Sur les différences de fonctions surhamoniques

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Théorie du potentiel et EDP non-linéaires

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- $B(t), t \geq 0$ a Brownian motion on $\mathbb{R}$
- $u : \mathbb{R} \longrightarrow \mathbb{R}$ a measurable function.

Then

1. $u(B(t)), t \geq 0$ is a rc martingale iff u is an affine function.

2. $u(B(t)), t \geq 0$ is a rc sub-martingale iff u is convex.

In particular, if u is a difference of convex functions then $u(B)$ is a semimartingale.
More generally

Meyer-Ito formula

If X is a real-valued semimartingale and u is the difference of two convex functions, then $u(X)$ remains a semimartingale and

$$u(X(t)) - u(X(0)) = \int_0^t u'(X(s))dX(s) + \frac{1}{2}\int_{\mathbb{R}} L_t^x \, \nu(dx)$$

General Markov processes

- $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ is a right Markov process on E.
- $P_t f(x) = \mathbb{E}^x f(X_t)$, $t \geq 0$; $P_t^\alpha := e^{-\alpha t} P_t$, $\alpha \geq 0$.
- $u : E \to [0, \infty]$ is called α-excessive if $P_t^\alpha u \leq u$ and $P_t^\alpha u \to u, t \to 0$.

Well known correspondedce

For $u : E \to \mathbb{R}_+$ and $\beta \geq 0$, are equivalent:

i) $(e^{-\beta t} u(X_t))_{t \geq 0}$ is a r.c. $\mathcal{F}_t$-supermartingale w.r.t. $\mathbb{P}^x$ for all $x \in E$.

ii) The function u is β-excessive.

Also, martingales correspond to harmonic functions: $P_t u = u, t \geq 0$

$$u(X)$$

$$\begin{array}{ccc} \text{martingales} & \subset & \text{supermartingales} \\ \updownarrow & & \updownarrow \\ \text{harmonic} & & \text{excessive} \end{array}$$

$$u(X)$$

$$\text{martingales} \subset \text{supermartingales} \subset \textsf{semimartingales}$$

$$\begin{array}{ccc} \text{martingales} & & \text{supermartingales} \\ \updownarrow & & \updownarrow \\ \textsf{harmonic} & & \textsf{excessive} \end{array}$$

$$u(X)$$

$$\begin{array}{ccccc} \text{martingales} & \subset & \text{supermartingales} & \subset & \text{semimartingales} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \text{harmonic} & & \text{excessive} & & \text{locally,} \\ & & & & \text{differences of excessive} \end{array}$$

[**E. Cinlar, J. Jacod, P. Protter, M. J. Sharpe**, Semimartingales and Markov Processes, Z. Wahrsch. verw. Gebiete, 1980]

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Folklore theorem

The following are equivalent:

1. f is the difference of two convex functions on each interval I.
2. The right-hand derivative of f exists, it is right-continuous and of bounded variation on each interval I.
3. The weak second derivative of f exists as a signed radon measure on $\mathbb{R}$.

! If f is locally the difference of two convex functions then $f(B_t)$ is a semimartingale.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Theorem [CiJaPrSh '80]

The following are equivalent:

1. f is the difference of two convex functions on each interval I.
2. The right-hand derivative of f exists, it is right-continuous and of bounded variation on each interval I.
3. The weak second derivative of f exists as a signed radon measure on $\mathbb{R}$.
4. $f(B)$ is a semimartingale.

Back to general Markov processes X

$u(X)$

martingales $\subset$	supermartingales $\subset$		$\subset$ semimartingales
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
harmonic	exc	exc - exc	locally, exc - exc

Aim: understand the differences of excessive functions!

$u(X)$

$$\begin{array}{ccccccc} \text{mart} & \subset & \text{supermart} & \subset & \text{quasimartingales} & \subset & \text{semimartingales} \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \text{harmonic} & & \text{exc} & & \text{exc - exc} & & \text{locally, exc - exc} \end{array}$$

L. Beznea, I. C., Quasimartingales of Markov processes, Transactions of the AMS (2018)

Our approach: quasimartingales

General definition: $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ An $\mathcal{F}_t$-adapted, right-continuous integrable process $(Z_t)_{t\geq 0}$ is called $\mathbb{P}$-quasimartingale if

$$Var^{\mathbb{P}}(Z) := \sup_{\tau} \mathbb{E}\{\sum_{i=1}^{n} |\mathbb{E}[Z_{t_i} - Z_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]| + |Z_{t_n}|\} < \infty,$$

over all partitions $\tau : 0 = t_0 \leq t_1 \leq \ldots \leq t_n < \infty$.

L^1-supermartingales $\subset$ quasimartingales $\subset$ semimartingales

In our case, $Z = u(X)$ and $\mathbb{P} = \mathbb{P}^x$...and employing the Markov property we have for all $x \in E$:

$$Var^{\mathbb{P}^x}(u(X)) = V(u)(x) := \sup_{\tau}\{\sum_{i=1}^{n} P_{t_{i-1}}|u - P_{t_i - t_{i-1}}u|(x) + P_{t_n}|u|(x)\}$$

Aim: find u s.t. $V(u) < \infty$!

$$V(u) = \sup_{\tau} \underbrace{\{\sum_{i=1}^{n} P_{t_{i-1}} |u - P_{t_i - t_{i-1}} u| + P_{t_n}|u|\}}_{V_\tau u}$$

Theorem (L. Beznea & I.C., 2018)

Let u be a finely continuous function with $P_t(|u|) < \infty$, $t \geq 0$. Then:

1. $[V(u) < \infty] = [\lim_n V_{\tau_n}(u) < \infty]$, where $\tau_n = \{\frac{k}{2^n}\}_{k=\overline{0,n2^n}}$.
2. On $[V(u) < \infty]$: $u = u_1 - u_2$ with u_1, u_2 excessive and $[V(u) < \infty] = [u_1 + u_2 < \infty]$.

Corollary 1

The following are equivalent:

- $u(X)$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$.
- $V(u) < \infty$
- $u = u_1 - u_2, \quad u_1, u_2$ excesive finite.

Corollary 2

If X is irreducible then the following are equivalent:

- $u(X)$ is a $\mathbb{P}^{x_0}$-quasimartingale for some $x_0 \in E$.
- $u(X)$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$, possibly except a polar set.

The "quasimartingale approach" can be performed at α-level, with $\alpha > 0$:

$$(e^{-\alpha t} u(X_t))_{t \geq 0} \text{ is a quasimartingale iff } V^{\alpha}(u) < \infty.$$

- Criteria to check that $V(u) < \infty$
- Doob-Meyer decomposition
- Quasimartingales under standard transformations of Markov processes

An L^p-criteria...

Let μ be a σ-finite measure on E s.t. $(P_t)_{t\geq 0}$ is strongly continuous on $L^p(\mu)$,$1 < p < \infty$;$\frac{1}{p} + \frac{1}{q}$=1.

Theorem (L. Beznea & I.C., 2018)

Let $u \in L^p(\mu)$. If there exists $0 < g_0 \in L^q(\mu)$, α-co-excessive such that

$$\int_E |P_t u - u| g_0 d\mu \lesssim t \quad \text{for small } t > 0$$

then $V^\beta(u) < \infty$ for all $\beta > \alpha$.

...in terms of the dual generator

Let $(\widehat{\mathrm{L}}, D(\widehat{\mathrm{L}}))$ denote the generator of the adjoint semigroup $(\widehat{P}_t)$ on $L^q(\mu)$.

Proposition (L. Beznea & I. C. '18)

Assume there exists $0 < g_0 \in L^q(\mu)$, α-co-excessive and bounded. If $u \in L^p(\mu)$ and

$$\left| \int_E u \, \widehat{\mathrm{L}} v \, d\mu \right| \leq c \|v\|_\infty \quad \text{for all bounded } v \in D(\widehat{\mathrm{L}}),$$

then $V^\beta(u) < \infty$ for all $\beta > \alpha$.

...using semi-Dirichlet forms

Theorem (L. Beznea & I. C. '18)

Let $(\mathcal{E}, \mathcal{F})$ be a lower-bounded semi-Dirichlet, $u \in \mathcal{F}$ and assume that there exists a "nest" $(F_n)_{n \geq 1}$ and constants c_n such that

$$|\mathcal{E}(u, v)| \leq c_n \|v\|_\infty \ \text{ for all } \ v \in \mathcal{F}_{b,F_n}.$$

Then $u(X)$ is a semimartingale.

The symmetric case has history

- **R. F. Bass, P. Hsu** (1990), The semimartingale structure of reflecting Brownian motion, PAMS.
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- **E. Pardoux, R. J. Williams** (1994), Symmetric Reflected Diffusions, AIHP.
- **Z. Q. Chen**, (1993) On reflecting diffusion processes and Skorokhod decompositions, PTRF.
- **Z. Q. Chen, P. J. Fitzsimmons, R. J. Williams** (1993), Reflecting Brownian motions: quasimartingales and strong Caccioppoli sets, Pot. Anal.

- **M. Fukushima**
  - (1996) Distorted Brownian motions and BV functions, Trends in probability and related analysis,
  - (1999) On semi-martingale characerizations of functionals of symmetric Markov processes, EJP,
  - (2000) BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space, JFA,
  - (2001) with **M. Hino**: On the space of BV functions and a related stochastic calculus in infinite dimensions JFA.

- (2012) **M. Röckner, R.-C. Zhu and X.-C. Zhu**, The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple, The Anals of Prob.

Common tools: Dirichlet forms and Fukushima decomposition!

Quasimartingales under killing.

Let $M := (M_t)_{t \geq 0}$ be a MF of X,
$E_M := \{x \in E : \mathbb{P}^x(M_0 = 1) = 1\}$.
$Q_t f(x) := \mathbb{E}^x\{f(X_t)M_t\}$,

Proposition

Let u be a real-valued $\mathcal{B}^u$-measurable function such that $P_t|u| < \infty$ for all $t \geq 0$. Then for all $x \in E$,

$$Var^{\mathbb{P}^x}(Mu(X)) = V^{(Q_t)}u(x).$$

If M is exact, then E_M is finely open and $Q_t|_{E_M}$ is the transition function of a right Markov process $(X_t^M)_{t \geq 0}$ on E_M; see Sharpe.

Theorem

Let u be finely continuous such that $Q_t|u| < \infty$ for all $t \geq 0$. Then for all $\alpha \geq 0$, $(e^{-\alpha t}M_t u(X_t))_t$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$ if and only if $u|_{E_M}$ is an α-quasimartingale function for X^M.

Quasimartingales under time change.

Let A be a perfect continuous additive functional of X (AF) and $F = supp(A)$ its fine support. Then the inverse τ_t of A_t defined

$$\tau_t(\omega) := \inf\{s : A_s(\omega) > t\},$$

Then $Y_t(\omega) := X_{\tau_t(\omega)}(\omega)$ is a right process on F and is called the time changed process of X w.r.t. A; see Sharpe.

Proposition

- If u is a quasimartingale function for X then $u|_F$ is a quasimartingale function for Y.
- Conversely, if $F = E$, then any quasimartingale function for Y is a quasimartingale function for X.

The α-quasimartingales are not preserved by time change, but:

Proposition

If u is an α-quasimartingale function of X for some $\alpha \geq 0$, then the process $(e^{-\alpha\tau_t}u(Y_t))_{t\geq 0}$ is a $\mathbb{P}^x$-quasimartingale for all $x \in F$.

Quasimartingales under Bochner subordination.

Assume that X is transient and let $\mu := (\mu_t)_{t\geq 0}$ be a vaguely continuous convolution semigroup of subprobability measures on $\mathbb{R}_+$. Define the *subordinate* $(P_t^\mu)_{t\geq 0}$ of $(P_t)_{t\geq 0}$ by

$$P_t^\mu f := \int_0^\infty P_s f \mu_t(ds) \quad \text{for all } f \in bp\mathcal{B},$$

whose resolvent is denoted by $\mathcal{U}^\mu := (U_\alpha^\mu)_{\alpha\geq 0}$. By [Lupascu 04], $(P_t^\mu)_{t\geq 0}$ is the transition function of a right process X^μ on E. Moreover, $E(\mathcal{U}) \subset E(\mathcal{U}^\mu)$, hence we have the following result.

Proposition

Any quasimartingale function for X is a quasimartingale function for X^μ.

Example.

Killing, time change, and Bochner subordination transformations do not commute in general: subordinate killed and killed subordinate Brownian motion. We follow [Song & Vondracek '03]; or [Hmissi & Jansen '14].

- Let $(B_t)_{t\geq 0}$ be a B.m. on $\mathbb{R}^d$ and $(\xi_t)_{t\geq 0}$ an α-stable subordinator starting at 0, $\alpha \in (0,1)$. Let $Y_t = B_{\xi_t}$ be the subordinate process, whose generator is $-(-\Delta)^\alpha$, the fractional power of the negative Laplacian. Let now $D \subset \mathbb{R}^d$ be a domain and denote by Y^D the killed upon leaving D.
- Changing the order of transformations, let Z be the right process obtained by first killing X upon leaving D and then subordinating the killed Brownian motion by means of μ. The generator of Z is $-(-\Delta|_D)^\alpha$.
- Z is S-subordinate to Y^D, hence:

Proposition

Any quasimartingale function for Y^D is a quasimartingale function for Z.