

Generation of hypohamiltonian graphs

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Definition

A graph G is *hypohamiltonian* if G is non-hamiltonian and $G - v$ is hamiltonian for every $v \in V(G)$.

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- Introduced by Sousselier in 1963.
- Smallest hypohamiltonian graph: Petersen graph.

Construction of hypohamiltonian graphs

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 - Generation algorithm for hypohamiltonian graphs.
 - Determined all hypohamiltonian graphs up to 17 vertices.

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# graphs	1	0	0	1	0	1	4	0

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- Combined with other results:
 - There exists a hypohamiltonian graph of order n
 $\iff n \in \{10, 13, 15, 16\}$ or $n \geq 18$.

Construction of planar hypohamiltonian graphs

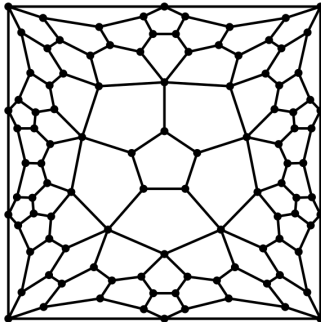
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- Grünbaum conjectured their nonexistence.
- Thomassen, 1976:
 - There are infinitely many planar hypohamiltonian graphs.



(Thomassen's smallest example has 105 vertices)

Search for the smallest planar hypohamiltonian graph:

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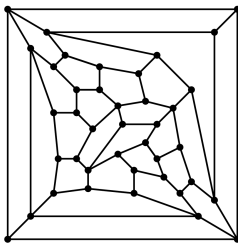
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(All by hand)

Smallest planar hypohamiltonian graph

Theorem (Jooyandeh, McKay, Östergård, Pettersson and Zamfirescu, 2016)

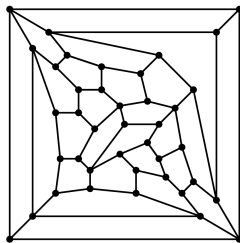
There exist at least 25 planar hypohamiltonian graphs on 40 vertices.



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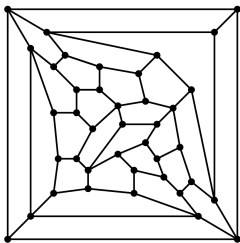


Lower bound smallest planar hypohamiltonian graph?

Smallest planar hypohamiltonian graph

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Lower bound smallest planar hypohamiltonian graph?

Theorem (Aldred, McKay, and Wormald, 1997)

The smallest planar hypohamiltonian graph has at least 18 vertices.

Our aim:

- Faster generation algorithm for hypohamiltonian graphs.
- Based on algorithm of Aldred, McKay, and Wormald (1997).
- Including support for:
 - Generation of hypohamiltonian graphs with a lower bound on the girth.
 - Generation of planar hypohamiltonian graphs.

Basic algorithm to generate all hypohamiltonian graphs with n vertices (simplified):

- Start from $C_{n-1} + v$
- Recursively add edges in all possible ways
 - Output hypohamiltonian graphs.

Properties of hypohamiltonian graphs

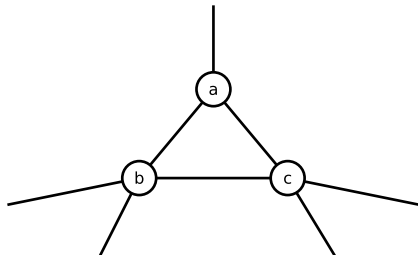
Lemma (Bondy, 1972)

*Let G be a hypohamiltonian graph containing a triangle T .
Then every vertex of T has degree at least 4.*

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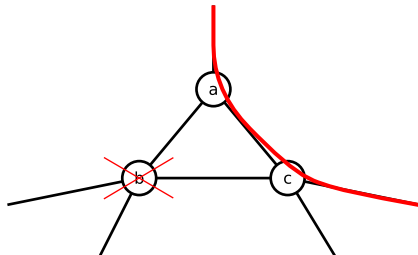
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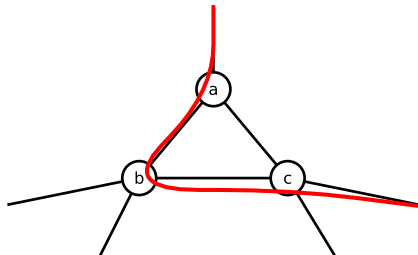
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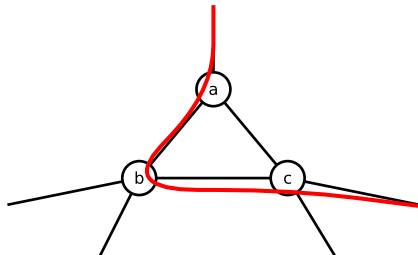
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- Other forbidden configurations...

Properties of hypohamiltonian graphs

Lemma (Aldred, McKay, and Wormald, 1997)

Given a hypocyclic graph G , for any partition (W, X) of the vertices of G with $|W| > 1$ and $|X| > 1$, we have that

$$p(G[W]) < |X| \quad \text{and} \quad k(G[W]) < |X|$$

Lemma (Aldred, McKay, and Wormald, 1997)

Let G be a hypocyclic graph, and consider a partition (W, X) of the vertices of G with $|W| > 1$ and $|X| > 1$ such that W is an independent set. Then we have $2n_2 + n_1 \geq 2|W|$ for every $v \in X$.

- This are called *type A, B and C obstructions*.

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- This are called *type A, B and C obstructions*.
- We refined these lemmas and added additional lemmas.

Basic algorithm to generate all hypohamiltonian graphs
with n vertices:

- Start from $C_{n-1} + v$
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Specialised construction algorithm

```
if Graph is non-hamiltonian AND not generated before then  
  if Graph contains triangle with vertex  $v$  of degree  $< 4$  then  
    Add a valid edge which increases the degree of  $v$   
    in all possible ways  
  else if Graph contains ... (i.e. try to apply other lemmas) then  
    Destroy ... in all possible ways  
  else  
    if Graph is hypohamiltonian then  
      Output graph  
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- Note: our algorithm never removes any edges.
Useful for planarity / girth !

Results – hypohamiltonian graphs

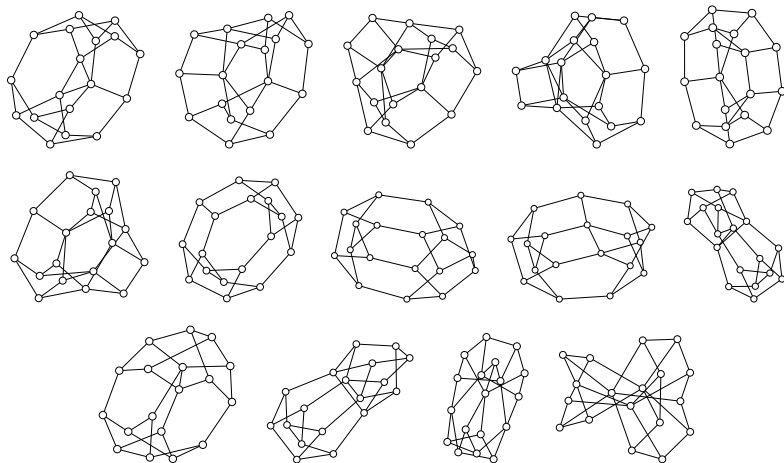
Order	# hypoham.	$g = 3$	$g = 4$	$g = 5$	$g = 6$	$g = 7$	$g \geq 8$
0 – 9	0	0	0	0	0	0	0
10	1	0	0	1	0	0	0
11	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0
13	1	0	0	1	0	0	0
14	0	0	0	0	0	0	0
15	1	0	0	1	0	0	0
16	4	0	0	4	0	0	0
17	0	0	0	0	0	0	0
18	14	1	5	8	0	0	0
19	34	0	0	34	0	0	0
20	?	?	?	4	0	0	0
21	?	?	?	85	0	0	0
22	?	?	?	420	0	0	0
23	?	?	?	?	0	0	0
24	?	?	?	?	0	0	0
25	?	?	?	?	1	0	0
26	?	?	?	?	0	0	0
27	?	?	?	?	?	0	0
28	?	?	?	?	≥ 1	1	0
29	?	?	?	?	?	0	0
30	?	?	?	?	?	0	0
31 – 35	?	?	?	?	?	?	0

Results – running times

Order	# hypoham.	Time (s)	Increase
16	4	9	
17	0	189	21.00
18	14	18 339	97.03
19	34		≈ 400

Table: Counts and generation times for hypohamiltonian graphs.

Results – hypohamiltonian graphs on 18 vertices

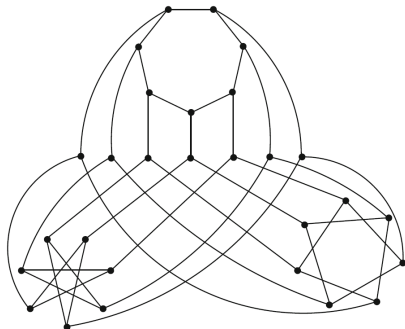
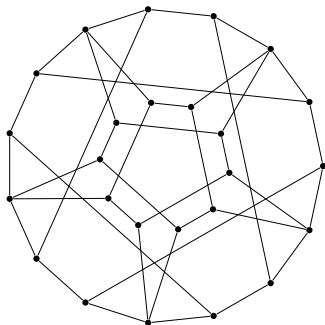


[Can be downloaded from the House of Graphs (<http://hog.grinvin.org>)]

Results – hypohamiltonian graphs

Theorem (JG and Zamfirescu)

The smallest hypohamiltonian graph of girth 6 (girth 7) has order 25 (order 28).



Results – planar hypohamiltonian graphs

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*The smallest **planar** hypohamiltonian graph has at least 23 vertices, and the smallest planar hypohamiltonian graph of girth 4 has at least 25 vertices.*

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Combined with upper bounds:

Girth 3: $23 \leq n \leq 216$ (Zamfirescu)

Girth 4: $25 \leq n \leq 40$ (Jooyandeh et al.)

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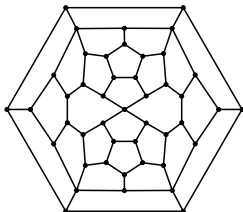
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Girth 5: $n = 45$ (Jooyandeh et al.)



Results – cubic planar hypohamiltonian graphs

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(Infinite family with $94 + 4k$ vertices.)*

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- All of these known graphs have girth 4...

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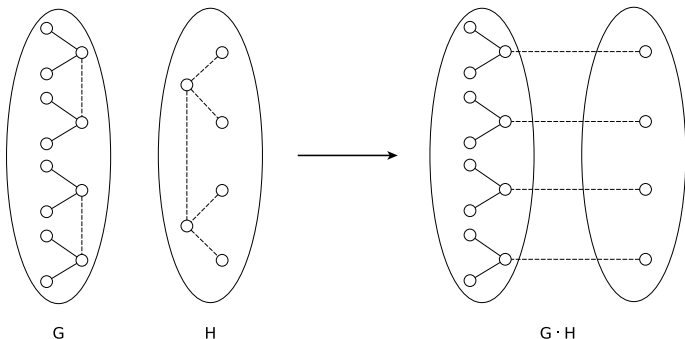
Question (McKay, 2016)

Are there infinitely many cubic planar hypohamiltonian graphs of girth 5?

Results – cubic planar hypohamiltonian graphs

Theorem (JG and Zamfirescu)

Let G be a cubic non-hamiltonian graph and let H be a cubic hypohamiltonian graph. If G contains a pair of **good edges**, then the **dot product** $G \cdot H$ applied to these good edges is a hypohamiltonian graph.



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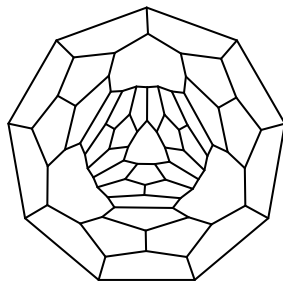
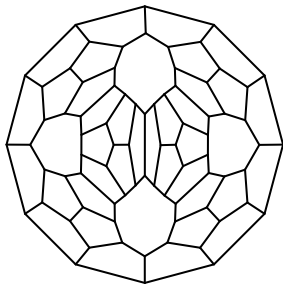
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- The smallest cubic planar hypohamiltonian graph has at least 54 vertices. (Recall: upper bound 70 vertices).

Results – cubic planar hypohamiltonian graphs

By using the program *plantri* we were able to show that:

- The smallest cubic planar hypohamiltonian graph has at least 54 vertices. (Recall: upper bound 70 vertices).
- There is exactly one cubic planar hypohamiltonian graph of order 78 and girth 5. It has D_{3h} symmetry.



Thanks for your attention!