Local Properties of Graphs and the Hamilton Cycle Problem

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Definitions

Background

NP-completeness of the HCP
- Locally connected (LC) graphs
- Locally traceable (LT) graphs
- Locally hamiltonian (LH) graphs
- Locally 2-nested hamiltonian (L2H) graphs
- Locally Hamilton-connected (LHC) graphs
- Locally Chvátal-Erdös graphs

Discussion

References
A graph $G$ is traceable if it has a path that visits every vertex.

A graph $G$ is hamiltonian if it has a cycle that visits every vertex.
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A graph $G$ is **hamiltonian** if it has a cycle that visits every vertex.
A graph $G$ is called locally $P$ if $\langle N(v) \rangle$ has the property $P$ for every $v \in V(G)$.

$M_3$ is locally traceable ($LT$).
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The Hamilton Cycle Problem (HCP) is the problem of determining whether a graph contains a Hamilton cycle. Oberly and Sumner (1979) Theorem: A connected, claw-free graph that is locally connected is hamiltonian. Speculation: A connected, locally hamiltonian graph is hamiltonian. Conjecture: A connected graph that is locally $k$-connected and $K_{1,k+2}$-free is hamiltonian.
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Oberly and Sumner (1979)

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Local properties to be investigated

locally connected
locally traceable
locally hamiltonian
locally 2-nested hamiltonian
locally Hamilton-connected
locally Chvátal-Erdős
closed locally Chvátal-Erdős
locally Ore
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Locally connected (LC) graphs

Smallest connected nonhamiltonian LC graph has order 5 and $\Delta = 4$.

HCP NP-complete for $\Delta = 5$ (and $\delta = 2$) (Irzhavski 2014)
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Locally traceable (LT) graphs

Smallest connected nonhamiltonian (LT) graph has order 7 and $\Delta = 5$ (van Aardt et al. 2016).

HCP NP-complete for $\Delta = 6$
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HCP NP-complete for $\Delta = 6$
The HCP for $LT$ graphs with maximum degree 6

Theorem:
The Hamilton Cycle Problem for $LT$ graphs with maximum degree 6 is NP-complete.

The HCP for cubic graphs is NP-complete. (Akiyama et al. 1980)

A nonhamiltonian locally traceable graph with maximum degree 5.
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A nonhamiltonian locally traceable graph with maximum degree 5.

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Graph \( G' \)

\[ z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \]

\( z_i \in V(G') \)

\( Z_i \) is the corresponding node in \( G \)

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Locally hamiltonian (LH) graphs

Smallest connected nonhamiltonian LH graph has order 11 and $\Delta = 8$ (Pareek et al. 1983).

HCP NP-complete for $\Delta = 9$.
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Hamilton Cycle Problem for $LH$ graphs

The HCP for maximally planar graphs is NP-complete (Chvátal 1985) if $\Delta \geq 12$.

Theorem (van Aardt et al. 2016)

If $G$ is a connected $LH$ graph with $\Delta(G) \leq 6$, then $G$ is hamiltonian.

There exist connected $LH$ graphs with maximum degree 8 that are nonhamiltonian.
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The Hamilton Cycle Problem for LH graphs

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Chvátal’s proof is valid for \( \Delta \geq 12 \).

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Locally Hamiltonian Graphs

Triangle identification

Theorem

Let $G_1$ and $G_2$ be two LH graphs, and let $G$ be a graph obtained from $G_1$ and $G_2$ by identifying suitable triangles. Then

(a) $G$ is LH.

(b) If $G_1$ and $G_2$ are planar, then so is $G$.

(c) If $G$ is hamiltonian, so are both $G_1$ and $G_2$. 

Johan de Wet et al. (UP, CoE)
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Theorem

The HCP for \( LH \) graphs with \( \Delta \geq 9 \) is NP-complete.

We will use the same approach as for \( LT \) graphs.

The graph \( H \) is locally hamiltonian and nonhamiltonian.
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Graph $H$ is combined with two copies of graph $D$ to create the graph $F$:
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\[ F_i \]
LH Graphs - the Hamilton Cycle Problem

Vertices and edges in $G'$

Nodes and borders in $G$
LH Graphs - the Hamilton Cycle Problem

Vertices and edges in $G'$

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Vertices and edges in \( G' \)

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Graph $G'$

Graph $G$

$z_i \in V(G')$

$Z_i$ is the corresponding node in $G$
LH Graphs - the Hamilton Cycle Problem

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Locally 2-nested hamiltonian (L2H) graphs

A graph $G$ is L2H if $G$ is LH and $\langle N(v) \rangle$ is LH for any $v \in V(G)$.

The smallest connected nonhamiltonian (L2H) graph has order 13 and $\Delta = 10$.

HCP NP-complete for $\Delta = 13$
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![Diagram of a graph with 13 vertices and 20 edges, illustrating the L2H property.](image-url)
A graph \( G \) is L2H if \( G \) is LH and \( \langle N(v) \rangle \) is LH for any \( v \in V(G) \).

- Smallest connected nonhamiltonian (L2H) graph has order 13 and \( \Delta = 10 \).

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L2H Graphs - the Hamilton Cycle Problem

Graph $G'$

$z_i \in V(G')$

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Graph $G$

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$Z_5$

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A graph $G$ is Hamilton-connected if there is a Hamilton path connecting any two vertices $u$ and $v$ in $V(G)$.

Smallest connected nonhamiltonian (LHC) graph has order 15 and $\Delta = 11$.

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The graph $G'$

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A graph $G$ is Chvátal-Erdös if $\alpha(G) \leq \kappa(G)$.
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- A graph is locally Chvátal-Erdős if $\alpha(\langle N(v) \rangle) \leq \kappa(\langle N(v) \rangle)$ for any $v \in V(G)$.
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- A graph is locally Chvátal-Erdös if $\alpha(\langle N(v) \rangle) \leq \kappa(\langle N(v) \rangle)$ for any $v \in V(G)$.
- A graph is closed-locally Chvátal-Erdös if $\alpha(\langle N[v] \rangle) \leq \kappa(\langle N[v] \rangle)$ for any $v \in V(G)$.
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- A cl-LCE graph is 1-tough (Chen et al. 2013).
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- A cl-LCE graph is 1-tough (Chen et al. 2013).
- It is not known if cl-LCE graphs are hamiltonian.
Table: The values of key parameters for various local properties.

<table>
<thead>
<tr>
<th></th>
<th>LC</th>
<th>LT</th>
<th>LH</th>
<th>L2H</th>
<th>LHC</th>
<th>cl-LCE</th>
<th>LCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum ( n(G) ) if ( G ) is not 1-tough</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Minimum ( \Delta(G) ) if ( G ) is not 1-tough</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>11</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>HCP is NP-complete for ( \Delta(G) ) at least</td>
<td>5</td>
<td>6</td>
<td>9*</td>
<td>13*</td>
<td>15*</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Minimum degree of local connectedness</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
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</tr>
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</table>

*It is not known whether these values are best possible.
Discussion

We can generalize the concept of L2H graphs to LkH graphs. Locally \((k + 1)\)-connected, the HCP for LkH graphs is NP-complete for any \(k \geq 1\). The important variable is the relationship between the local connectivity and local independence number.
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The important variable is the relationship between the local connectivity and local independence number.
Oberly-Sumner Conjecture: A connected graph that is locally $k$-connected and $K_1, k+2$-free is hamiltonian.

Saito's Conjecture: A connected graph that is locally Chvátal-Erdős is hamiltonian.
Oberly-Sumner Conjecture: A connected graph that is locally \(k\)-connected and \(K_{1,k+2}\)-free is hamiltonian.
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For a $\text{cl-LCE}$ graph, $\alpha(\langle N(v) \rangle) \leq \kappa(\langle N(v) \rangle) + 1$, where $v \in V(G)$.

For a graph meeting Oberly-Sumner condition, $\alpha(\langle N(v) \rangle) \leq \kappa(\langle N(v) \rangle) + 1$, where $v \in V(G)$.

Saito's conjecture is stronger than the Oberly-Sumner Conjecture.
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Saito's conjecture is stronger than the Oberly-Sumner Conjecture.
For a cl-LCE graph, \( \alpha(⟨N(ν)⟩) \leq κ(⟨N(ν)⟩) + 1 \), where \( ν \in V(G) \).

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Saito’s conjecture is stronger than the Oberly-Sumner Conjecture.
Unanswered questions

- Is the Oberly-Sumner Conjecture correct?
- Is Saito's Conjecture correct?
- Is the HCP NP-complete for LH graphs with maximum degree 8?
- Can a local condition slightly weaker than cl-LCE be usefully defined?
- Can either conjecture be proved for a smaller local independence number?
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References


9. D. Oberly and P. Sumner, Every locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph Theory 3 (1979) 351-356.
