Searching Uniquely Hamiltonian Planar Graphs with Minimum Degree 3

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Motivation

Definition (Uniquely Hamiltonian Graph (UHG))

If a graph contains exactly one hamiltonian cycle it is called a *uniquely hamiltonian graph* (UHG).

Theorem (Fleischner 2014)

There exists an infinite family of uniquely hamiltonian simple graphs with minimum degree 4.

Motivation - Planar Graphs

Theorem (Bondy and Jackson 1998)

Every planar uniquely hamiltonian graph has at least two vertices of degree two or three.

Conjecture by Bondy and Jackson

Every planar uniquely hamiltonian graph has at least two vertices of degree two.

Goal

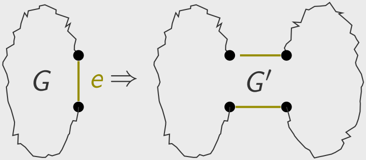
Search for a simple planar uniquely hamiltonian graph with minimum degree 3, which would disprove the conjecture of Bondy and Jackson.

Transformation - Fixed Edge Cycle

Definition (FE-cycle)

A *FE-cycle* is a pair (e, C) where C is a cycle and e an edge in C. A FE-cycle (e, C) is called *unique* if there is no other cycle C' with $V(C') = V(C)$ that also contains the edge e.

Fixing one Edge



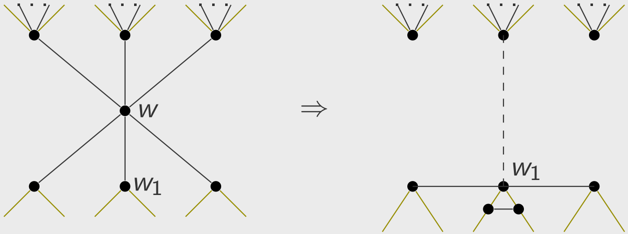
New Goal

Search for a simple planar graph with minimum degree 3 that does contain a unique hamiltonian FE-cycle.

Definition (Dominating Maximal FE-cycle)

A FE-cycle (e, C) is called *maximal* if there is no other cycle C' with $V(C') \supseteq V(C)$ that also contains the edge e.
A FE-cycle (e, C) is called *dominating* if C is edge dominating.

Removing unvisited vertices

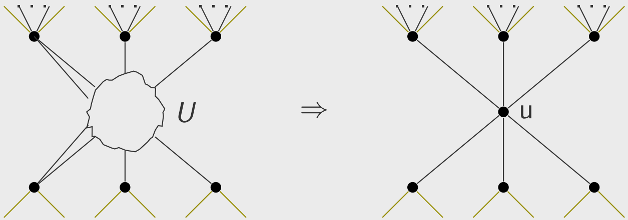


New Goal

Search for a simple planar graph with minimum degree 3 that does contain a uniquely dominating maximal FE-cycle.

Let G be 2-connected.

Transforming non-dominating cycles into dominating cycles



Observation: If G is 2-connected, the new vertex u has degree at least 2 and in this case we can still apply the previous transformation.

New Goal

Find a simple planar 2-connected graph with minimum degree 3 which contains a uniquely maximal ~~dominating~~ FE-cycle.

1. Generate (all) simple planar 2-connected graphs (with a fixed number of vertices) with minimum degree 3.
2. Check for each graph if it contains a uniquely (dominating) maximal FE-cycle

Problem (Uniquely Dominating Maximal FE-cycle (UDFEC))

Given a simple planar 2-connected graph. Does it contain a uniquely dominating maximal FE-cycle?

Problem (Uniquely Maximal FE-cycle (UMFEC))

Given a simple planar 2-connected graph. Does it contain a uniquely maximal FE-cycle?

Simple Approach

For each edge e in G repeat the following steps until no new maximal FE-cycle with e as the fixed edge could be found:

1. Find a new maximal FE-cycle with e as the fixed edge.
2. Check if the FE-cycle is unique.

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ILP model for Finding a Maximal FE-cycle

Input

A graph $G = (V, E)$, an edge $e_0 = i_0 j_0 \in E$ and a set $\mathcal{C}$ of all maximal FE-cycles with e as the fixed edge found until now.

Variables

- $(x_v)_{v \in V}$ $\ldots x_v = 1$ iff $v \in V$ is used in the cycle
- $(y_e)_{e \in E}$ $\ldots y_e = 1$ iff $e \in E$ is used in the cycle

ILP model for Finding a Maximal FE-cycle cont.

Objective:

$$\max \sum_{i \in V} x_i$$

Constraints:

$$\sum_{j \in N(i)} y_{ij} = 2x_i \quad \forall i \in V \tag{1}$$

$$y_{i_0 j_0} = 1 \tag{2}$$

$$\sum_{i \in V \setminus C} x_i \geq 1 \quad \forall C \in \mathcal{C} \tag{3}$$

$$\sum_{e \in \delta(V')} y_e \geq 2x_i \quad \forall \emptyset \neq V' \subseteq V \setminus \{i_0\}, i \in V' \tag{4}$$

$$y_e \in \{0,1\} \quad \forall e \in E \tag{5}$$

$$x_i \in \{0,1\} \quad \forall i \in V \tag{6}$$

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ILP model for Checking Uniqueness of FE-cycle

Input

A graph $G = (V, E)$ and a maximal FE-cycle (e, C).

Variables

- $(y_e)_{e \in E_C}$ ... $y_e = 1$ iff $e \in E$ is used in the cycle

ILP model for Checking Uniqueness of FE-cycle cont.

No objective (only feasibility interesting)

Constraints:

$$\sum_{j \in N(i)_C} y_{ij} = 2 \quad \forall i \in V_C \tag{7}$$

$$y_{i_1 i_2} = 1 \tag{8}$$

$$\sum_{e \in \delta(V')} y_e \geq 2 \quad \forall \emptyset \neq V' \subseteq V_C \setminus \{i_1\}, k \in V' \tag{9}$$

$$\sum_{ij \in E_C \setminus E(C)} y_{ij} \geq 2 \tag{10}$$

$$y_e \in \{0, 1\} \quad \forall e \in E_C \tag{11}$$

Improvement - Reducing the number of ILPs to solve

Split into two phases

1. Store for each set of vertices V' and for each edge e a list of all cycles found until now using the edge e and the vertices V'. Search maximal (dominating) cycles with a not yet found vertex set / edge combination.
2. Check uniqueness of all found cycles. We do not have to check FE-cycles for which there are already two cycles containing the fixed edge.

Improvement - Reusing ILP-states

- Idea: Reuse ILP-tree after maximal cycle got found.
- Use callback to store every found cycle in $\mathcal{C}$ and add the constraint (3) for every cycle.
  - The found cycles don't have to be maximal!
  - The constraint (3) ensures that afterwards only larger cycles or not comparable cycles get found
  - If a larger cycle gets found remove all smaller cycles from the datastructure $\mathcal{C}$ and the according constraints from the model, since they get dominated from the new constraint.
- If no new cycle got found, all cycles in the datastructure are maximal and no other maximal cycle exists
- The ILP terminates as infeasible, since all work happens in the collection of the cycles during the callback.

Minimal Counter Example

Goal

Find properties for a minimal planar 2-connected graph with minimum degree 3 that contains a unique maximal FE-cycle. Reduce the number of graphs to test drastically by only testing candidates for a minimal counter example.

By minimal we mean minimal according to the following relation.

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then we say $G_1 \leq G_2$ iff

$$|V_1| < |V_2| \vee (|V_1| = |V_2| \wedge |E_1| \leq |E_2|) .$$

Definition

A vertex with degree 3 or less is called a *small* vertex and otherwise a *large* vertex.

Let $G = (V, E)$ be a minimal counter example with the unique FE-cycle (e, C):

- C is dominating
- G is 3-connected
- Every neighbor of a large vertex is in $V(C)$
- Every arc between large vertices is in $E(C)$
  - No large vertex has 3 or more large neighbors
  - There is no cycle consisting only of large vertices in G
- $|E| \leq |V| + n_3 - \delta$ where $n_3 = |\{v \in V : \deg(v) = 3\}|$ and δ is the number of small vertices incident to e
- G does not contain any triangles
- $|E| \leq 2|V| - 4$

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Construction of Candidate Graphs

- We use plantri to construct planar graphs
- We fix a number of vertices and give an upper bound for the number of edges ($|E| \leq 2n - 4$). Then we filter the results by the other properties of a minimal counter example.
- Disadvantages:
  - The upper bound for the edges is only a filter and therefore not efficient
  - All filters together filter out most of the generated graphs, only a small part is really interesting (especially the property that the graph has no triangles)

Construction of Candidate Graphs through Dual Graphs

New idea: Generate dual graphs with plantri

- Use the edge upper bound to get an upper bound for the faces:

$$|F| = |E| - |V| + 2 \leq 2|V| - 4 - |V| + 2 \leq |V| - 2$$

- The dual graph of a 3-connected graph is also 3-connected
- The dual graph has minimum degree 4 since the original graph contained no triangles

To get all relevant graphs with at most n vertices we construct all dual graphs with the above properties with at most $n - 2$ vertices and build the dual graphs of them.

Computational Results $|V| \leq 22$

#Faces	#Tested Graphs	#Found UMFE-cycles	Runtime (s)
6	1	0	< 1
7	1	0	< 1
8	4	0	< 1
9	14	0	39
10	66	0	1
11	427	0	256
12	3483	0	1471
13	31 253	0	9243
14	280 242	0	81 385
15	1 762 507	0	264 307
16	5 350 843	0	542 157
17	5 776 730	0	215 990
18	919 394	0	18 068
19	10 565	0	39 590
20	7	0	201 645

There does not exist any planar graph with at most 22 vertices with minimum degree three that has a uniquely maximal FE-cycle