A spanning Bipartite Quadrangulation of a Triangulation

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Joint work with

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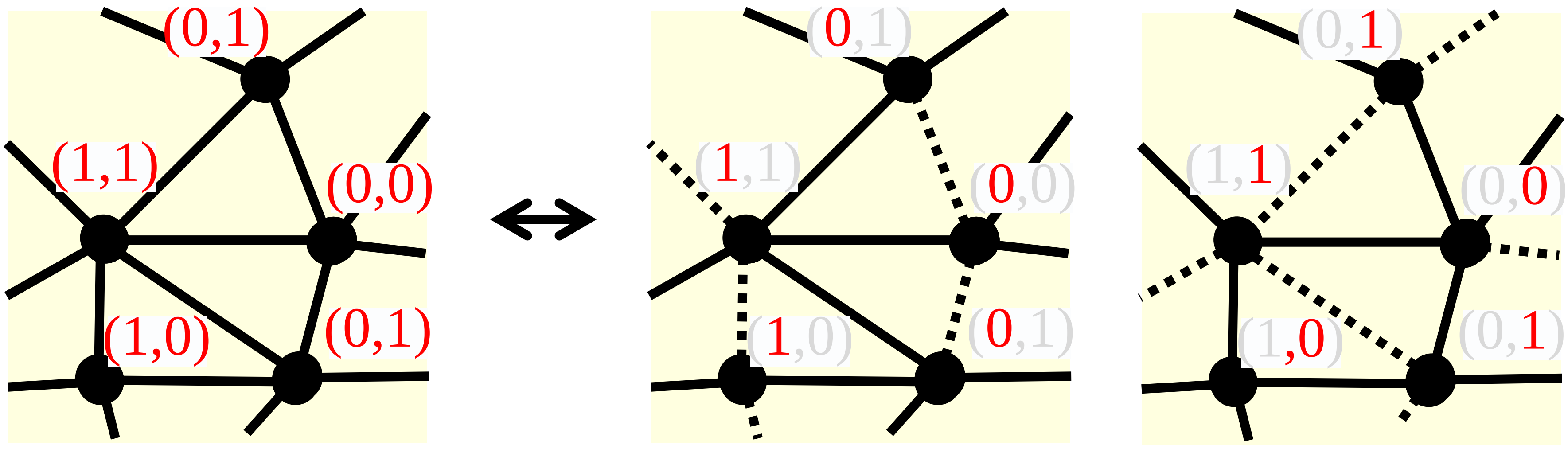
Spanning bipartite quadrangulation

✓ (folklore) G : triangulation (of any surface)

$\exists$ 4-coloring in G

$\Leftrightarrow$ $\exists$ 2 spanning bipartite quadrangulations covering $E(G)$



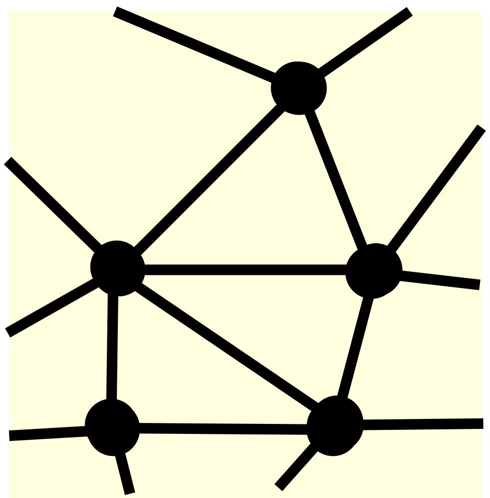
Find a sp. bip. quad. in triangulations

Spanning bipartite quadrangulation

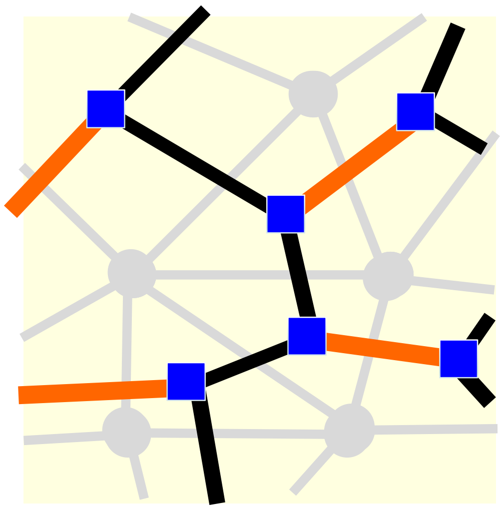
Prop.

Bipartite or non-bipartite?

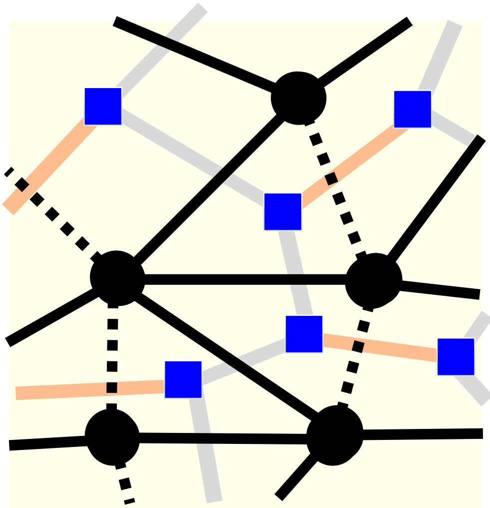
G : triangulation of a surface $\Rightarrow$ $\exists$ a spanning quadragulation



G : triangulation



The dual G^* has a perfect matching



Any PM gives a sp. quad. of G

Spanning bipartite quadrangulation

Prop.

G : triangulation of a surface $\Rightarrow$ $\exists$ a spanning quadragulation

Bipartite or non-bipartite?

Note: Any quadrangulation of the plane is bipartite.

Cor.

G : triangulation of the plane

$\Rightarrow$ $\exists$ a spanning bipartite quadragulation

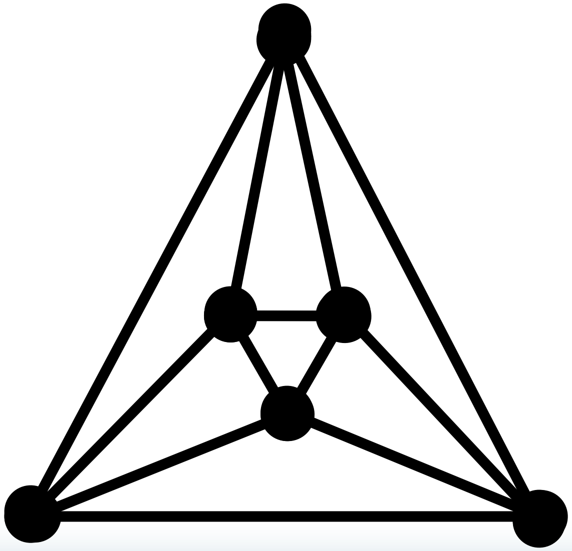
What about the case of non-spherical surfaces?

Spanning bipartite quadrangulation

The general cases seem difficult.

→ Our target : Eulerian triangulation

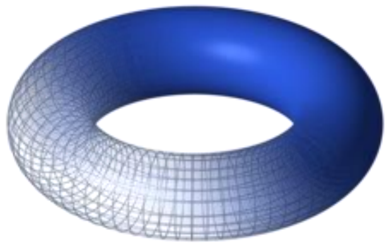
($\forall$vertex has even degree)

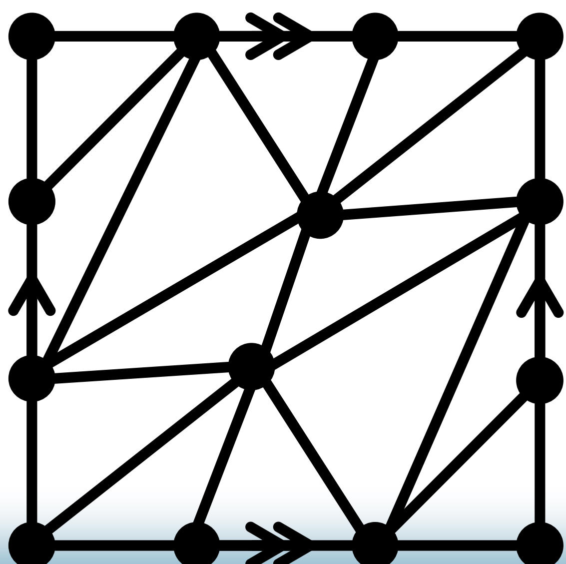


Not all (Eulerian) triangulations have a sp. bip. quadrangulation

The toroidal case

✓ K_7 on the torus has NO sp. bip. quadrangulation





$\because$ K_7 on the torus has 7 vertices, 21 edges, and 14 faces

To obtain a sp. bip. quad., we delete exactly 14/2 = 7 edges

But, $\nexists$ bip. graph on 7 vertices and 21 – 7 = 14 edges. $\square$

The toroidal case

✓ K_7 on the torus has NO sp. bip. quadrangulation

Main Thm.

G : Eulerian triangulation of the torus

$\exists$ a sp. bip. quadrangulation in G

$\Leftrightarrow$ G does NOT have K_7 as a subgraph

✓ Kundgen & Thomassen (`17) gave a weaker sufficient condition

✓ Later, I will show an idea of the proof.

The existence of sp. bip. quad.

✓ Eulerian triangulation

Plane	Torus	
○	○ $\Leftrightarrow \nexists K_7$	

The projective planar case

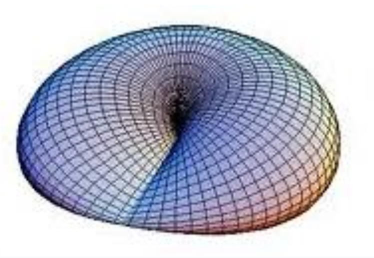
Main Thm. 2

G : Eulerian triangulation of the projective plane

$\Rightarrow$ $\exists$ a sp. bip. quadrangulation in G

Furthermore, if G : 3-colorable,

$\Rightarrow$ ALL sp. quadrangulations in G are bipartite



✓ Kundgen & Thomassen (`17) proved the same,

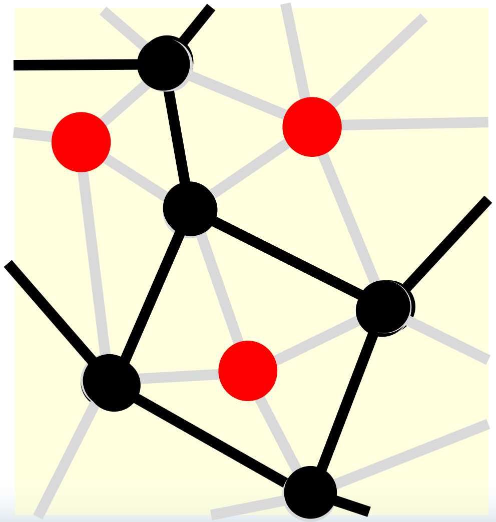
but our proof is shorter

The projective planar case

Main Thm. 2

G : Eulerian triangulation of the projective plane

$\Rightarrow \exists$ a sp. bip. quadrangulation in G



$\because$ (Mohar `02)

$\forall$ Eulerian triangulation of the projective plane is the face subdivision of an even embedding

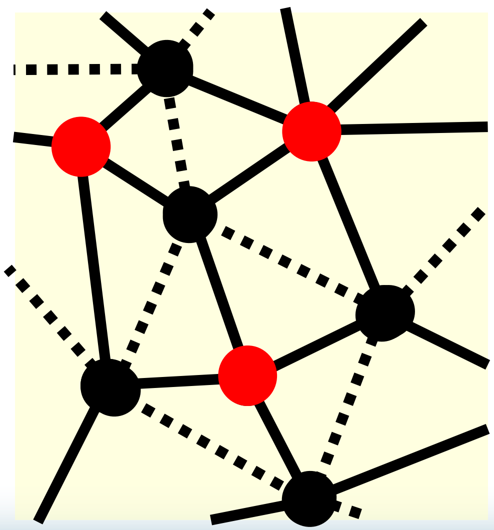
$\forall$ facial cycle is even length

The projective planar case

Main Thm. 2

G : Eulerian triangulation of the projective plane

$\Rightarrow \exists$ a sp. bip. quadrangulation in G



$\because$ (Mohar `02)

$\forall$ Eulerian triangulation of the projective plane

is the face subdivision of an even embedding

$\forall$ facial cycle is even length

Delete all edges in the even embedding $\square$

The projective planar case

Main Thm. 2

G : Eulerian triangulation of the projective plane

If G : 3-colorable,

$\Rightarrow$ ALL sp. quadrangulations in G are bipartite

$\because$ (Youngs `96)

$\forall$ quadrangulation of the projective plane is

either bipartite or non-3-colorable (3-chromatic is impossible)

If G : 3-colorable, then all sp. quad.s are 3-colorable, so bipartite $\square$

The projective planar case

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Furthermore, if G : 3-colorable,

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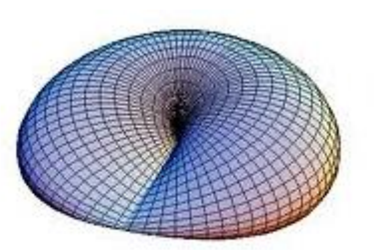


✓ Kundgen & Thomassen (`17) proved the same, but our proof is shorter

The existence of sp. bip. quad.

✓ Eulerian triangulation

Plane	Torus	Projective plane	
○	○ $\Leftrightarrow \nexists\, K_7$	○	

The case of other surfaces

Main Thm. 3

G : Eulerian triangulation of non-spherical surface

If edge-width of G is large enough,

$\Rightarrow \exists$ a sp. bip. quadrangulation in G

- ✓ Edge-width : the length of shortest essential cycle
- ✓ Shown by using the following result;

(Hutchinson, Richter, and Seymour `02)

(Archdeacon, Hutchinson, Nakamoto, Negami, and Ota `99)

$\forall$ Eulerian triangulation G with large edge-width is 4-colorable, unless G is the face subdivision of an even embedding

The existence of sp. bip. quad.

✓ Eulerian triangulation

Plane	Torus	Projective plane	Others
○	○ $\Leftrightarrow \nexists K_7$	○	○ if edge-width large

✓ General triangulation

Only little is known: Plane ○

``Dense'' triangulations ✗

e.g. complete graph

The toroidal case

Main Thm.

G : Eulerian triangulation of the torus

$\exists$ a sp. bip. quadrangulation in G

$\Leftrightarrow$ G does NOT have K_7 as a subgraph

- ✓ $\Leftarrow$ is an easy part, while we need some arguments
- ✓ $\Rightarrow$ is the main part

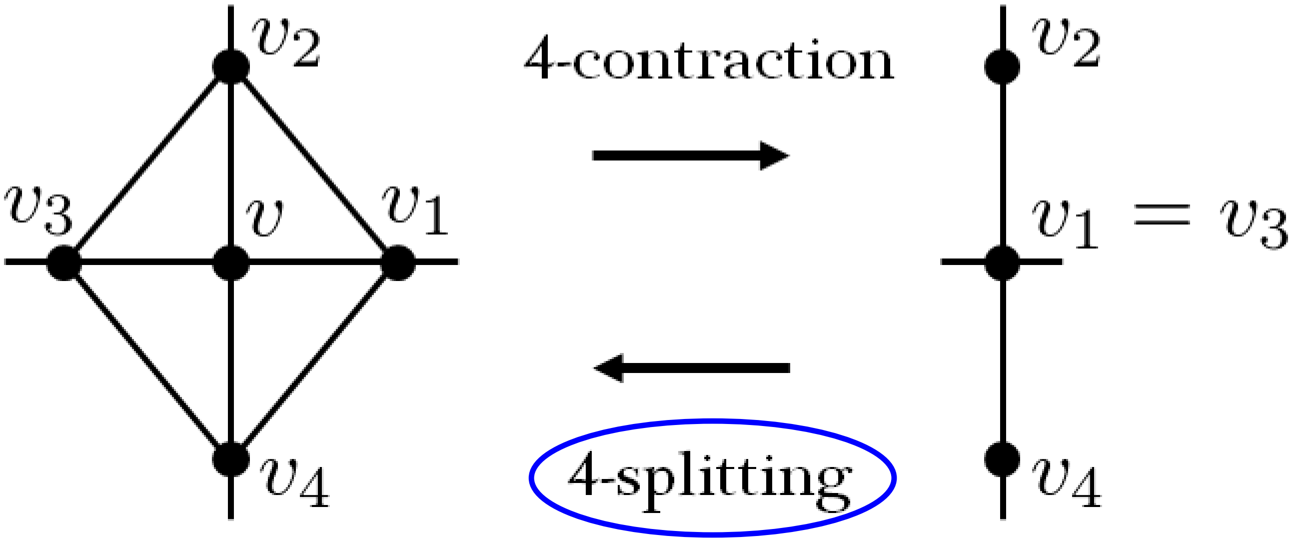
The toroidal case

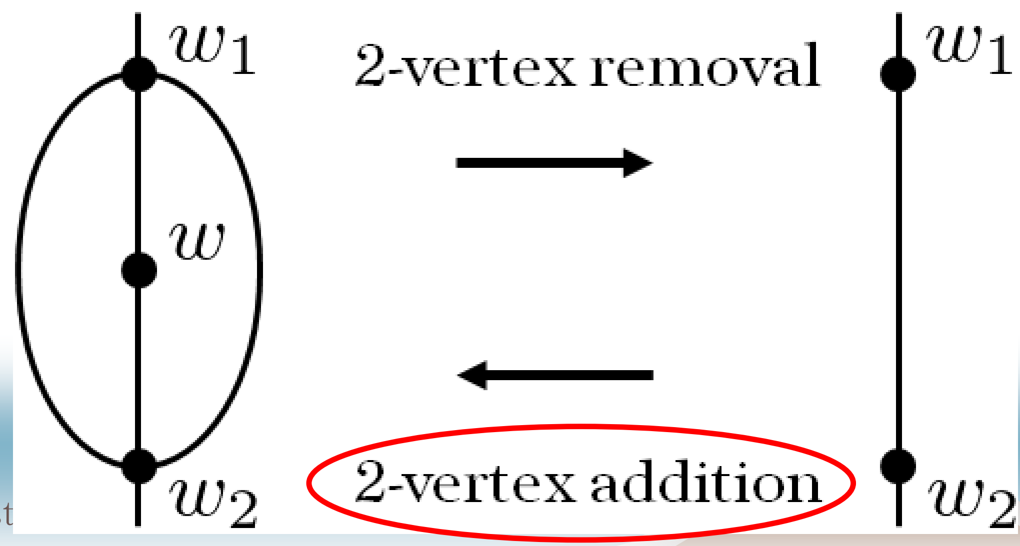
- ✓ Use generating thm., allowing multiple edges

Thm. (Matsumoto, Nakamoto, and Yamaguchi, `18)

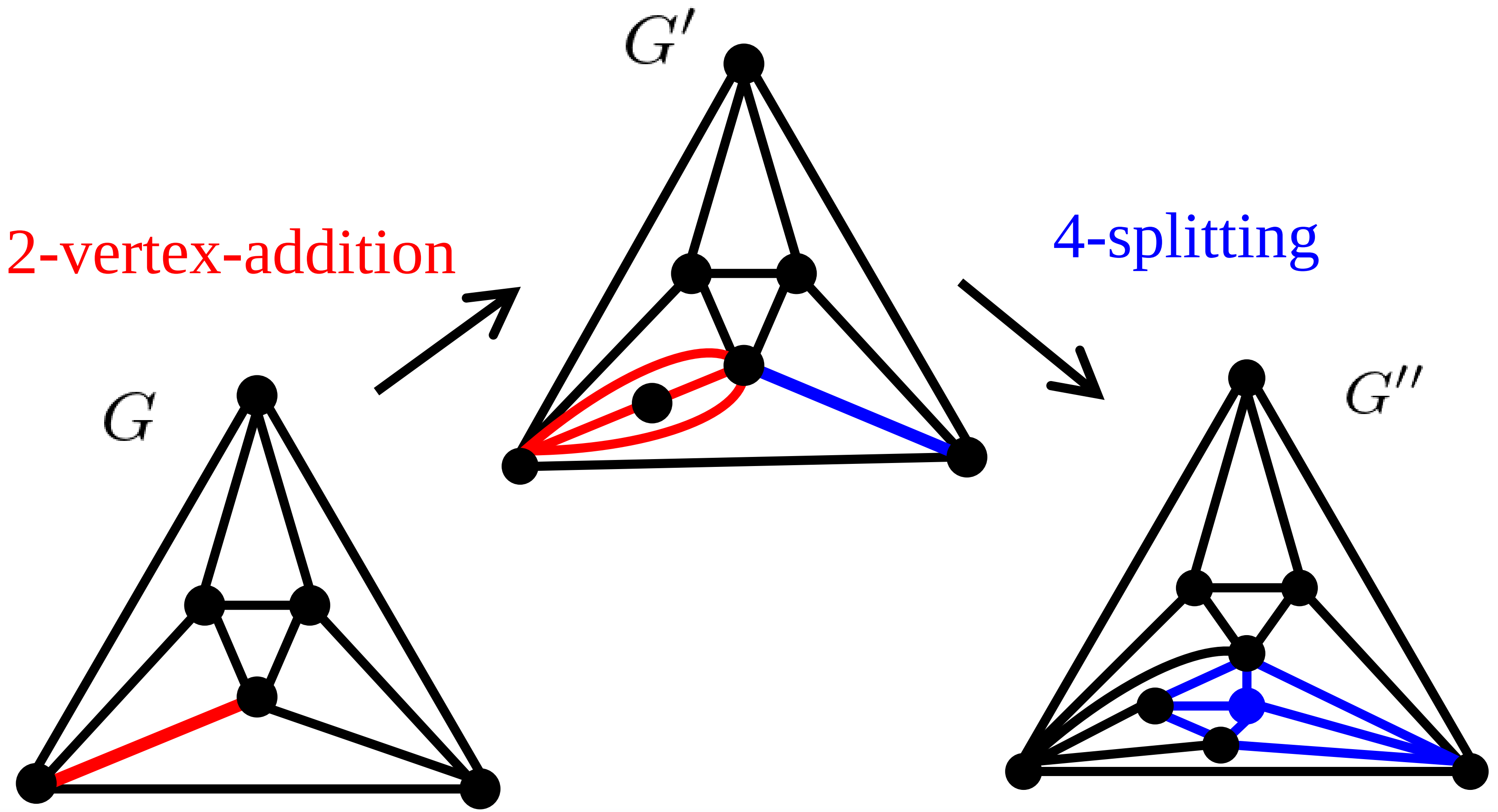
$\forall$ Eulerian multi-triangulation of the torus
is generated from 27 base graphs or 6-regular ones
by a sequence of 4-splittings and 2-vertex additions

4-splittings and 2-vertex-addition

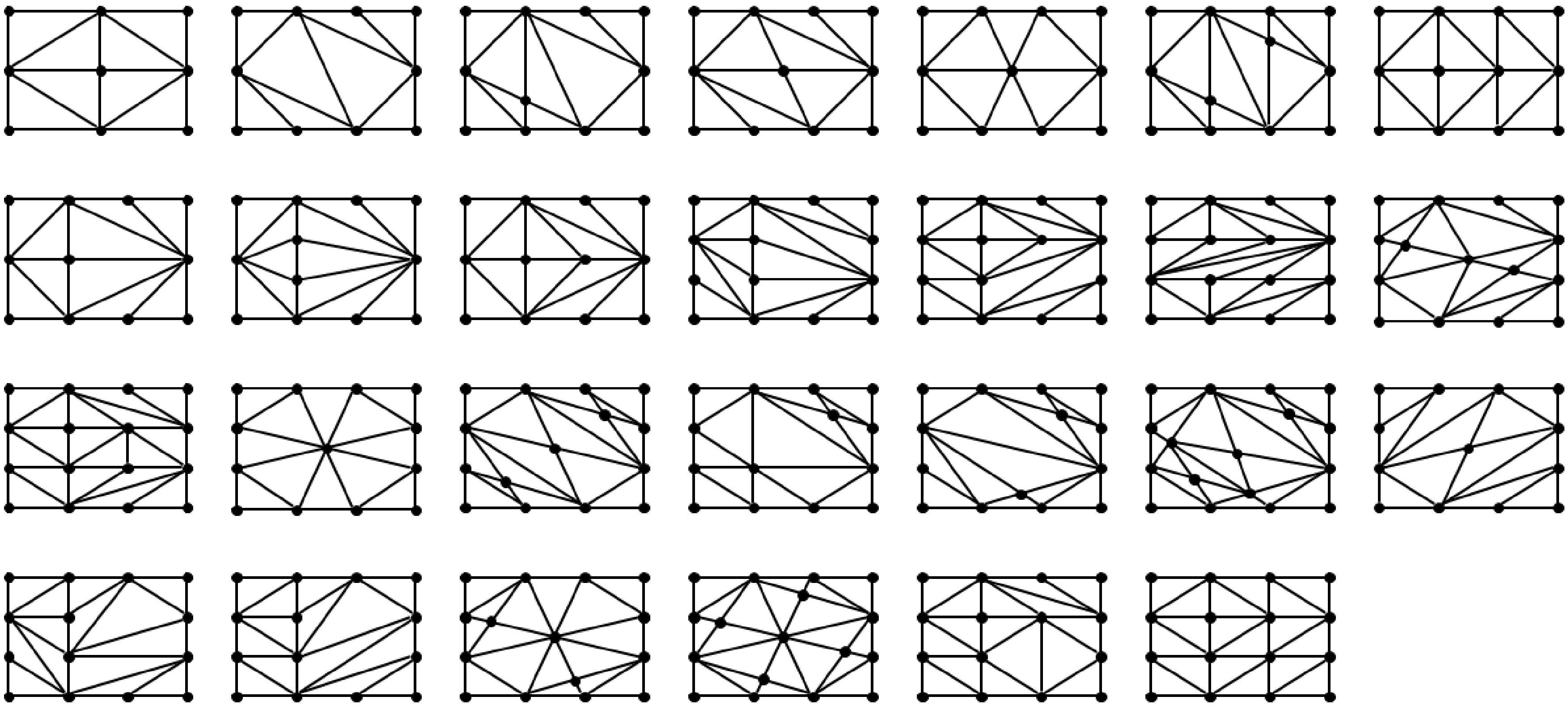




4-splittings and 2-vertex-addition



27 base graphs

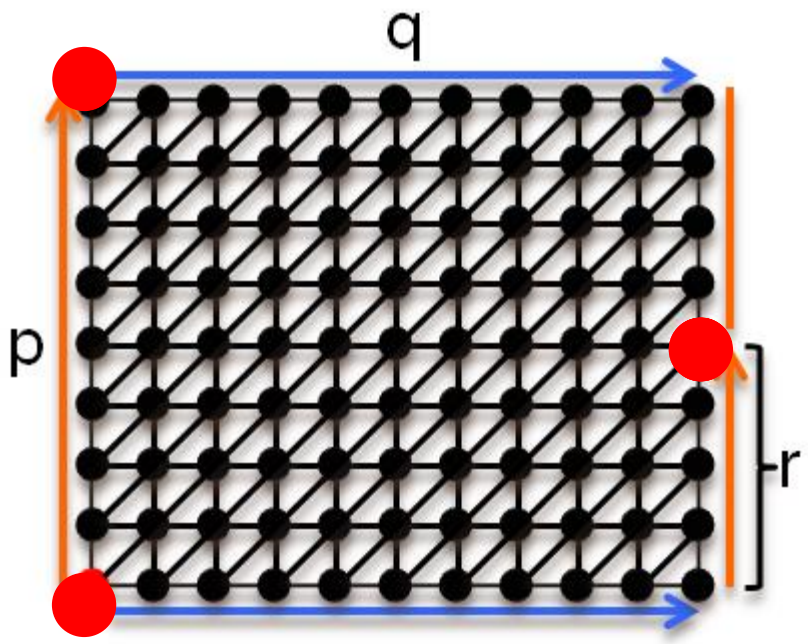


6-regular triangulations

Thm. (Altschuler, `73)

∀ 6-reguler multi-triangulation of the torus

is represented as follows:



(Yeh and Zhu, `03)

Characterize by p, q, r,

all non-4-colorable

triangulations on the torus

The toroidal case

Main Thm.

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G does NOT have K_7 as a subgraph $\Rightarrow$ $\exists$ a sp. bip. quad. in G

Thm. (Matsumoto, Nakamoto, and Yamaguchi, `18)

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is generated from 27 base graphs or 6-regular ones

by a sequence of 4-splittings and 2-vertex additions

The toroidal case

Main Thm.

G : Eulerian triangulation of the torus

G does NOT have K_7 as a subgraph $\Rightarrow$ $\exists$ a sp. bip. quad. in G

- ✓ Show that for all the 27 base graphs and 6-regular ones.
- ✓ Suppose H' is obtained from a triangulation H by 4-splitting and 2-vertex addition. Then show that
  - ➢ If H has a sp. bip. quad., then so is H'.
  - ➢ If H has K_7 as a subgraph, then either so does H' or H' has a sp. bip. quad.

The existence of sp. bip. quad.

✓ Eulerian triangulation

Plane	Torus	Projective plane	Others
○	○ $\Leftrightarrow \nexists K_7$	○	○ if edge-width large

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Only little is known: Plane ○

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The existence of sp. bip. quad.

✓ Eulerian triangulation

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○	○ $\Leftrightarrow \nexists K_7$	○	○ if edge-width large

For the existence of sp. NON-bip. quadrangulation

Plane	Torus	Projective plane	Others
✕	○	○ $\Leftrightarrow$ Not 3-colorable	○ if edge-width large

Thank you for your attention