| > f(N), \\ \frac{1}{2}\left(\frac{\lambda_n}{n\pi}+1\right)a_n + \frac{1}{2}\left(\frac{\lambda_n}{n\pi}-1\right)a_{-n}, & |n| \leqslant f(N). \end{cases} \tag{8}$$

Construction of the biorthogonals

We construct and evaluate an explicit biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq |n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ in the following way:

1. We construct an entire function P_m, with the property that $P_m(\lambda_n)=\delta_{mn}$.

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2. We obtain an optimal estimate of the product P_m on the real axis.

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1. We construct an entire function P_m, with the property that $P_m(\lambda_n)=\delta_{mn}$.
2. We obtain an optimal estimate of the product P_m on the real axis.
3. We construct a smart multiplier M_m with rapid decay on the real axis such that $P_m M_m \in L^1(R)\cap L^2(R)$ and $M_m(\lambda_n)=\delta_{mn}$.

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3. We construct a smart multiplier M_m with rapid decay on the real axis such that $P_m M_m \in L^1(R)\cap L^2(R)$ and $M_m(\lambda_n)=\delta_{mn}$.
4. The Fourier transform of the entire function $\psi_m(z) := P_m(z)M_m(z)$ gives the element θ_m of a biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$.

The sequence of biorthogonals

We consider the following discretization of the initial condition (u^0, u^1) given by

$$U_h^0 = \begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\le j\le N} = \sum_{1\leqslant |n|\leqslant f(N)} a_{hn}^0 \Phi_h^n(x), \tag{9}$$

where $f(N) \leqslant N$ represents the range of filtration, and

$$\Gamma(f) := \limsup_{N\to\infty} \frac{f(N)}{N} \in [0,1], \tag{10}$$

where $f : \mathbb{N}^* \to \mathbb{N}^*$ is the range of filtration.

The main result

Theorem (Lissy, Rovenţa, Math. Comp. 2019)

Let $(u^0, u^1) \in L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$. *Then, for any* $T > \frac{4}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$, *there exists a control* $v_h \in C^0([0,T],\mathbb{C})$ *bringing the solution of* (31) *(with initial condition* U_h^0*) to (0,0) such that the sequence* $(v_h)_{h>0}$ *is bounded in* $C^0([0,T],\mathbb{C})$.

We define our control as follows:

$$v_h(t) = \sum_{1\leq |n|\leq N} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_n \frac{T}{2}} a_n(h)\theta_n\left(t-\frac{T}{2}\right),$$

where $a_n(h)$ was defined in (8).

The key estimate

For any $T > \frac{4}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$ we are able to prove that

$$||\theta_m||_\infty \leqslant C. \tag{11}$$

Conclusions and open problems about the wave

- By filtering in an optimal way the initial condition we obtain that the minimal time of control is $T = 4$.

- The optimal range of filtration is localized in the area where the gap between the eigenvalues of the discrete model becomes small.

- Beyond this range, the gap is altered by the numerical discretization.
- By adding a vanishing viscosity term the problem appearing in the high frequencies has been solved (Micu, SIAM Cont. Opt. ('08))

Approximation of the controls for the beam equation with vanishing viscosity

We consider a finite difference semi-discrete scheme for the approximation of the boundary controls of a 1-D equation modeling the transversal vibrations of a hinged beam.

The continuous model for the hinged beam

The boundary controlled transversal vibrations of a 1–D beam with hinged boundary conditions are modelled by the following equation

$$\begin{cases} u''(t,x) + u_{xxxx}(t,x) = 0 & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = u(t,1) = u_{xx}(t,0) = 0 & t \in (0,T) \\ u_{xx}(t,1) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,1) \\ u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{12}$$

The vector $\begin{pmatrix} u \\ u' \end{pmatrix}$ represents the state and v is the control acting on the extremity $x = 1$ of the beam.

The null controllability

Given $T > 0$ we say that (12) is *null–controllable in time T* if, for every initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H} := H_0^1(0,1) \times H^{-1}(0,1)$, there exists a control $v \in L^2(0,T)$ such that the corresponding solution of (12) verifies

$$u(T,x) = u'(T,x) = 0 \qquad (x \in (0,1)). \tag{13}$$

From mathematical point of view: Bad news.

Bad news

It is known that, due to the high frequency numerical spurious oscillations, the uniform (with respect to the mesh-size) controllability property of the semi-discrete model fails in the natural setting.

The convergence of the approximate boundary controls corresponding to initial data in the finite energy space cannot be guaranteed.

Good news

Good news

We solve this deficiency by adding a vanishing numerical viscosity term, which will damp out these high frequencies.

We prove that, by adding a vanishing numerical viscosity, the uniform controllability property and the convergence of the scheme is ensured.

The semi-discrete model

- Let $N \in \mathbb{N}^*$, $x_0 = 0 < x_1 = h < ... < x_j = jh < ... < x_{N+1} = 1$, where the mesh-size is $h = \frac{1}{N+1}$ and let two external points $x_{-1} = x_0 - h$ and $x_{N+2} = x_{N+1} + h$.
- The semi–discretization of (12), for each $1 \leq j \leq N$, $t \in (0, T)$ is

$$\begin{cases} u_j''(t) + \frac{u_{j+2}(t) - 4u_{j+1}(t) + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4} = 0 \\ u_0(t) = 0, \quad u_{N+1}(t) = 0 \\ u_{-1}(t) = -u_1(t), \quad u_{N+2}(t) = h^2 v_h(t) - u_N(t) \\ u_j(0) = u_j^0(x), \quad u_j'(0) = u_j^1(x). \end{cases} \tag{14}$$

- The quantities $u_j(t)$ approximate $u(t, x_j)$,
- If the initial data are regular, we shall choose

$$u_j^0 = u^0(jh), \quad u_j^1 = u^1(jh) \qquad (1 \leq j \leq N). \tag{15}$$

The null controllability of the discrete model

We consider the following controllability property for (14): *given* $T > 0$ *and* $\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N}$, *we look for a control* $v_h \in L^2(0,T)$ *such that the corresponding solution* $\begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{1\leq j\leq N}$ *of* (14) *verifies*

$$u_j(T) = u_j'(T) = 0 \qquad (1 \leq j \leq N). \tag{16}$$

If this property is verified for every initial data $\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N}$, we say that (14) is *null-controllable in time T.*

Non uniformly observability and controllability

The null–controllability property for (14) holds in any time $T > 0$.

Leon and Zuazua (ESAIM COCV, 2002)

For any $h > 0$, there exists a constant $C = C(T, h)$ such that

$$\left\| \begin{pmatrix} \varphi_j \\ \varphi_j' \end{pmatrix}_{1\leq j\leq N} (0) \right\|_{1,-1}^2 \leq C \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt, \qquad (17)$$

for any $\begin{pmatrix} \varphi_j^0 \\ \varphi_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N}$ and $\begin{pmatrix} \varphi_j \\ \varphi_j' \end{pmatrix}_{1\leq j\leq N}$ solution of the correspoding backward equation but

$$\lim_{h\to 0} \sup_{(\varphi,\, \varphi')\text{ solution}} \frac{\left\| \begin{pmatrix} \varphi_j \\ \varphi_j' \end{pmatrix} (0) \right\|_{1,-1}^2}{\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt} = \infty. \qquad (18)$$

How was solved this deficiency

In order to obtain a uniform observability inequality, two possibilities have been proposed and analyzed in Leon and Zuazua (ESAIM COCV, (2002)):

- the class of solutions has been restricted to a space in which the high frequencies have been filtered out. Under this assumption, the corresponding observability inequality becomes uniform and, consequently, the projection of the solution of (14) over this filtered space is controlled to zero uniformly;

- the observed quantity in the right side of (17) has been reinforced by introducing an extra term. This shows that an additional boundary control, which vanishes in limit, makes the system uniformly controllable.

The third possibility: the vanishing viscosity method

The bad spurious high frequencies introduced by the discretization process are responsible for the bad controllability properties of (14).

The third possibility

We introduce in the discrete equation (14) a numerical viscosity which vanishes in the limit.

Since this term damps out the high frequencies which are responsible for (18), we can expect that it will also help us to restore the desired uniform observability inequality and to improve the convergence properties of the discrete controls.

The new proposed perturbed problem

More precisely, for each $1 \leq j \leq N$ and $t \in (0,T)$ we consider

$$\begin{cases} u_j''(t) + \frac{u_{j+2}(t)-4u_{j+1}(t)+6u_j(t)-4u_{j-1}(t)+u_{j-2}(t)}{h^4} = \varepsilon \frac{u_{j+1}'(t)-2u_j'(t)+u_{j-1}'(t)}{h^2} \\ u_{-1}(t) = -u_1(t), \quad u_{N+2}(t) = h^2 v_h(t) - u_N(t) \\ u_j(0) = u_j^0(x), \quad u_j'(0) = u_j^1(x) \end{cases}$$

The viscosity term

The ratio $\varepsilon \frac{u'_{j+1}(t) - 2u'_j(t) + u'_{j-1}(t)}{h^2}$ represents a viscous term and he parameter ε depends on the step size h and verifies

$$\lim_{h \to 0} \varepsilon(h) = 0. \qquad (19)$$

Note that the parameter ε should be chosen small, in order to preserve the convergence and the accuracy of the numerical scheme, but also sufficiently large to improve the observability properties of the system, damps out the spurious high frequencies.

More about vanishing viscosity

The artificial viscosity is a common tool in many numerical schemes.

- the uniform stabilization of discrete hyperbolic equations is achieved: Tebou and Zuazua (Numer. Math. (2003)), , Zuazua (SIAM Rev. (2005)), Münch and Pazoto (ESAIM COCV (2007)), Zuazua (SIAM Rev. (2005)).
- uniform Strichartz's estimates for the discrete Schrödinger equation: Ignat and Zuazua (SIAM J. Numer. Anal., (2009))
- control problems for the wave equation: Micu (SICON (2008))
- a uniform controllability result in arbitrarily small time for Schrodinger equation: Micu and Roventa (ESAIM COCV (2012) and JOTA (2014))
- The corresponding result for the semi-discrete wave equation is still an open problem.

The discrete problem

We study the null–controllability of the following finite-difference space discretization equation:

$$\begin{cases} u_j''(t) + \frac{u_{j+2}(t)-4u_{j+1}(t)+6u_j(t)-4u_{j-1}(t)+u_{j-2}(t)}{h^4} = \varepsilon \frac{u_{j+1}'(t)-2u_j'(t)+u_{j-1}'(t)}{h^2} \\ u_0(t) = 0, \quad u_{N+1}(t) = 0 \\ u_{-1}(t) = -u_1(t), \quad u_{N+2}(t) = h^2 v_h(t) - u_N(t) \\ u_j(0) = u_j^0(x), \quad u_j'(0) = u_j^1(x), \end{cases} \tag{20}$$

where $1 \leq j \leq N,\ t \in (0,T)$.

The eigenvalues of the adjoint problem without viscosity

The eigenvalues of A_h are given by

$$\mu_n = \frac{4}{h^2} \sin^2\left(\frac{n\pi h}{2}\right) \qquad (1 \leq n \leq N), \tag{21}$$

with the corresponding eigenvectors

$$\phi_h^n = (\sin(knh\pi))_{1 \leq k \leq N} \in \mathbb{R}^N \qquad (1 \leq n \leq N). \tag{22}$$

The eigenvalues of the adjoint problem with viscosity

Lemma

The eigenvalues of the operator $\mathcal{A}_h$ are given by

$$\lambda_n = \mu_{|n|} \frac{\varepsilon + i\, sgn(n)\sqrt{4-\varepsilon^2}}{2} \qquad (1 \leq |n| \leq N), \tag{23}$$

and the corresponding eigenvectors are

$$\Phi_h^n = \frac{1}{\sqrt{\mu_{|n|}}} \begin{pmatrix} 1 \\ \lambda_n \end{pmatrix} \phi_h^{|n|} \qquad (1 \leq |n| \leq N), \tag{24}$$

where $\phi_h^{|n|}$ are given by (22). Moreover, the vectors $(\Phi_h^n)_{1\leq|n|\leq N}$ form a basis in $\mathbb{C}^{2N}$.

Uniformly boundedness of the sequence of controls

Theorem (Bugariu, Micu, Rovenţa, Math. Comp., 2016)

Let $T>0$. There exist $h_0, c_0>0$ such that for any $h\in(0,h_0)$, $\varepsilon\in\left(c_0h^2\ln\frac{1}{h},h\right)$ and any initial data

$$\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} = \sum_{1\leq|n|\leq N} a_{nh}^0\Phi_h^n,$$

such that there exists a constant $C>0$ independent of h and ε with the property

$$\left\|(a_{nh}^0)_n\right\|_{\ell^\infty} < C, \tag{25}$$

there exists a control $v_h\in L^2(0,T)$ such that the family $(v_h)_h$ is uniformly bounded in $L^2(0,T)$.

Proof of controllability result

For any initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ and $t \in (0, T)$ the control for (**??**) is given by

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^{n+1} e^{-\overline{\lambda}_n \frac{T}{2}}}{2\cos\left(\frac{n\pi h}{2}\right)} \cdot \left(\mathrm{sgn}(n)\, i\, \sqrt{4-\varepsilon^2}\, a_{nh}^0 + \varepsilon a_{nh}^0 + \varepsilon a_{-nh}^0 \right) \theta_n \left(t - \frac{T}{2}\right). \quad (26)$$

Proof of controllability result

By using the the biorthogonal estimates we have that

$$\int_0^T |v_h(t)|^2\,dt \leq C \sum_{|n|=1}^{N} \left| \frac{1}{\cos\left(\frac{n\pi h}{2}\right)} \right|^2 e^{-\frac{T}{2}|\Re(\lambda_n)|} \leq C,$$

where C is a positive constant independent of h, ε and m. Note that, the last inequality takes place since

$$e^{-\frac{T}{4}|\Re(\lambda_N)|} < h^2.$$

The main controllability result

Theorem (Bugariu, Micu, Rovenţa, Math. Comp., 2016)

Let $T > 0$ *and* $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$. *There exist* $h_0, c_0 > 0$ *such that for any for any* $h \in (0, h_0)$, $\varepsilon \in \left(c_0 h^2 \ln \frac{1}{h}, h\right)$ *and any initial data* $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ *with the property*

$$(a_{nh}^0)_n \overset{*}{\rightharpoonup} (a_n^0)_n \text{ in } \ell^\infty \text{ when } h \to 0, \tag{27}$$

there exists a family of exact controls $(v_h)_h \subset L^2(0,T)$ *for problem* (20) *which converges to a null-control for in* $L^2(0,T)$ *for the continuous problem.*

Interior controllability case

Until now we have treated the boundary controllability problem and we have seen that the negative results are due to the bad numerical approximation of the high eigenmodes.

It is interesting to note that, in the case when the control acts in the interior of the domain, the uniform controllability property is ensured automatically for any initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ in the space

$$H^2(0,1) \cap H_0^1(0,1) \times L^2(0,1).$$

What about clamped beam equation?

The controllability problem associated to the beam clamped at both extremities

For every initial data $(y_0, y_1) \in L^2(0,1) \times H^{-2}(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of

$$\begin{cases} \ddot{y}(x,t) + \partial_x^4 y(x,t) = 0, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ \partial_x y(0,t) = 0, \quad \partial_x y(1,t) = v(t), & t \in (0,T) \\ y(x,0) = y_0(x), \quad \dot{y}(x,0) = y_1(x), & x \in (0,1), \end{cases} \tag{28}$$

verifies

$$y(x,T) = \dot{y}(x,T) = 0 \qquad (x \in (0,1)).$$

The main result

Theorem (Cindea, Micu, Roventa, SIAM Cont. Optim., 2017)

We prove that even in this case the control does not allow us to prove that the discrete system is uniformly observable (is uniformly observable if the initial data is filtered in the range δN).

- since the discrete finite-difference operator is no more the square of finite-differences discrete Laplacian, the eigenvalues cannot be explicitly computed.
- for the eigenvales: algebraic computations combined with Rouché's theorem
- for the eigenvectors: asymptotic estimates and a discrete multiplier method.

Example 1. In this example we take $T = 1.7$ and the initial data to be controlled are given by

$$u^0(x) = \sin(\pi x), \quad u^1(x) = 0 \qquad (x \in (0,1)).$$

Note that this particular initial condition belongs to $\mathcal{D}(\mathcal{A})$.

Two approximations of the control are presented for $N = 100$ and two different values of the viscosity parameter: $\varepsilon = 0$ and $\varepsilon = h$.

Sufficiently smooth initial data can be uniformly controlled, even if $\varepsilon = 0$.

Numerical results

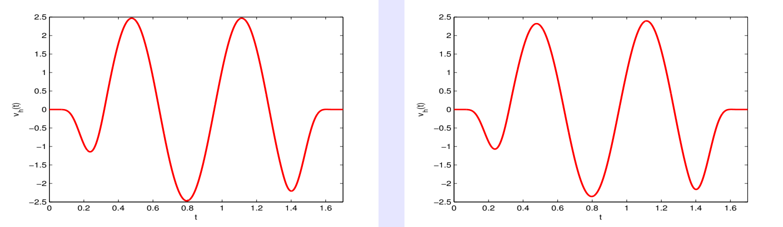


Figure: Example 1 - Two approximations of the control v_h: with $\varepsilon = 0$ (left) and with $\varepsilon = h$ (right).

Numerical results

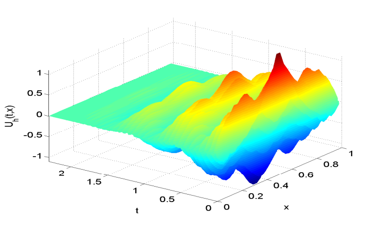
Example 2. In this example we take $T = 2.3$ and the following initial data to be controlled

$$u^0(x) = \begin{cases} 16x^3 & \text{if } x \leq \frac{1}{2} \\ 16(1-x)^3 & \text{if } x > \frac{1}{2}, \end{cases} \qquad u^1(x) = 0 \qquad (x \in (0,1)).$$

Note that the initial data belong to $\mathcal{H} = H_0^1(0,1) \times H^{-1}(0,1)$ but not to $\mathcal{D}(\mathcal{A})$.

The space $\mathcal{H}$ is the largest space of initial data which can be controlled with L^2-controls.

Numerical results



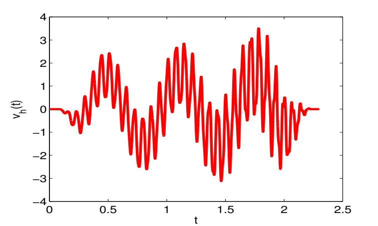


Figure: Example 2 - Controlled solution and the approximation of the control with $N = 100$ and $\varepsilon = h$.

Numerical results

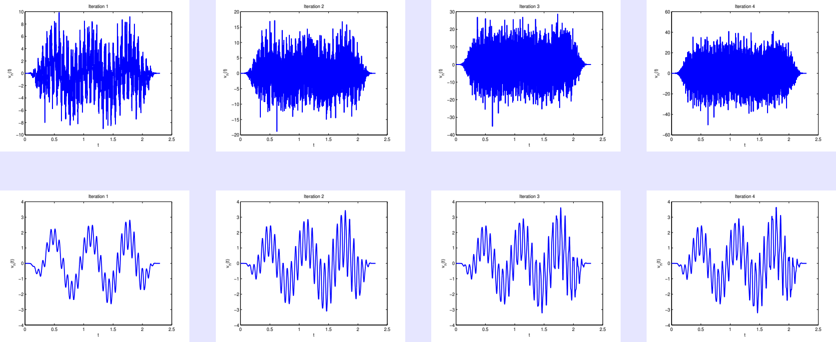


Figure: Example 2 - The first four iterations of the conjugate gradient method for the approximation of v_h with $N = 100$ and $\varepsilon = 0$ (up) or $\varepsilon = h$ (down).

Example 3. In this example we take $T = 3$ and the initial data to be controlled are the following

$$u^0(x) = 1 - |2 - |4x - 1|| \qquad u^1(x) = 0 \qquad (x \in (0,1)).$$

The initial data belong to $\mathcal{H}$ but not to $\mathcal{D}(\mathcal{A})$.

Numerical results

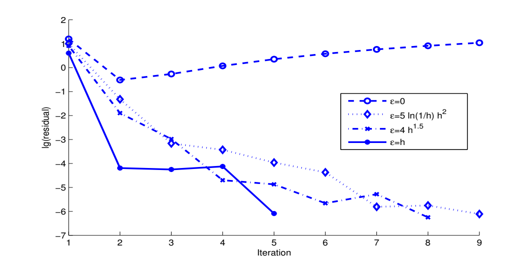


Figure: Example 3 - Error evolution in the conjugate gradient method with four different values of ε.

N	25	50	100	200	400
$\|v_h\|_{L^2}$ with $\varepsilon = h$	0.5376	1.1518	1.6301	1.9209	2.099
$\|v_h\|_{L^2}$ with $\varepsilon = 4h^{1.5}$	0.7635	1.5728	1.9908	2.2008	2.3175
$\|v_h\|_{L^2}$ with $\varepsilon = 5h^2 \log(1/h)$	0.9746	1.7655	2.1201	2.2929	2.3819

Table: Example 3 - Numerical results for $\|v_h\|_{L^2}$

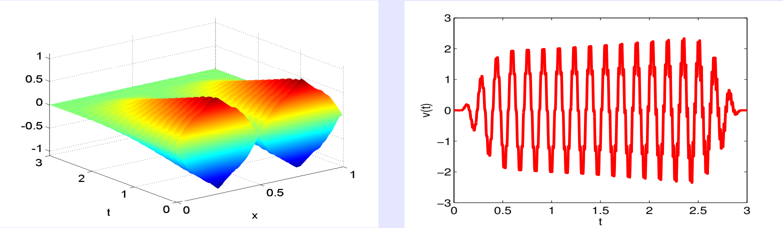


Figure: Example 3 - Controlled solution and the approximation of the control with $N = 400$ and $\varepsilon = h$.

Conclusions

- the lack of convergence of the algorithm if $\varepsilon = 0$ and the initial data are not smooth enough.

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- the lack of convergence of the algorithm if $\varepsilon = 0$ and the initial data are not smooth enough.
- a larger viscosity parameter ε helps the convergence of the scheme in the non smooth case, but produces a slower convergence rate in the regular case.
- the amount of dissipation introduced in the system through the parameter ε should be decided by taking into account the regularity of initial data to be controlled.

Between the beam and the wave

- the approximation of controls for the wave equation:

$$\lambda_n = \frac{2}{h} \sin \frac{n\pi h}{2}$$

- the approximation of controls for the beam equation:

$$\lambda_n = \frac{4}{h^2} \sin^2 \frac{n\pi h}{2}$$

- What is between the BEAM and WAVE?
- We consider the approximation of controls a problem with fractional Laplacian:

$$\lambda_n = \left(\frac{2}{h} \sin \frac{n\pi h}{2} \right)^{\alpha}, \quad \alpha \in (1, 2).$$

Internal controls for a problem with fractional Laplacian using finite-difference method

Let us consider $\alpha > 1$, $T > 0$ and the following equation:

$$\begin{cases} u''(t,x) + (-\Delta_D)^\alpha u_{xx}(t,x) = f(x)v(t) & t \in (0,T),\ x \in (0,1), \\ u(t,0) = 0 & t \in (0,T), \\ u(t,1) = 0 & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{29}$$

where Δ_D is the one-dimensional Dirichlet-Laplace operator on (0,1) with domain $H^2(0,1) \cap H^1_0(0,1)$, the profile f lies in $C^0(0,1)$ (this regularity is needed in order to give a precise meaning to the discretization) and the control v lies in $L^2(0,T)$. The operator $(-\Delta_D)^\alpha$ is defined using the spectral definition.

For $k \in \mathbb{Z}^*$, we consider

$$f_k = \int_0^1 f(x) \sin(k\pi x) dx. \tag{30}$$

We assume that $f \in L^2(0,T)$ is chosen such that $f_k(x) \neq 0, \forall k \in \mathbb{Z}^*$. The space of initial conditions is given by

$$\mathcal{H} := \left\{ (u^0, u^1) = \sum_{k \in \mathbb{Z}^*} \left(\frac{1}{k^\alpha i}, -1 \right) a_k \sin(k\pi x) \mid \sum_{k \in \mathbb{Z}^*} \frac{|a_k|^2}{|f_k|^2} < \infty \right\}.$$

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. For any $T > 0$, we consider the following semi-discrete space approximation:

$$\left\{ \begin{array}{ll} u_j''(t) + A_h^\alpha u = f(jh)v(t) & 1 \leq j \leq N,\ t > 0, \\ u_0(t) = 0 & t \in (0,T), \\ u_{N+1}(t) = 0 & t \in (0,T), \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \leq j \leq N, \end{array} \right. \tag{31}$$

where $A_h \in \mathcal{M}_N(\mathbb{R})$ is the discrete Laplace operator given by

$$A_h := \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Given $T > 0$, $h > 0$ and $((u_j^0, u_j^1))_{1\le j\le N} \in \mathbb{C}^{2N}$, we study the existence of a control function $v_h \in L^2([0,T])$ such that the solution of the equation (31) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (j = 1, 2, ..., N).$$

More precisely, our aim is to study the existence of a uniformly bounded sequence of controls $(v_h)_{h>0}$ with respect to the mesh size h, by using the moment method.

Theorem (Lissy, Roventa, 2019)

Let $(u_j^0, u_j^1)_{1\leq j\leq N} = \sum_{1\leq|n|\leq f(N)} a_n \Phi_h^n$, *where* $(a_n)_{1\leq|n|\leq N}$ *verifies*

$$\sum_{1\leq|n|\leq f(N)} \left|\frac{a_n}{f_n}\right|^2 < \infty.$$

There exists a uniformly bounded sequence of controls $(v_h)_{h>0}$ *for the discrete control problem* (31).

Theorem (Lissy, Roventa, 2019)

Let $(u^0, u^1) = \sum_{1 \le |n| \le f(N)} a_n \Phi^n$ *be the initial datum for the continuous problem* (29) *and let us consider the initial datum of the semi-discrete problem* (31)

$$(u_j^0, u_j^1)_{1 \le j \le N} = \sum_{1 \le |n| \le f(N)} a_n \tilde{\Phi}^n(h), \tag{32}$$

where $\tilde{\Phi}^n(h)$ *is the discretization of the eigenfunction* Φ^n *of the form*

$$\tilde{\Phi}^n(h) = \left(\left(\frac{1}{n^\alpha i} \sin(jn\pi h) \right)_{1 \le j \le N}, (-\sin(jn\pi h))_{1 \le j \le N} \right)^T.$$

Let $v \in L^2([0,T])$ *the weak limit of the sequence of controls* $(v_h)_{h>0}$ *given by Theorem 7, with initial datum given in* (32). *Then* v *is a control for continuous problem* (29).

Open problems

- the approximations of controls for 2D wave equation
- to consider non-uniform meshes
- to obtain general controllability results in terms of the gap between the eigenvalues


Thank you