Spanning a tough graph

Adam Kabela

Toughness of a graph

The *toughness* of a graph G is

the minimum of $\frac{|S|}{c(G-S)}$ taken over all $S \subseteq V(G)$ such that $c(G-S) \geq 2$, where $c(G-S)$ denotes the number of components of $G-S$.

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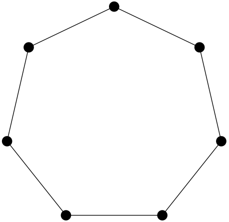

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The toughness of a complete graph is defined to be ∞.

A graph is *t-tough*

if its toughness is at least t.

Conjecture (Chvátal, 1973)

There exists t such that every t-tough graph (on at least 3 vertices) is Hamiltonian.

Chvátal's Conjecture remains open. Many related results are to be found in the survey of Bauer, Broersma, and Schmeichel (2006).

Spanning a tough enough graph

Theorem (Win, 1989)

For $k \geq 3$, every $\frac{1}{k-2}$-tough graph has a spanning tree of maximum degree at most k.

Theorem (Enomoto, Jackson, Katerinis, Saito, 1985)

For $k \geq 1$, every k-tough graph (on n vertices such that $n \geq k+1$ and kn is even) has a k-factor.

Conjecture (Tkáč, Voss, 2002)

For $k \geq 2$, there exists t_k such that every t_k-tough graph (on at least 3 vertices) has a 2-connected spanning subgraph of maximum degree at most k.

Tough enough $K_{1,k}$-free graphs

Proposition

For $\ell \geq 3$, every k-connected $K_{1,\ell}$-free graph is $\frac{k}{\ell-1}$-tough.

Conjecture (Matthews, Sumner, 1984)

Every 4-connected $K_{1,3}$-free graph is Hamiltonian.

Question (Jackson, Wormald, 1990)

If $k \geq 4$, is every k-connected $K_{1,k}$-free graph Hamiltonian?

Question

For $\ell \geq 4$, is there k such that every k-connected $K_{1,\ell}$-free graph is Hamiltonian?

Partial results on Chvátal's t-tough conjecture

Conjecture (Chvátal, 1973)

There exists t such that every t-tough graph (on at least 3 vertices) is Hamiltonian.

- 1-tough interval graphs (Keil, 1985)
- $\frac{3}{2}$-tough split graphs (Kratsch, Lehel, Müller, 1996)
- $\frac{3}{2}$-tough spider graphs (Kaiser, Král', Stacho, 2007)
- 2-tough multisplit graphs (Broersma, K., Qi, Vumar, 2018+)
- chordal planar graphs of toughness greater than 1 (Böhme, Harant, Tkáč, 1999)
- k-trees of toughness greater than $\frac{k}{3}$ (for $k \geq 2$) (K., 2018+)
- 10-tough chordal graphs (K., Kaiser, 2017)
- 25-tough $2K_2$-free graphs (Broersma, Patel, Pyatkin, 2014)

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10-tough chordal graphs

Theorem (K., Kaiser, 2017)

Every 10-tough chordal graph is Hamilton-connected.

- We view a chordal graph as an intersection graph of subtrees of a tree.
- We use the hypergraph extension of Hall's theorem (Aharoni, Haxell, 2000).

Corollary of Hall's theorem for hypergraphs

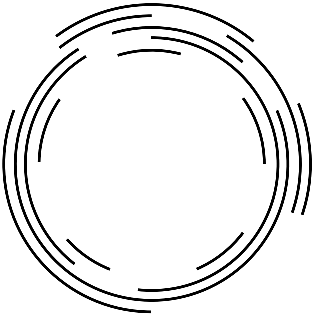
Let $\mathcal{A}$ be a family of hypergraphs of rank at most n. If for every $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$, then there exists a system of disjoint representatives for $\mathcal{A}$.

Intersection representation and Hall's theorem for hypergraphs

Note

Every 4-tough circular arc graph (on at least 3 vertices) is Hamiltonian.

Idea of the proof:



Thank you for your attention.

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