Local Properties of Graphs and the Hamilton Cycle Problem

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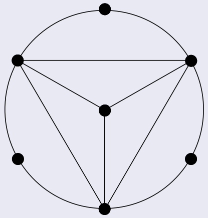

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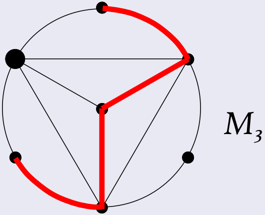

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Conjecture: A connected graph that is locally k-connected and $K_{1,k+2}$-free is hamiltonian.

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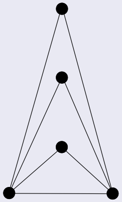

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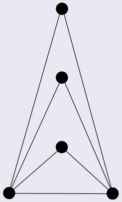

- HCP NP-complete for $\Delta = 5$ (and $\delta = 2$) (Irzhavski 2014)

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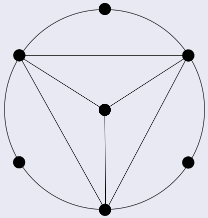

- HCP NP-complete for $\Delta = 6$

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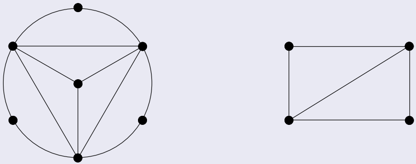
A nonhamiltonian locally traceable graph with maximum degree 5.



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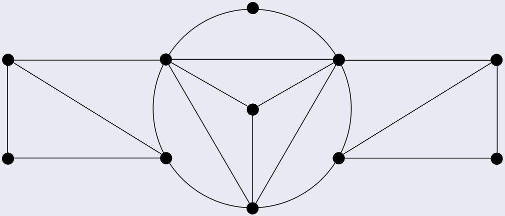

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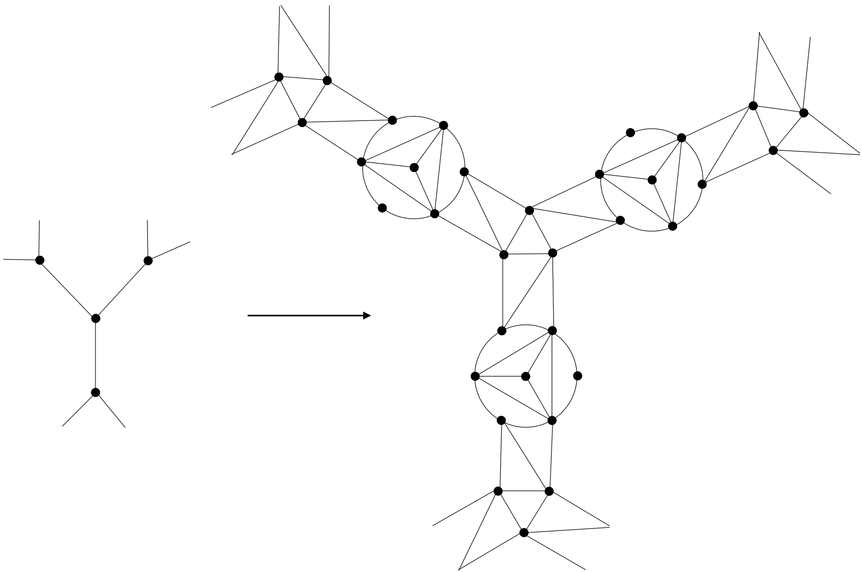

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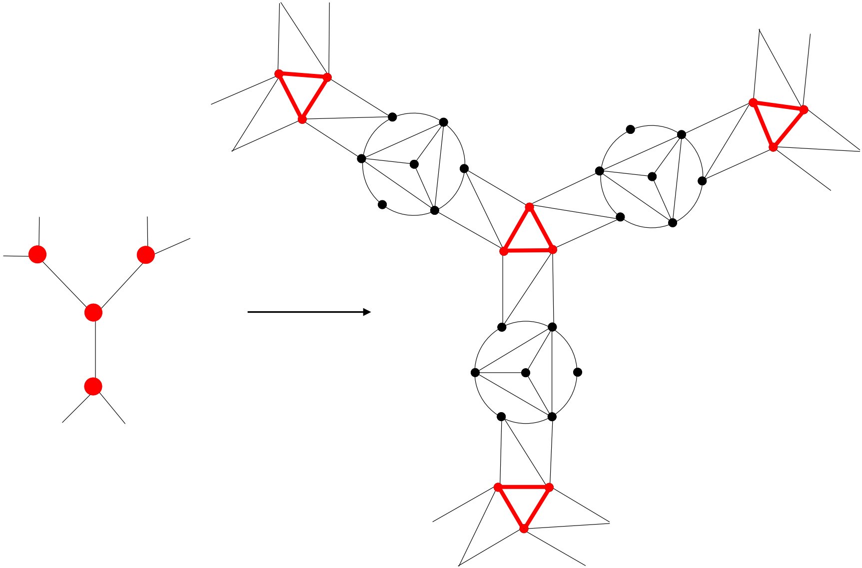

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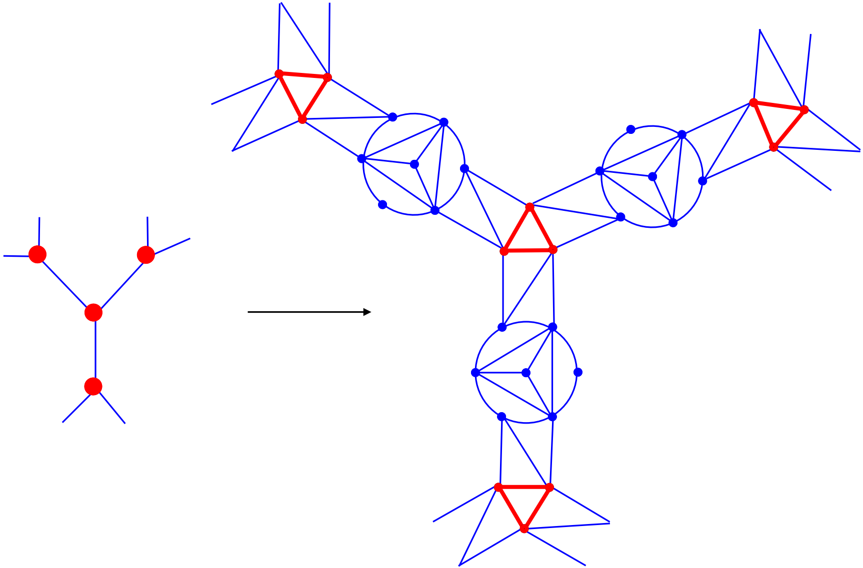

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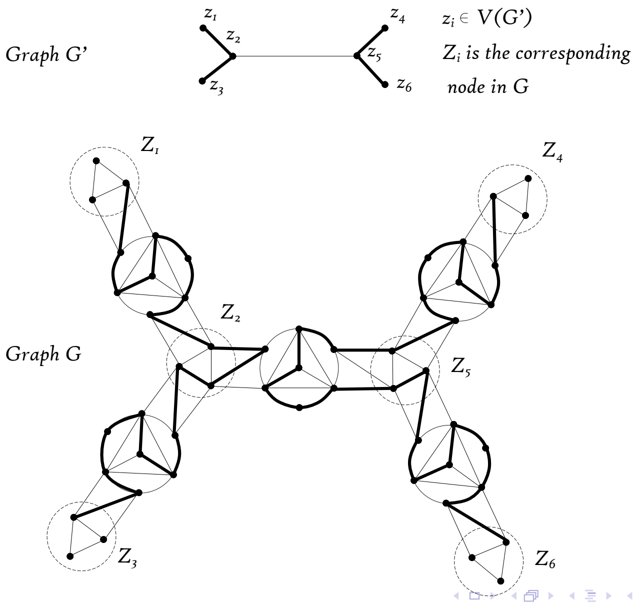

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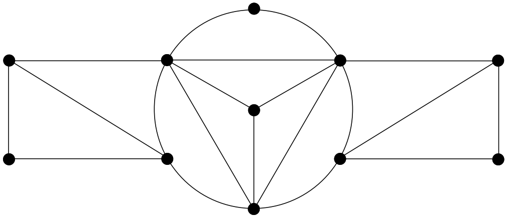

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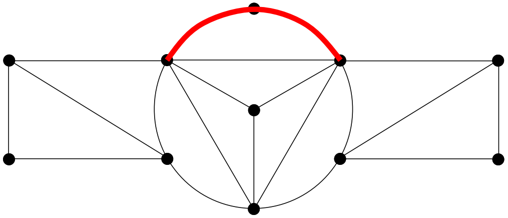

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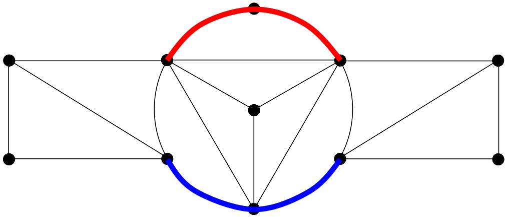

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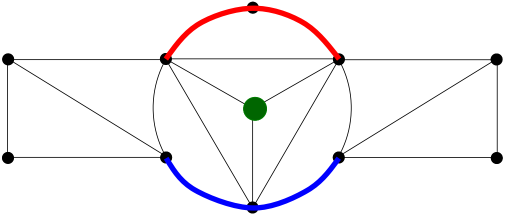

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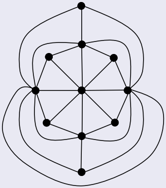

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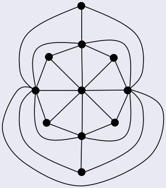

- HCP NP-complete for $\Delta = 9$

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Theorem (van Aardt et al. 2016)

If G is a connected LH graph with $\Delta(G) \leq 6$, then G is hamiltonian.

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Theorem (van Aardt et al. 2016)

If G is a connected *LH* graph with $\Delta(G) \leq 6$, then G is hamiltonian.

There exist connected *LH* graphs with maximum degree 8 that are nonhamiltonian.

Locally Hamiltonian Graphs

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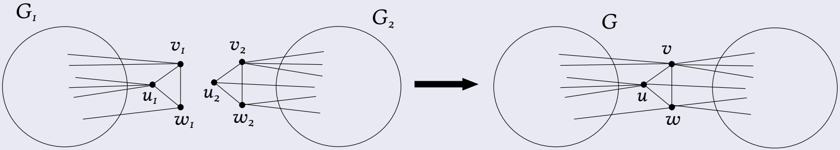
Triangle identification



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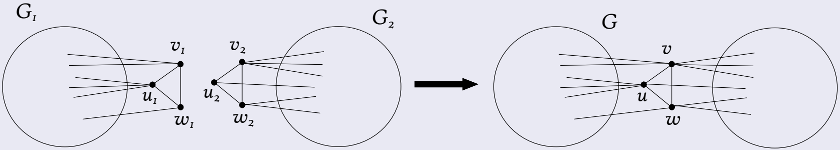

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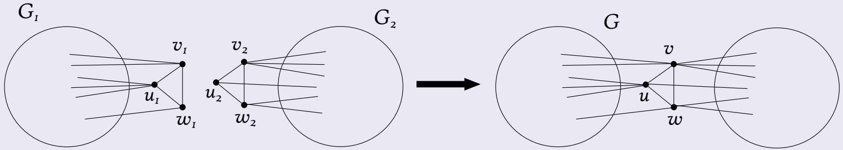

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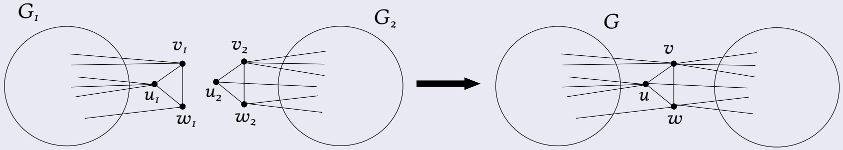

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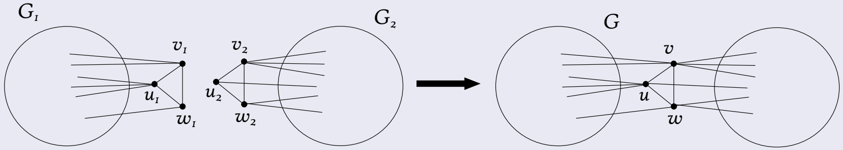

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(c) If G is hamiltonian, so are both G_1 and G_2.

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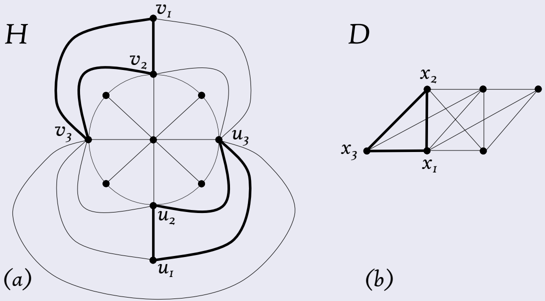

The graph *H* is locally hamiltonian and nonhamiltonian.

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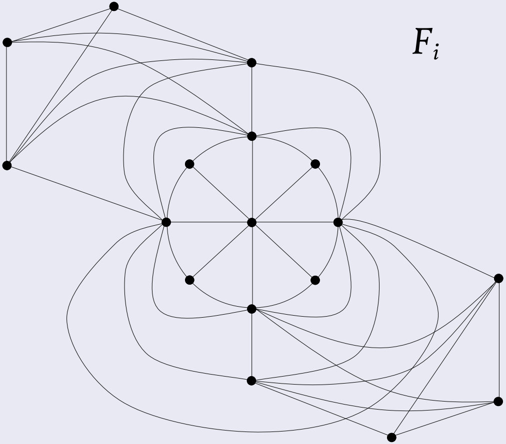

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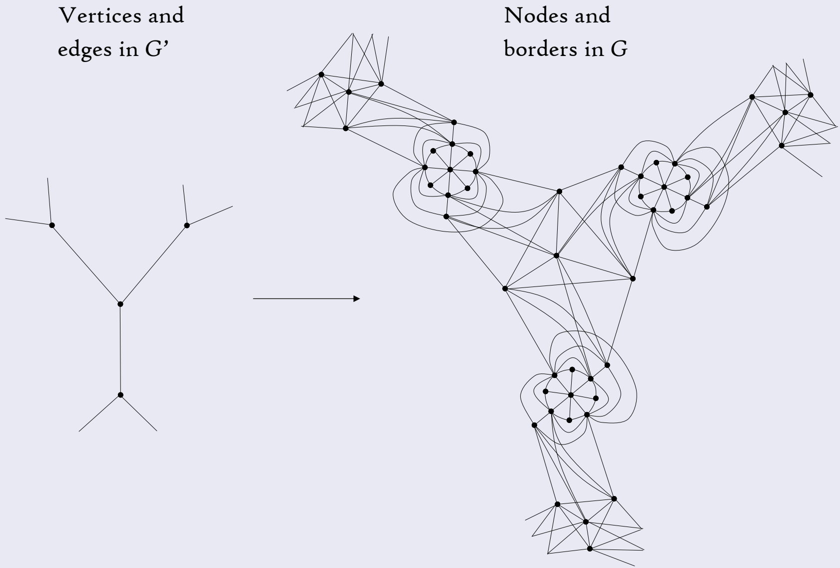
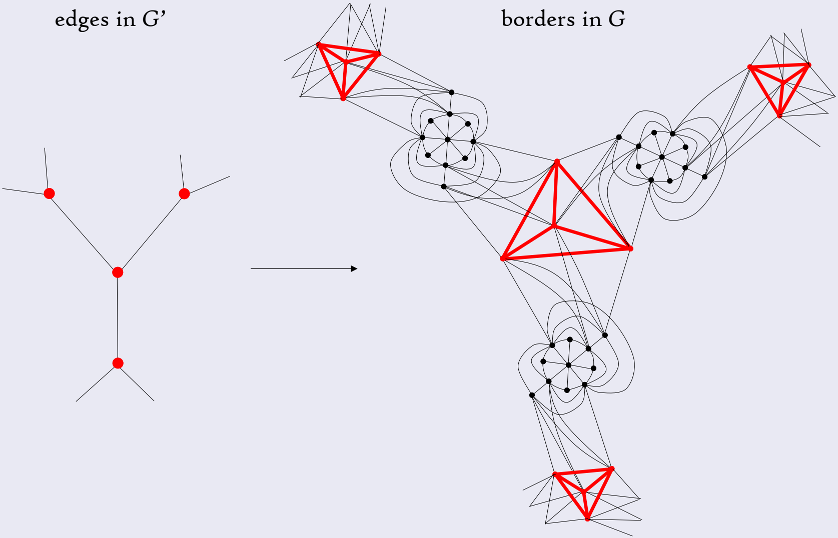


LH Graphs - the Hamilton Cycle Problem

Vertices and edges in G'

Nodes and borders in G



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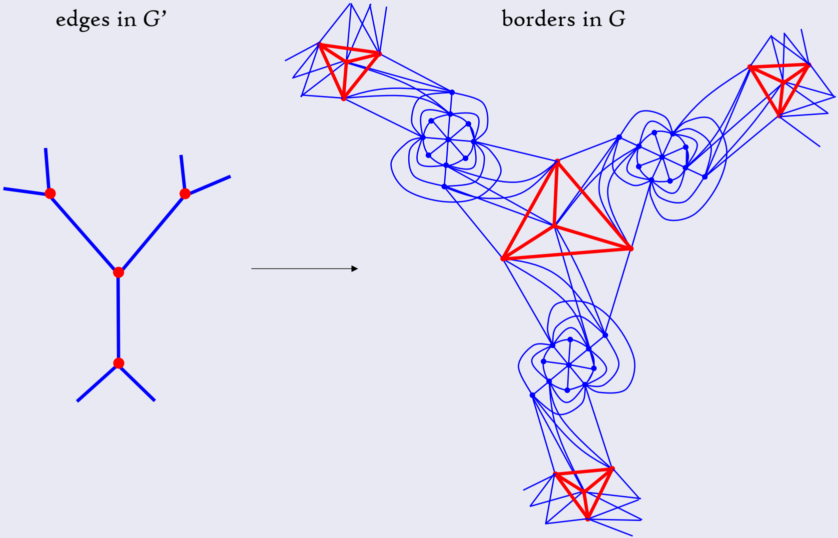

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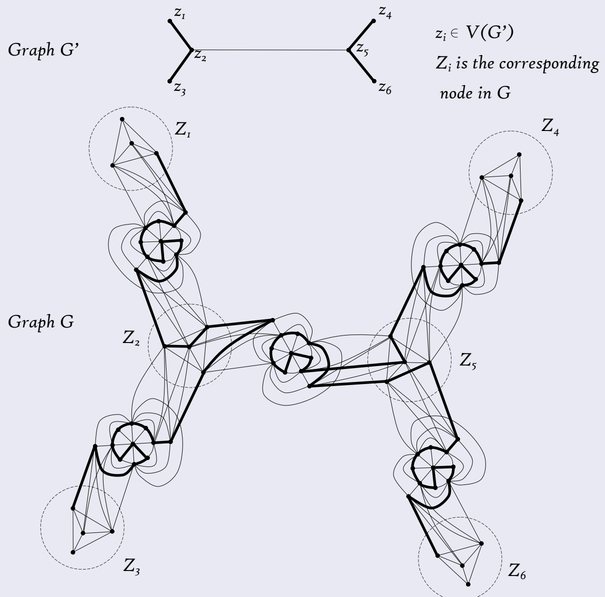

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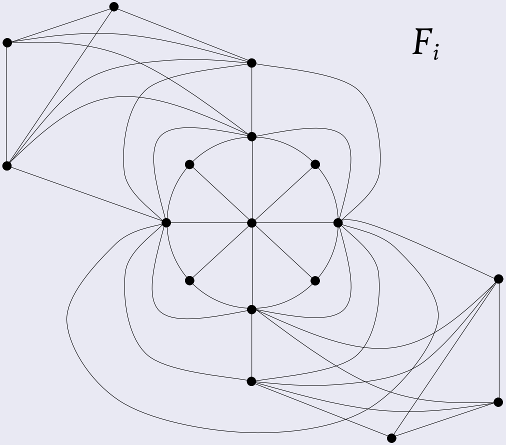

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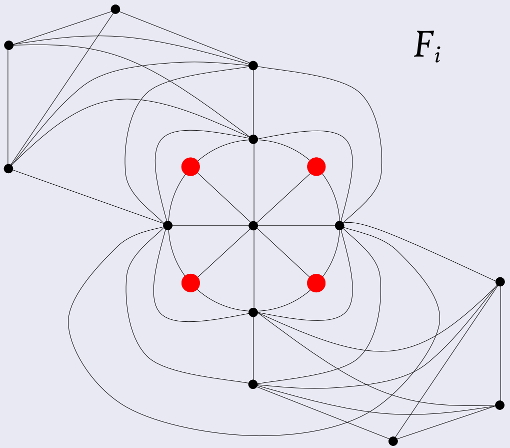

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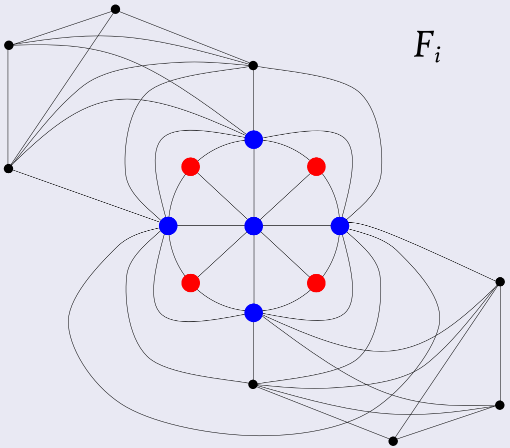

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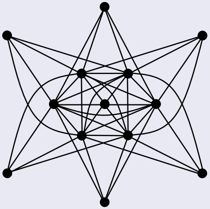

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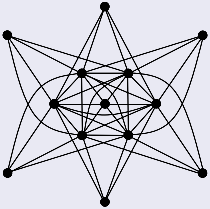

- HCP NP-complete for $\Delta = 13$

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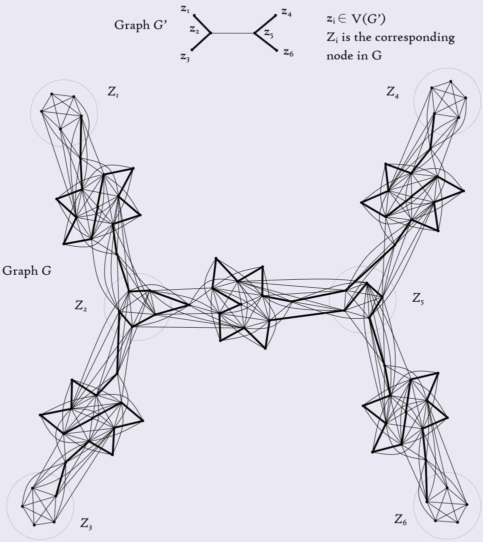

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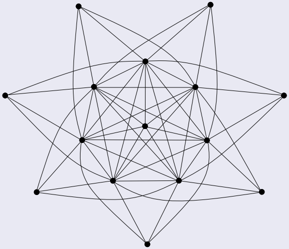

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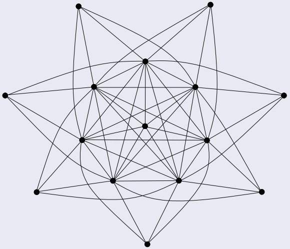

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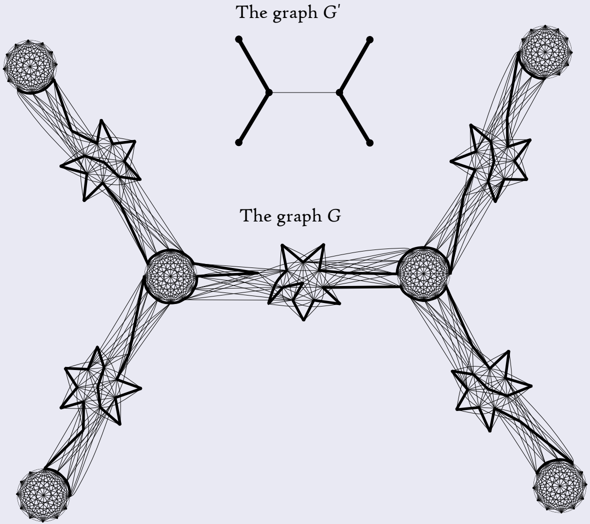

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- It is not known if cl-LCE graphs are hamiltonian.

Discussion

Table: The values of key parameters for various local properties.

	LC	LT	LH	L2H	LHC	cl-LCE	LCE
Minimum $n(G)$ if G is not 1-tough	5	7	11	13	15	N/A	N/A
Minimum $\Delta(G)$ if G is not 1-tough	4	5	8	10	11	N/A	N/A
HCP is NP-complete for $\Delta(G)$ at least	5	6	9*	13*	15*	?	?
Minimum degree of local connectedness	1	1	2	3	3	1	2

*It is not known whether these values are best possible.

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The important variable is the relationship between the local connectivity and local independence number.

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- Oberly-Sumner Conjecture: A connected graph that is locally k-connected and $K_{1,k+2}$-free is hamiltonian.

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- Oberly-Sumner Conjecture: A connected graph that is locally k-connected and $K_{1,k+2}$-free is hamiltonian.
- Saito's Conjecture: A connected graph that is locally Chvátal-Erdös is hamiltonian.

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- Saito's conjecture is stronger than the Oberly-Sumner Conjecture.

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- Can either conjecture be proved for a smaller local independence number?

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