PhD Second Year Scientific Progress Report
01/2018 – 01/2019

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–7 pages–

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Doctoral School of Mathematics: University of Bucharest and "Simion Stoilow" Institute of Mathematics of the Romanian Academy

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Former title of the thesis: Qualitative and quantitative study of some evolution equations

Communications

(1) *"The Schrodinger equation on star-graphs under general coupling conditions" (Poster)*, Inverse and Spectral Problems for (Non)-Local Operators, September 10–14, 2018, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany
(2) *"The Schrodinger equation on star-graphs under general coupling conditions"*, XIV-eme Colloque Franco-Roumain de Mathematiques Appliquees, August 27–31, Institut de Mathematiques de Bordeaux, Bordeaux, France
(3) *"The Schrodinger equation on a star-shaped graph under general coupling conditions"*, Summer School on Nonlocal Interactions in PDEs and Geometry, May 21–25, 2018, Mittag-Leffler Institute, Stockholm, Sweden
(4) *Limits of sequences of real numbers. Limits of functions*, Admission Lessons Faculty of Mathematics and Computer Science, March 10, 2018, University of Bucharest, Bucharest, Romania

Attendances at workshops, summer-schools, conferences

(1) *Unique Continuation and Uncertainty Principles*, November 26–30, 2018, Basque Center for Applied Mathematics, Bilbao, Spain (supported by the Organizers)
(2) *Workshop of Analysis, Differential Equations and Mechanics*, November 9, 2018, University of Bucharest, Bucharest, Romania
(3) *Inverse and Spectral Problems for (Non)-Local Operators*, September 10–14, 2018, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany (partially supported by the Organizers)
(4) *XIV-eme Colloque Franco-Roumain de Mathematiques Appliquees*, August 27–31, Institut de Mathematiques de Bordeaux, Bordeaux, France (partially supported by the Organizers)
(5) *Regional Romanian-French Summer School in Applied Mathematics, 5th Edition*, July 2–10, 2018, Sinaia, Romania (supported by University of Bucharest and Agence Universitaire de la Franchophonie)
(6) *Summer School on Fractional and Other Nonlocal Models*, May 28–31, 2018, Basque Center for Applied Mathematics, Bilbao, Spain (supported by the Organizers and University of Bucharest)
(7) *Summer School on Nonlocal Interactions in PDEs and Geometry*, May 21–25, 2018, Mittag-Leffler Institute, Stockholm, Sweden (supported by the Organizers)
(8) *Workshop for Young Researchers in Mathematics*, May 17–18, 2018, Bucharest, Romania
(9) *Atelier Transitions de Phase et Equations Nonlocales*, 25-27 April 25–27, 2018, Bucharest, Romania

Attendances at weekly scientific seminars

(1) *Rencontres hebdomadaires du Groupe de Travail Equations aux Dérivés Partielles* (IMAR - FMI), Bucharest, Romania

Co-organizer

(1) Atelier de travail en Equations aux Derivees Partielles, December 13-14, 2018, "Simion Stoilow" Institute of Mathematics of the Romanian Academy, Bucharest, Romania

Fellowships

(1) *BITDEFENDER Junior Research Fellowship*, October-December, 2018 ("Simion Stoilow" Institute of Mathematics of the Romanian Academy)

Teaching activity

(1) *Laboratory of Numerical Analysis and Numerical Methods (L3)*, University of Bucharest, Faculty of Mathematics and Computer Science, October, 2018–January, 2019 (Teaching activity in hourly payment regime)
(2) *Laboratory of General Mechanics (L2)*, University of Bucharest, Faculty of Mathematics and Computer Science, February –June, 2018 (Teaching activity in hourly payment regime

Publications

(1) *The Schrödinger equation on a star-shaped graph under general coupling conditions*, Andreea Grecu, Liviu I. Ignat (*published in* **J. Phys. A: Math. Theor. 52** *(2019) 035202, https://doi.org/10.1088/1751-8121/aaf3fc*)

Brief description of the results

(1) **The Schrödinger equation on a star-shaped graph under general coupling conditions**, Andreea Grecu, Liviu I. Ignat (*published in* **J. Phys. A: Math. Theor. 52** *(2019) 035202, https://doi.org/10.1088/1751-8121/aaf3fc*):

Previous results:

We consider the linear Schrödinger equation:

$$\begin{cases} \mathrm{i}u_t(t,x) + \Delta(A,B)u(t,x) = 0, & t \neq 0, \quad x \in \mathcal{G}, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathcal{G}, \end{cases} \tag{1}$$

where $\mathcal{G}$ is a metric graph given by a finite number $n \in \mathbb{N}^*$ of infinite length edges attached to a common vertex (a so-called *star-graph*), having each edge identified with the positive real axis:

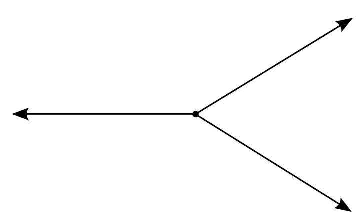

Figure 1. Star-graph $\mathcal{G}$ (one vertex, n edges)

By u_t we denote the time derivative of u, the Laplace operator $\Delta(A,B)$ with domain

$$D(\Delta(A,B)) = \{u \in H^2(\mathcal{G}) : A\underline{u} + B\underline{u'} = 0\},$$

acts as the second derivative along the edges. A and B are $\mathbb{C}^{n\times n}$ matrices which express the coupling conditions at the vertex (we read $\underline{u} = (u_j(t,0_+))_{j=\overline{1,n}}$, $\underline{u'} = (u_j'(t,0_+))_{j=\overline{1,n}}$, respectively) are assumed to satisfy:

(H1) The horizontally concatenated matrix (A, B) has maximal rank;

(H2) $AB^\dagger$ is self-adjoint;

ensuring the self-adjointness of the corresponding Laplacian.

For $q \in [2,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, we obtain $L^p(\mathcal{G}) - L^q(\mathcal{G})$ dispersion properties and space-time Strichartz estimates for the solution of (1).

The proof of the dispersive properties is based on an explicit form of the solution obtained via spectral theory, which is further estimated using classical results for oscillatory integrals. Once the dispersive properties are obtained, the Strichartz estimates follow by a result of Keel and Tao for space-time estimates. Moreover, these estimates help to establish well-posedness in $L^2(\mathcal{G})$ for the nonlinear Schrödinger equation for a class of power nonlinearities, more precisely, the sub-critical case:

$$\begin{cases} \mathrm{i}u_t(t,x) + \Delta(A,B)u(t,x) + |u|^{p-1}u = 0, & t \neq 0, \quad x \in \mathcal{G}, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathcal{G}, \end{cases} \tag{2}$$

where $p \in (1,5)$. The latter is based on contraction mapping principle: first we prove that there exists locally in time a unique mild solution, based on fixed point argument in some space-time balls, and then, after we show the conservation of its L^2–norm, we obtain the global well-posedness.

Recent results:

(a) Well-posedness in $L^2(\mathcal{G})$ of the nonlinear equation (2), in the case of general coupling conditions expressed by matrices A and B satisfying $(H1)$ and $(H2)$, dropping the further assumption of nonexistence of positive real eigenvalues for the matrix $AB^\dagger$;

(b) Moreover, if the initial data u_0 is more regular, more precisely, if it belongs to the domain $D(\mathcal{E})$ of the form $\mathcal{E}$ corresponding to the Hamiltonian $-\Delta(A,B)$, we obtain existence and uniqueness os a solution $u \in C(\mathbb{R}, D(\mathcal{E})) \cap C^1(\mathbb{R}, D(\mathcal{E})^*)$ of (2), where $D(\mathcal{E})$ stands for the dual space of $D(\mathcal{E})$. Since generally the nonlinearity does not keep invariant the form domain, we cannot rely on a fixed-point method. To overcome this difficult, we regularize the nonlinearity and we show that the global solution corresponding to the new equation converges strongly in în $C([-T,T], L^2(\mathcal{G}))$ to the solution $u \in L^2(\mathcal{G})$. Finally we prove that the limit solution inherits the regularity and the conservation of energy.

(2) **The nonlinear Schrödinger equation with white noise dispersion on quantum graphs**, Iulian Cîmpean, Andreea Grecu (*Preprint*):

We consider the stochastic Schrödinger equation:

$$\begin{cases} dX = -\dfrac{1}{2}\Delta_\Gamma^2 X\,dt + \mathrm{i}\Delta_\Gamma X\,d\beta + \mathrm{i}|X|^{2\sigma}X\,dt, \quad t > 0 \\ X(0) = X_0, \end{cases} , \tag{3}$$

where Γ is a quantum metric graph, $(\beta(t))_{t\geq 0}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ and X_0 is a random initial data. A *quantum graph* is a metric graph $\Gamma = (V, E, \partial)$, with a set of vertices V, E the set of adjacent edges to the vertices and a map $\partial : E \to V \times V$ which establishes the orientation on the graph, together with a self-adjoint Laplacian Δ_Γ, which acts as the second derivative along the edges. Moreover, we assume that the deterministic Schrödinger group $\mathrm{e}^{\mathrm{i}t\Delta_\Gamma}$ satisies the following dispersive propery:

$$\text{(H3)} \qquad \|\mathrm{e}^{\mathrm{i}t\Delta_\Gamma}\|_{L^1(\Gamma)\to L^\infty(\Gamma)} \lesssim \frac{1}{\sqrt{|t|}}, \quad t \neq 0.$$

Based on an article of De Bouard and Debussche, we obtain stochastic Strichartz-type estimates, and the existence and uniqueness of a solution for the nonlinear stochastic equation (3).

(3) **Carleman estimates and the unique continuation property for the Schrödinger equation on star-shaped graphs**, Aingeru F. Bertolin, Andreea Grecu, Liviu I. Ignat (*in preparation*):

We consider the linear Schrödinger equation:

$$\begin{cases} iu_t(t,x) + \Delta(A,B)u(t,x) = V(t,x), & t \in (0,1], \quad x \in \mathcal{G} \\ u(0,x) = u_0(x), & \qquad\qquad\quad x \in \mathcal{G} \end{cases}, \tag{4}$$

where $\mathcal{G}$ is a star-shaped metric graph as in Figure 1, $\Delta(A,B)$ is the self-adjoint Laplacian with domain described in(**??**), with the coupling matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix},$$

corresponding to Kirchhoff-type boundary conditions. The potential $V : [0,1] \times \mathcal{G} \to \mathbb{C}^n$ is such that $V \in L^\infty([0,1] \times \mathcal{G})$ and $V_x \in L^1([0,1]; L^\infty(\mathcal{G}))$. We study unique continuation properties for the strong solution of(4). More precisely, if the solution u satisfies

$$\|\mathrm{e}^{\alpha x^2} u(0)\|_{L^2(\mathcal{G})} < \infty$$

and

$$\|\mathrm{e}^{\alpha x^2} u(1)\|_{L^2(\mathcal{G})} < \infty,$$

for $\alpha\beta > \gamma$, where γ is a critical exponent depending only on the number of the star-graph's edges, $\mathcal{G}$, then $u \equiv 0$. The proof of the unique continuation properties relies on obtaining first some Carleman estimates for the operator $\mathrm{i}\partial_t + \Delta(A,B)$.

(4) **Nonlocal elliptic problems, optimal control and nonlocal inverse problems**, Andreea Grecu, Liviu I. Ignat, Diana Stan (*in preparation*):

We consider the nonlocal elliptic equation:

$$
\begin{cases} (-\Delta^s)u(x) = f(x), & x \in \Omega \\ u(x) = v(x), & x \in \Gamma_2 \\ \mathcal{N}_s u(x) = g(x), & x \in \Gamma_1 \end{cases}, \tag{5}
$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $\Gamma_2, \Gamma_1 \subset \mathbb{R}^n$ is such that $\Gamma_1 = \mathbb{R}^n \setminus (\Omega \cup \Gamma_2)$, the fractional Laplacian

$$
(-\Delta)^s u(x) = \int_{\mathbb{R}^n} (u(x) - u(y)\gamma(x,y)\, dy
$$

and

$$
\mathcal{N}_s u(x) = -\int_{\mathbb{R}^n} (u(x) - u(y))\gamma(x,y)\, dy,
$$

with $\gamma(x,y) = c_{n,s} \dfrac{1}{|x-y|^{n+2s}}$, $s \in (0,1)$ and the corresponding truncated equation

$$
\begin{cases} (-\Delta^s)^R u(x) = f(x), & x \in \Omega \\ u(x) = v(x), & x \in \Gamma_2 \\ \mathcal{N}_s^R u(x) = g(x), & x \in \Gamma_{1R} \end{cases}, \tag{6}
$$

whre $R > 0$ is large enough such that $\Gamma_2 \subset \Omega_I^R := \{x \in \mathbb{R}^n, d(x,\Omega) \leq R\}$, and $\Gamma_{1R} = \Gamma_1 \cap \Omega_I^R$. We study existence and uniqueness of weak solution for u and u^R for the euqations (5) and (6), respectively, as well as the convergence of $u^R \to u$ in the fractional Sobolev space $H^s(\Omega \cap \Omega_I^R)$, when $R \to \infty$.

Moreover, we consider the nonlocal equation with over abundant conditions on Γ_1

$$
\begin{cases} (-\Delta)^s u(x) = 0, & x \in \Omega \\ u(x) = \varphi(x), & x \in \Gamma_1 \\ \mathcal{N}_s u(x) = g(x), & x \in \Gamma_1 \end{cases}, \tag{7}
$$

and the corresponding truncated equation. We show that

$$
\inf_{v \in H^s(\Gamma_2)} J(v) = 0,
$$

where

$$
J(v) := \frac{1}{2} \int_{\Gamma_1} |u_{(g,v)-\varphi}|\, dx,
$$

with $u_{(g,v)}$ the weak solution of

$$
\begin{cases} (-\Delta^s)^R u(x) = 0, & x \in \Omega \\ u(x) = v(x), & x \in \Gamma_2 \\ \mathcal{N}_s^R u(x) = g(x), & x \in \Gamma_{1R} \end{cases}.
$$

An open problem which we study is if the same minimization result holds true in the case of the truncated equation, together with its numerical analysis and simulations.

Evaluation of the scientific progress by the guidance committee

(1) On May 17, 2018, the guidance committee consisting of Prof. Dr. Mihai Mihăilescu, Conf. Dr. Cristian Bereanu and Lect. Dr. Cristian Cazacu evaluated the PhD student's scientific progress. The committee ascertained that the student had conducted a scientific activity conforming with the research doctoral program plan and it recommended the continuation of the doctoral studies.

Bibliography

[1] T. Cazenave, *Semilinear Schrödinger Equations*, American Mathematical Society, Vol. 10, (2003)

[2] V. Kostrykin, R. Schrader, *Laplacians on metric graphs: Eigenvalues, resolvents and semigroups*, Quantum Graphs and Their Applications, Vol. 415, (2006).

[3] V. Kostrykin, R. Schrader, *The inverse scattering problem for metric graphs and the traveling salesman problem*, e-print arXiv:math-ph/0603010, (2006).

[4] M. Keel, T. Tao, *Endpoint Strichartz Estimates*, American Journal of Mathematics, Vol. 120, p. 955-980, (1998).