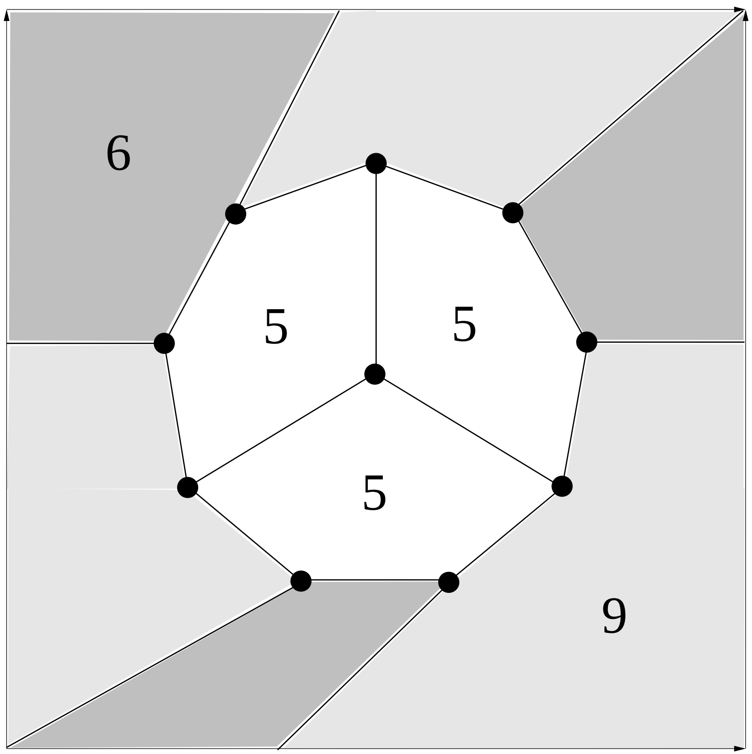
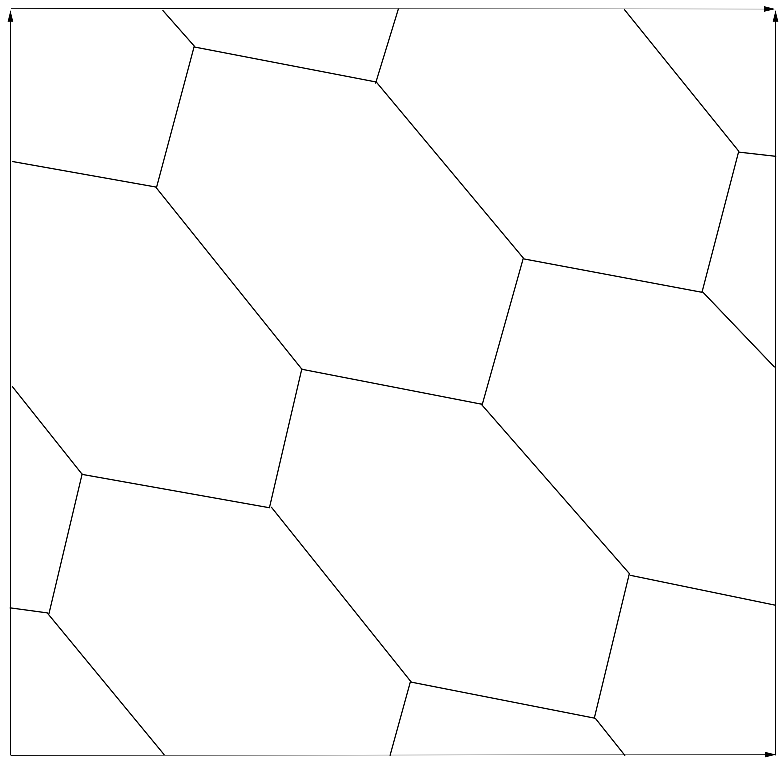
On Grinberg's Criterion





Gunnar Brinkmann and Carol T. Zamfirescu

Grinberg's Criterion (Grinberg, 1968)

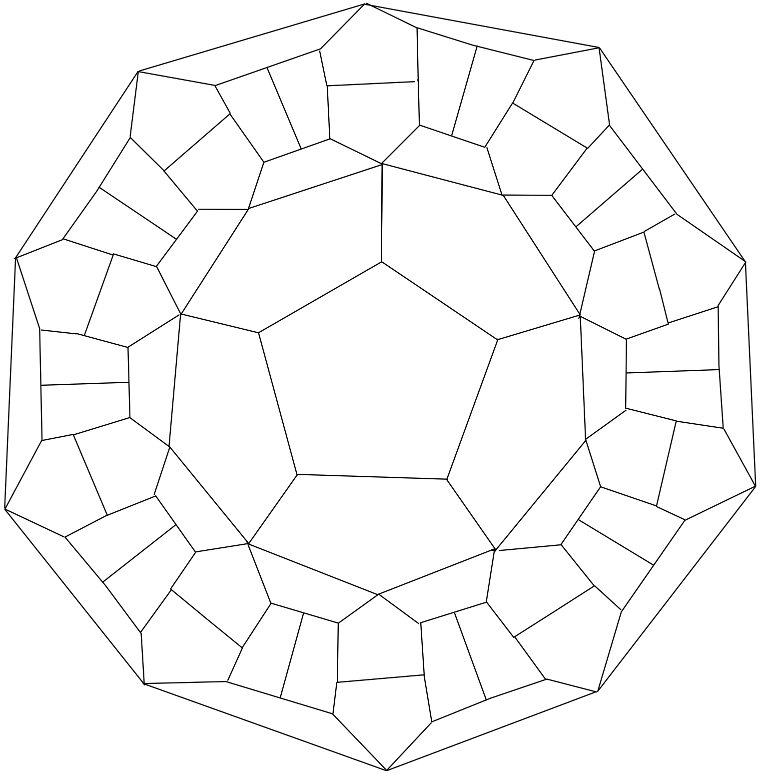
Given a plane graph with a hamiltonian cycle S and f_k (f'_k) faces of size k inside (outside) of S, we have

$$\sum_{k\geq 3}(k-2)(f'_k - f_k) = 0.$$

Or – with $s(f)$ the size of a face f:

$$\sum_{f \text{ inside } S}(s(f)-2) = \sum_{f \text{ outside } S}(s(f)-2).$$

This graph G is hypohamiltonian (Thomassen (1976)):



One 10-gon, all other faces pentagons.

Hamiltonicity of vertex-deleted subgraphs:
just give a Hamiltonian cycle!

Non-hamiltonicity of G:

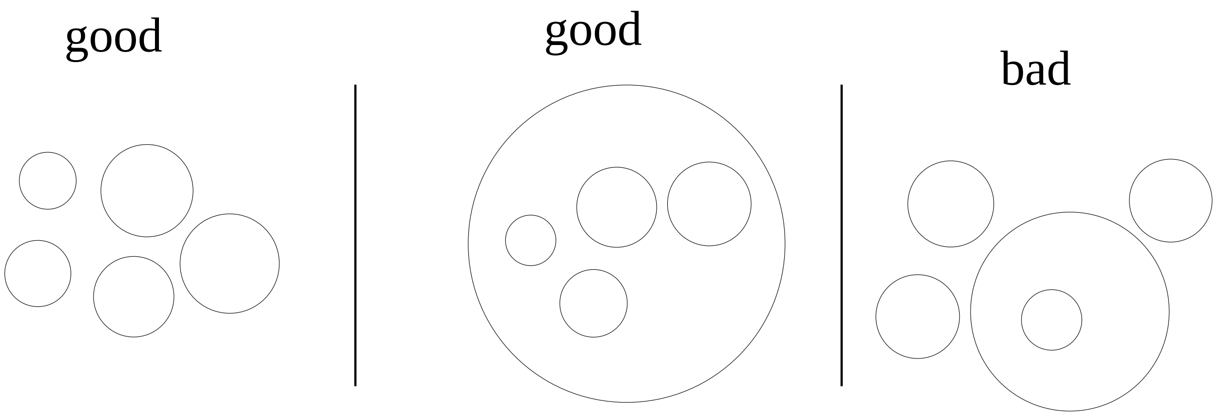
One 10-gon, all other faces pentagons, so

$$\sum_{f \text{ inside } S} (s(f)-2)(\text{mod} 3) \neq \sum_{f \text{ outside } S} (s(f)-2)(\text{mod} 3).$$

One side 0 – the other not.

Generalizations by Gehner (1976), Shimamoto (1978), and finally Zaks (1982):

Let $C_1, \ldots, C_n$ be disjoint cycles in a plane graph, so that "no cycle separates two others".



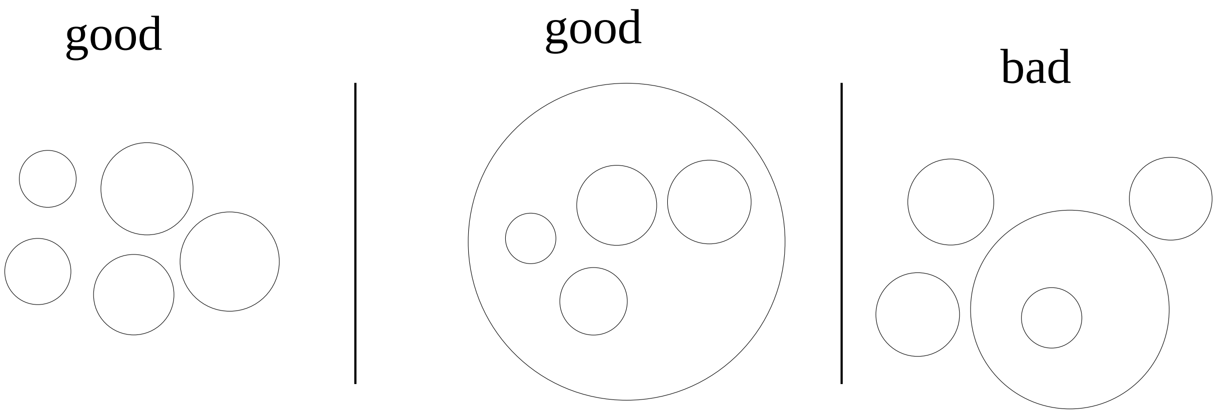
If v_i vertices are strictly inside the cycles and v_o vertices strictly outside, then

$$\sum_{k\geq 3}(k-2)(f'_k - f_k) = 4(n-1) + 2(v_o - v_i).$$

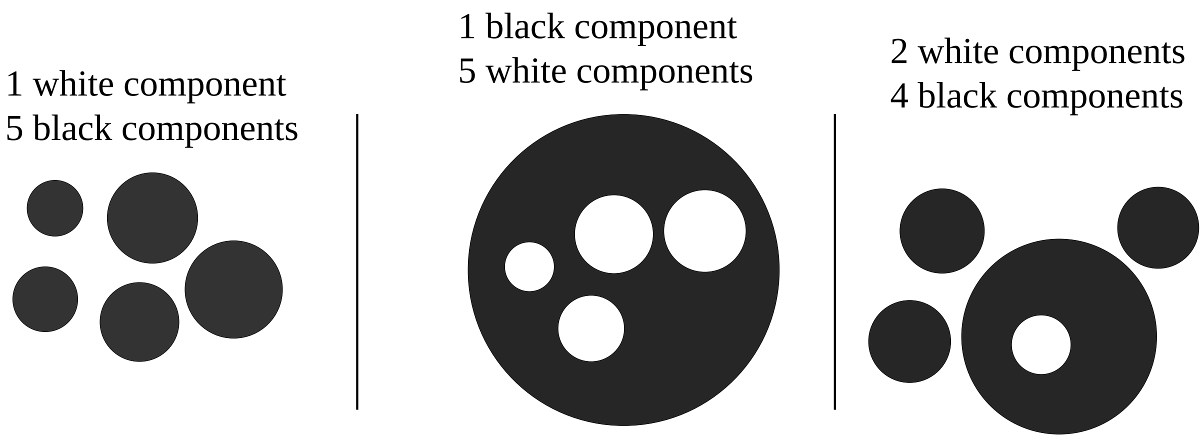
Or:

$$\sum_{f \text{ inside } S}(s(f)-2) - 2v_i + 4\cdot 1 = \sum_{f \text{ outside } S}(s(f)-2) - 2v_o + 4n.$$

Inside and outside are vague. . .



better talk about black and white:

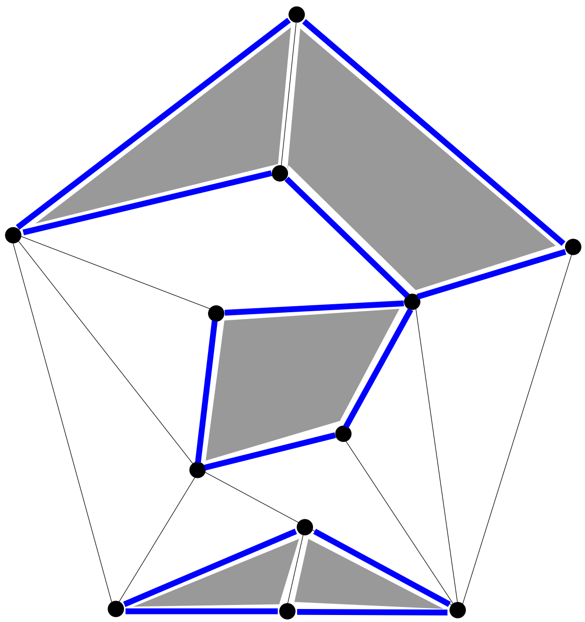


The **minimum requirement** to talk about an equality for two sets of faces is to be able to **distinguish** the two sets. . .

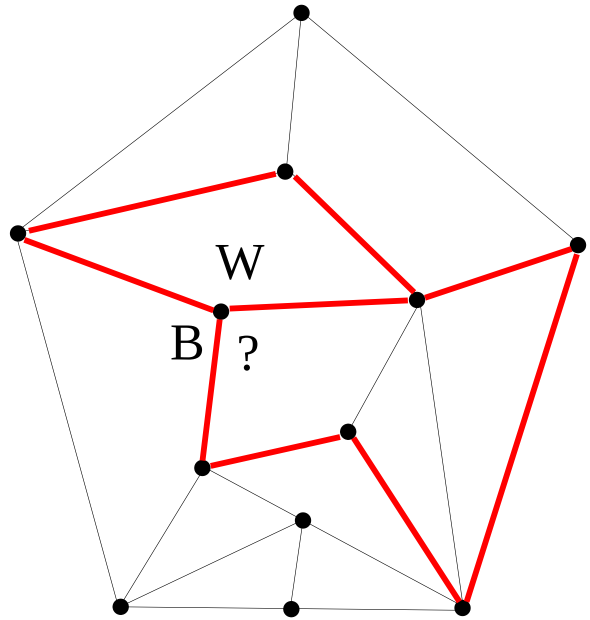
Partitioning subgraph S:

a subgraph of an embedded graph G that **allows to colour the faces black and white** so that the edges of S are exactly those between the black and the white faces.

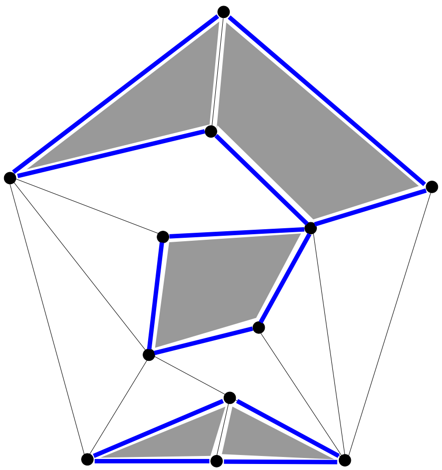
partitioning

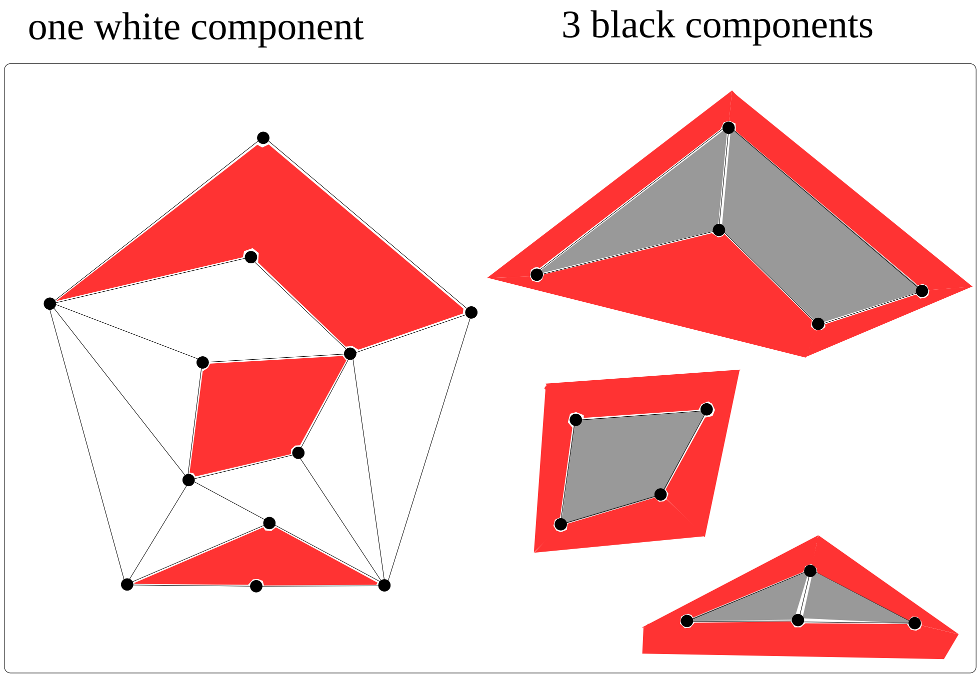


not partitioning



black/white component: induced by (b/w) faces sharing an edge





The white component has 3 faces that are originally no white faces (marked in red).
Some are originally no faces **at all**.

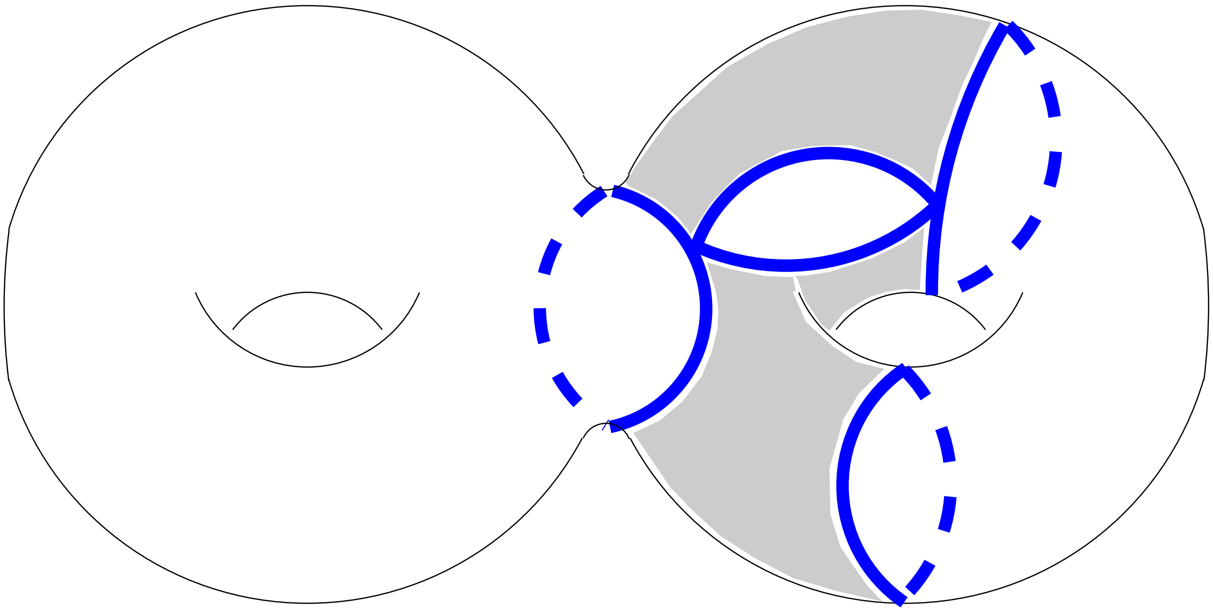
If S is a Hamiltonian cycle in a plane graph:

- one white and one black component
- both components are outerplanar graphs
- both components have one new (red) face: the outer face

1 black component with genus 0

2 white components with genus 0

1 white component with genus 1



Now apply the Euler formula to each component C:

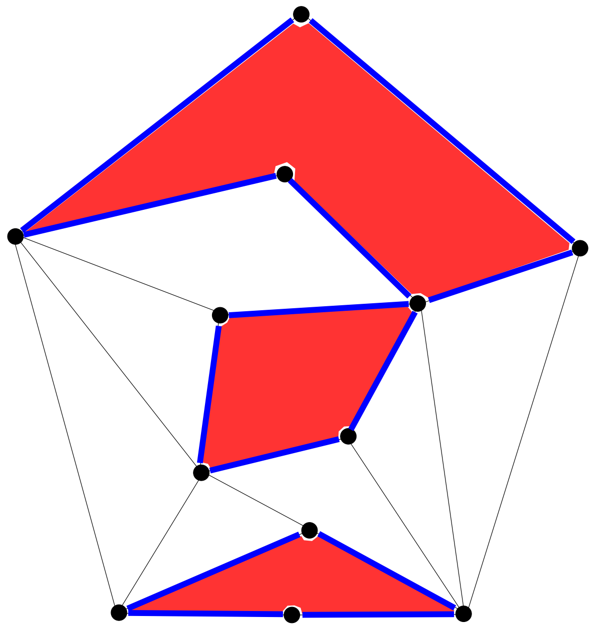
$$\underbrace{2 - 2\gamma(C) = |V_C| - |E_C| + |F_C|} = |V_C| - \frac{\sum_{f \in F_C}(s(f) - 2)}{2}$$

Introduce all kinds of parameters and determine the number of edges in $C \cap S$:

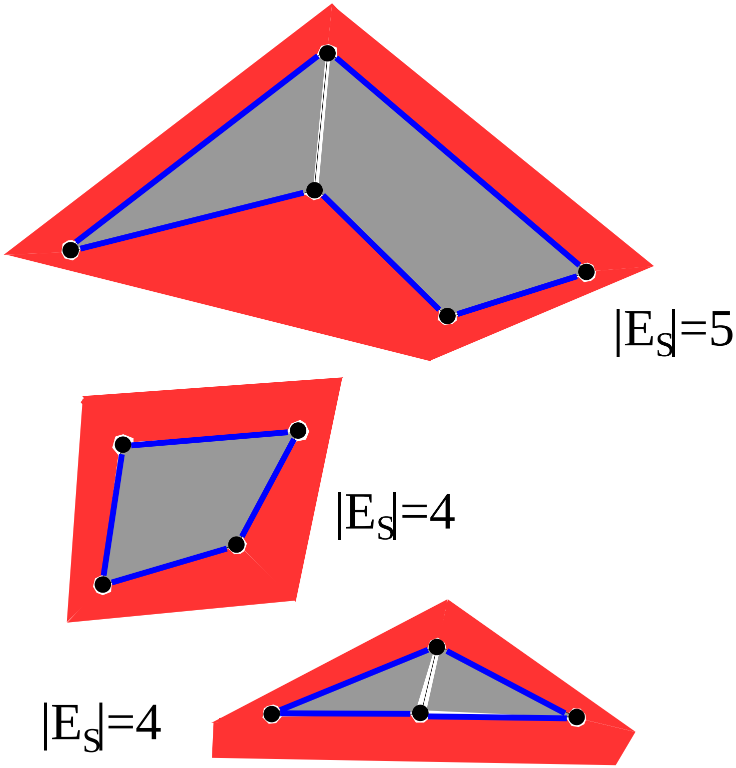
$$|E_{C,S}| = \sum_{f \in F_{C,i}} (s(f) - 2) - 2|V_{C,i}| + 4 - 4\gamma(C) - 2|B_{C,S}| + 2d_C$$

one white component

3 black components



$|E_S|=13$



Then sum up over all (e.g. black) components and get

$$|E_S| = \underbrace{\sum_{f \in F_b} (s(f) - 2)}_{\text{Grinberg}} \underbrace{-2|V_b| + 4|C_b| - 4 \sum_{C \in C_b} \gamma(C) - 2|B_b| + 2d_b}_{\text{correction term}}$$

V_b: set of black vertices not in S

C_b: set of black components

B_b: set of red faces in black components

d_b: sum over all black components C of $|E_C \cap E_S| - |V_C \cap V_S|$

Theorem:

$$\sum_{f \in F_b} (s(f)-2) - 2|V_b| + 4|C_b| - 4 \sum_{C \in C_b} \gamma(C) - 2|B_b| + 2d_b$$

$$= |E_S| =$$

$$\sum_{f \in F_w} (s(f)-2) - 2|V_w| + 4|C_w| - 4 \sum_{C \in C_w} \gamma(C) - 2|B_w| + 2d_w$$

This is ugly!

So best check when the correction terms

$$-2|V_b| + 4|C_b| - 4\sum_{C \in C_b} \gamma(C) - 2|B_b| + 2d_b$$

$$-2|V_w| + 4|C_w| - 4\sum_{C \in C_w} \gamma(C) - 2|B_w| + 2d_w$$

(almost) cancel out!

Corollary:

Let G be plane and let S be connected and spanning (and of course partitioning. . .). Then

$$\sum_{f\in F_b}(s(f)-2)+2|C_b| = \sum_{f\in F_w}(s(f)-2)+2|C_w|$$

C_b: set of black components

Corollary:

(Combinatorial generalization of Grinberg's theorem)

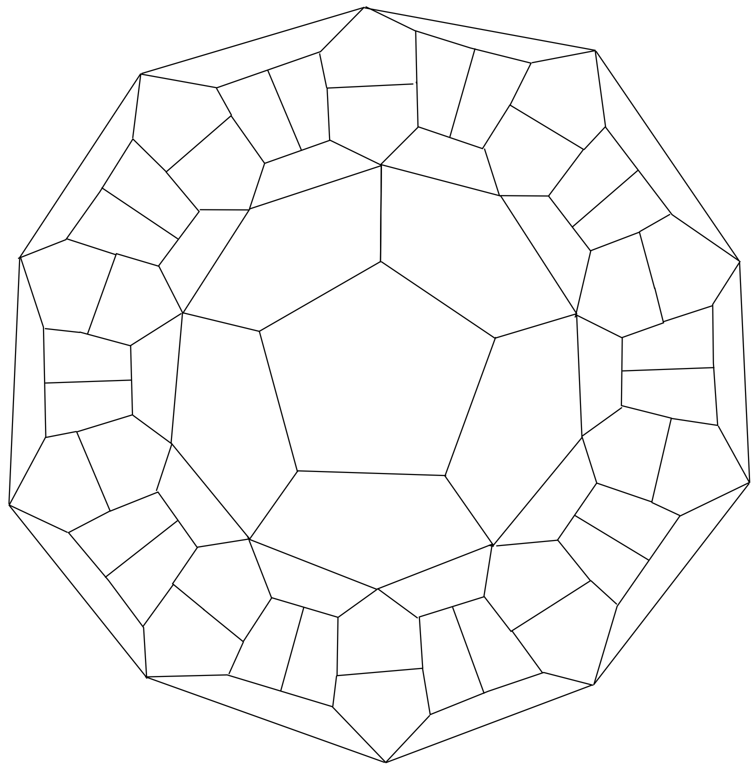
Let G be plane and let S be connected and spanning with $|C_b| = |C_w|$. Then Grinberg's original formula is valid:

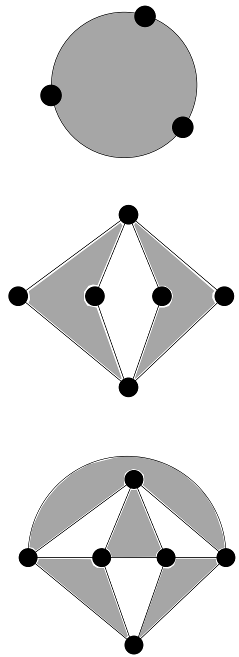
$$\sum_{f \in F_b} (s(f) - 2) = \sum_{f \in F_w} (s(f) - 2)$$

Grinberg's theorem is just the special case

$|C_b| = |C_w| = 1$

Example:





This graph has no spanning subgraph that is isomorphic to a cycle (Thomassen), but also not one isomorphic to a subdivided $K_{2,4}$ or a subdivided Octahedron...

We had for some plane graphs:

Grinberg's theorem is just the special case $|C_b| = |C_w| = 1$

Let's now fix $|C_b| = |C_w| = 1$

but allow higher genera.

Corollary:

Let G be an embedded graph of arbitrary genus and S be a partitioning 2-factor with $|C_b| = |C_w| = 1$. Then

$$\sum_{f \in F_b} (s(f) - 2) - 4\gamma(C_b) = \sum_{f \in F_w} (s(f) - 2) - 4\gamma(C_w)$$

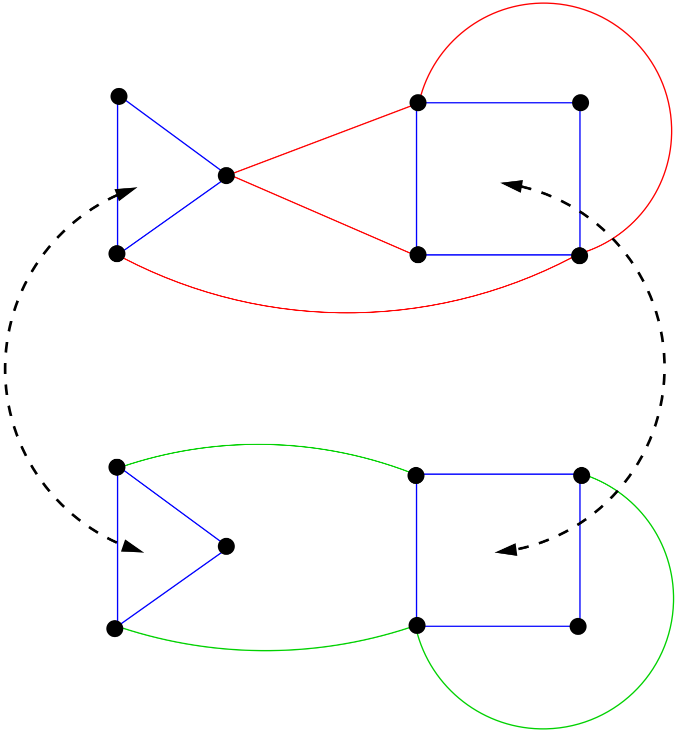
Planarizing 2-factor:

A partitioning 2-factor with $|C_b| = |C_w| = 1$ and $\gamma(C_b) = \gamma(C_w) = 0$.

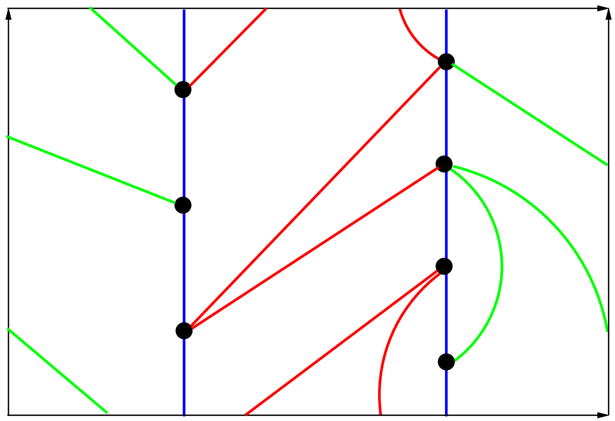
Informally: Obtained by identifying 2-factors consisting of faces of two plane graphs.

Hamiltonian cycle in plane graph: obtained by identifying the boundaries of two outerplanar graphs.

two plane graphs



1 toroidal graph



Corollary:

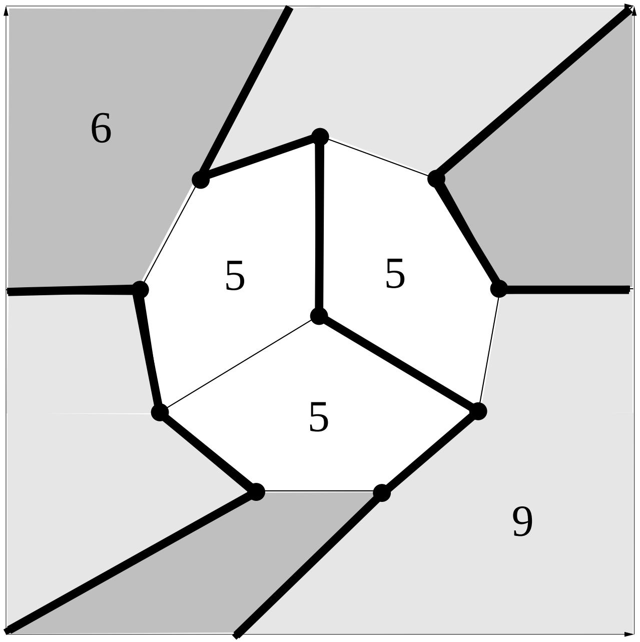
(Topological generalization of Grinberg's theorem)

Let G be an embedded graph of arbitrary genus and S be a planarizing 2-factor. Then

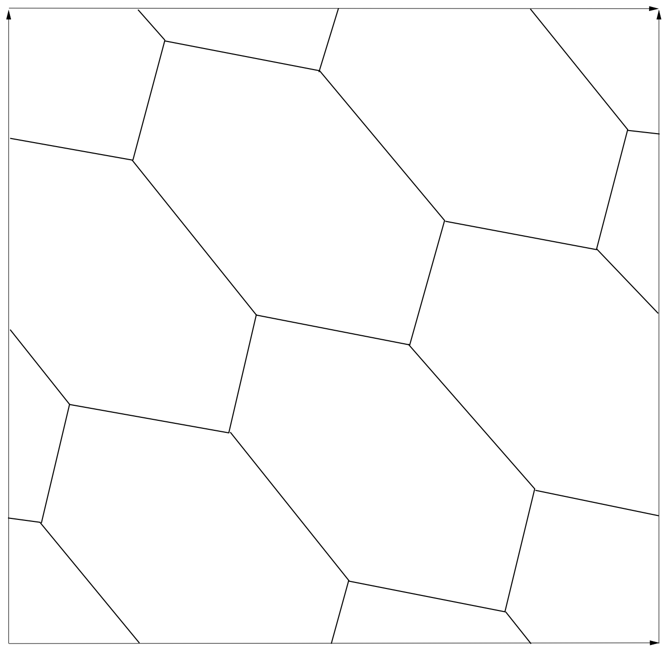
$$\sum_{f \in F_b} (s(f) - 2) = \sum_{f \in F_w} (s(f) - 2)$$

Grinberg's theorem is just the special case that $\gamma(G) = 0$.

Example applications:



3,3,3,4,7



4,4,4,4,4,4,4

- Find a planarizing 2-factor of the Petersen graph.
- The Heawood graph has no planarizing 2-factor.
- Any hamiltonian cycle in the toroidal embedding of the Heawood graph is not null-homotopic.

Further impact:

- An easy proof of a theorem of Lewis on the length of spanning walks.
- A generalization of a theorem by Bondy and Häggkvist on the decomposability of a graph into two hamiltonian cycles.

Conclusion

- We have proven a very general formula generalizing Grinberg's theorem.

- As a consequence even Grinberg's original formula in all its simplicity can be generalized to larger classes of graphs.

- Theorems entirely or at least essentially based on Grinberg's formula can be proven in a more general context.

Thanks!