A spanning Bipartite Quadrangulation of a Triangulation

Kenta Ozeki

(Yokohama National University, Japan)

Joint work with

A. Nakamoto (YNU), and K. Noguchi (Tokyo U. of Science)
Spanning bipartite quadrangulation

✓ (folklore) $G$: triangulation (of any surface)

∃ 4-coloring in $G$

$\iff$ ∃ 2 spanning bipartite quadrangulations covering $E(G)$

Find a sp. bip. quad. in triangulations
Spanning bipartite quadrangulation

Prop. Bipartite or non-bipartite?

\( G : \) triangulation of a surface \( \Rightarrow \exists \) a spanning quadrangulation

1. \( G : \) triangulation
2. The dual \( G^* \) has a perfect matching
3. Any PM gives a sp. quad. of \( G \)

17th August, 2018 Bucharest Graph Theory Workshop
Spanning bipartite quadrangulation

Prop.
$G$: triangulation of a surface $\Rightarrow \exists$ a spanning quadragulation

Bipartite or non-bipartite?

Note: Any quadrangulation of the plane is bipartite.

Cor.
$G$: triangulation of the plane
$\Rightarrow \exists$ a spanning bipartite quadragulation

What about the case of non-spherical surfaces?
Spanning bipartite quadrangulation

The general cases seem difficult.

→ Our target: **Eulerian triangulation**

(∀ vertex has even degree)

Not all (Eulerian) triangulations have a sp. bip. quadrangulation
The toroidal case

✓ \( K_7 \) on the torus has NO sp. bip. quadrangulation

\[
\therefore K_7 \text{ on the torus has } 7 \text{ vertices, } 21 \text{ edges, and } 14 \text{ faces}
\]

To obtain a sp. bip. quad., we delete exactly \( 14/2 = 7 \) edges.

But, \( \nexists \) bip. graph on 7 vertices and \( 21 - 7 = 14 \) edges.
The toroidal case

✓ $K_7$ on the torus has NO sp. bip. quadrangulation

Main Thm.

$G$: Eulerian triangulation of the torus

∃ a sp. bip. quadrangulation in $G$

$\iff$ $G$ does NOT have $K_7$ as a subgraph

✓ Kundgen & Thomassen (\textsuperscript{17}) gave a weaker sufficient condition

✓ Later, I will show an idea of the proof.
The existence of sp. bip. quad.

- Eulerian triangulation

<table>
<thead>
<tr>
<th>Plane</th>
<th>Torus</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Circle" /></td>
<td>⊕ ( K_7 )</td>
</tr>
</tbody>
</table>
The projective planar case

**Main Thm. 2**

\[ G : \text{Eulerian triangulation of the projective plane} \]

\[ \Rightarrow \exists \text{ a sp. bip. quadrangulation in } G \]

Furthermore, if \( G : 3\text{-colorable} \),

\[ \Rightarrow \text{ALL sp. quadrangulations in } G \text{ are bipartite} \]

✓ Kundgen & Thomassen (\(^{17}\)) proved the same, but our proof is shorter
The projective planar case

Main Thm. 2

\[ G : \text{Eulerian triangulation of the projective plane} \implies \exists \text{ a sp. bip. quadrangulation in } G \]

\[ \therefore (\text{Mohar `02}) \quad \forall \text{Eulerian triangulation of the projective plane is the face subdivision of an even embedding} \]

\[ \forall \text{facial cycle is even length} \]
The projective planar case

Main Thm. 2

$G$: Eulerian triangulation of the projective plane

$\Rightarrow$ $\exists$ a sp. bip. quadrangulation in $G$

\[\therefore\] (Mohar `02)

$\forall$ Eulerian triangulation of the projective plane is the face subdivision of an even embedding

$\forall$ facial cycle is even length

Delete all edges in the even embedding
The projective planar case

Main Thm. 2

\[ G : \text{Eulerian triangulation of the projective plane} \]

If \( G \) : 3-colorable,

\[ \Rightarrow \text{ALL sp. quadrangulations in } G \text{ are bipartite} \]

\[ \therefore \text{ (Youngs `96)} \]

\[ \forall \text{ quadrangulation of the projective plane is } \]

either bipartite or non-3-colorable (3-chromatic is impossible)

If \( G \) : 3-colorable, then all sp. quad.s are 3-colorable, so bipartite \( \Box \)
The projective planar case

Main Thm. 2

$G$: Eulerian triangulation of the projective plane

$\Rightarrow \exists$ a sp. bip. quadrangulation in $G$

Furthermore, if $G$: 3-colorable,

$\Rightarrow$ ALL sp. quadrangulations in $G$ are bipartite

✓ Kundgen & Thomassen (\textasciitilde17) proved the same,

but our proof is shorter
The existence of sp. bip. quad.

✓ Eulerian triangulation

<table>
<thead>
<tr>
<th>Plane</th>
<th>Torus</th>
<th>Projective plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>O ↔ \overline{\mathbb{A}} K_7</td>
<td>O</td>
</tr>
</tbody>
</table>
The case of other surfaces

Main Thm. 3

\[ G : \text{Eulerian triangulation of non-spherical surface} \]
If edge-width of \( G \) is large enough,

\[ \Rightarrow \exists \text{ a sp. bip. quadrangulation in } G \]

✓ Edge-width : the length of shortest essential cycle
✓ Shown by using the following result;
  (Hutchinson, Richter, and Seymour `02)
  (Archdeacon, Hutchinson, Nakamoto, Negami, and Ota `99)

∀ Eulerian triangulation \( G \) with large edge-width is 4-colorable,
unless \( G \) is the face subdivision of an even embedding
The existence of sp. bip. quad.

- Eulerian triangulation

<table>
<thead>
<tr>
<th>Plane</th>
<th>Torus</th>
<th>Projective plane</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>☑</td>
<td>☑ $\Leftrightarrow$ $\exists \ K_7$</td>
<td>☑</td>
<td>☑ if edge-width large</td>
</tr>
</tbody>
</table>

- General triangulation

Only little is known:

- Plane: ☑
- "Dense" triangulations: ×

E.g. complete graph
The toroidal case

Main Thm.

\[ G : \text{Eulerian triangulation of the torus} \]

\[ \exists \text{ a sp. bip. quadrangulation in } G \]

\[ \iff G \text{ does NOT have } K_7 \text{ as a subgraph} \]

✓ \[ \iff \text{is an easy part, while we need some arguments} \]

✓ \[ \Rightarrow \text{is the main part} \]
The toroidal case

✓ Use generating thm., allowing multiple edges

Thm. (Matsumoto, Nakamoto, and Yamaguchi, `18)

∀ Eulerian multi-triangulation of the torus is generated from 27 base graphs or 6-regular ones by a sequence of 4-splittings and 2-vertex additions
4-splittings and 2-vertex-addition
4-splittings and 2-vertex-addition
27 base graphs
6-regular triangulations

Thm. (Altschuler, `73)

∀ 6-regular multi-triangulation of the torus

is represented as follows:

(Yeh and Zhu, `03)

Characterize by p, q, r,

all non-4-colorable

triangulations on the torus
The toroidal case

Main Thm.

\[ G : \text{Eulerian triangulation of the torus} \]
\[ G \ \text{does NOT have } K_7 \ \text{as a subgraph} \Rightarrow \exists \ \text{a sp. bip. quad. in } G \]

Thm. (Matsumoto, Nakamoto, and Yamaguchi, `18)

\forall \ \text{Eulerian multi-triangulation of the torus}

is generated from 27 base graphs or 6-regular ones

by a sequence of 4-splittings and 2-vertex additions
The toroidal case

Main Thm.

\[ G : \text{Eulerian triangulation of the torus} \]

\[ G \text{ does NOT have } K_7 \text{ as a subgraph } \implies \exists \text{ a sp. bip. quad. in } G \]

✓ Show that for all the 27 base graphs and 6-regular ones.

✓ Suppose \( H' \) is obtained from a triangulation \( H \)
  by 4-splitting and 2-vertex addition. Then show that
  ➢ If \( H \) has a sp. bip. quad., then so is \( H' \).
  ➢ If \( H \) has \( K_7 \) as a subgraph,
    then either so does \( H' \) or \( H' \) has a sp. bip. quad.
The existence of sp. bip. quad.

- Eulerian triangulation

<table>
<thead>
<tr>
<th>Plane</th>
<th>Torus</th>
<th>Projective plane</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>☑</td>
<td>☐</td>
<td>☑</td>
<td>☑</td>
</tr>
</tbody>
</table>

- General triangulation

Only little is known:

Plane: 

```
```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```

```
The existence of sp. bip. quad.

- Eulerian triangulation

<table>
<thead>
<tr>
<th>Plane</th>
<th>Torus</th>
<th>Projective plane</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

For the existence of sp. NON-bip. quadrangulation

<table>
<thead>
<tr>
<th>Plane</th>
<th>Torus</th>
<th>Projective plane</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

if edge-width large
Thank you for your attention