ANNEXE 3.1

INSTITUTUL DE MATEMATICA "SIMION STOILOW" AL ACADEMIEI ROMANE

AUF Agence UNIVERSITAIRE DE LA FRANCOPHONIE

CENTRE FRANCOPHONE EN MATHEMATIQUES BUCAREST

Atelier de travail en *Equations aux Dérivées Partielles*

Stabilization of a Boussinesq system of Benjamin-Bona-Mahony type

Ademir Feranado Pazoto

(Instituto de Matematico, Universidade Federal de Rio de Janeiro)

Abstract: We study the stability properties of a family of Boussinesq systems proposed by J. L. Bona, M. Chen and J.-C. Saut describe the propagation of water waves on the surface of a canal, under the effect of a damping term. By means of spectral analysis a Fourier expansion, we prove that the solutions of the linearized system may have an uniform decay to zero or not. In the uniform dec case, we show that the same property holds for the nonlinear system. Joint work with Sorin Micu from University of Craiova

Miercuri 10 mai 2017, ora 10:00, sala 306 -307 "C. Bănică"

Uniform Stabilization of a family of Boussinesq systems

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Agence Universitaire de la Francophonie

Outline

- Description of the model: a family of Boussinesq systems
- Setting of the problem: stabilization of a coupled system of two Benjamin-Bona-Mahony **(BBM)** equations
- Main results
- Main Idea of the proofs
- Remarks and open problems

The Benjamin-Bona-Mahony (BBM) equation

The BBM equation

$$u_t + u_x - u_{xxt} + uu_x = 0, \quad (1)$$

was proposed as an alternative model for the Korteweg-de Vries equation (KdV)

$$u_t + u_x + u_{xxx} + uu_x = 0, \quad (2)$$

to describe the propagation of one-dimensional, unidirectional small amplitude long waves in nonlinear dispersive media.

- $u(x,t)$ is a real-valued functions of the real variables x and t.

In the context of shallow-water waves, $u(x,t)$ represents the displacement of the water surface at location x and time t.

The Boussinesq system

J. L. **Bona**, M. **Chen**, J.-C. **Saut** - J. Nonlinear Sci. 12 (2002).

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0 \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{3}$$

The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

η is the elevation of the fluid surface from the equilibrium position;

$w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid;

a, b, c, d, are parameters required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which velocity the variable w represents.

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Stabilization Results: $E(t) \leq cE(0)e^{-\omega t}$, $\omega > 0$, $c > 0$

The Boussinesq system posed on a bounded interval:

- A. Pazoto and L. Rosier, Stabilization of a Boussinesq system of KdV-KdV type, System and Control Letters 57 (2008), 595-601.
- R. Capistrano Filho, A. Pazoto and L. Rosier, Control of Boussinesq system of KdV-KdV type on a bounded domain, Preprint.

The Boussinesq system posed on the whole real axis: $(-\eta_{xx}, -w_{xx})$

- M. Chen and O. Goubet, Long-time asymptotic behavior of dissipative Boussinesq systems, Discrete Contin. Dyn. Syst. Ser. 17 (2007), 509-528.

The Boussinesq system posed on a periodic domain:

- S. Micu, J. H. Ortega, L. Rosier and B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete Contin. Dyn. Syst. 24 (2009), 273-313.

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Controllability and Stabilization

• S. Micu, J. H. Ortega, L. Rosier, B.-Y. Zhang - Discrete Contin. Dyn. Syst. 24 (2009).

$$b, d \geq 0, a \leq 0, c \leq 0 \quad \text{or} \quad b, d \geq 0, a = c > 0.$$

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = f(x,t) \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = g(x,t) \end{cases}$$

where $0 < x < 2\pi$ and $t > 0$, with boundary conditions

$$\frac{\partial^r \eta}{\partial x^r}(0,t) = \frac{\partial^r \eta}{\partial x^r}(2\pi,t), \quad \frac{\partial^r w}{\partial x^r}(0,t) = \frac{\partial^r w}{\partial x^r}(2\pi,t)$$

and initial conditions

$$\eta(x,0) = \eta^0(x), \quad w(x,0) = w^0(x).$$

• f and g are locally supported forces.

Periodic boundary conditions

For $b,\ d > 0$ and $\beta_1,\ \beta_2, \alpha_1,\ \alpha_2 \geq 0$, we consider the system

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} + (\eta w)_x + \beta_1 M_{\alpha_1}\eta = 0, \\ &w_t + \eta_x - dw_{txx} + ww_x + \beta_2 M_{\alpha_2} w = 0, \end{aligned} \tag{4}$$

with periodic boundary conditions

$$\begin{aligned} &\eta(0,t) = \eta(2\pi,t);\ \eta_x(0,t) = \eta_x(2\pi,t), \\ &w(0,t) = w(2\pi,t);\ w_x(0,t) = w_x(2\pi,t), \end{aligned}$$

and initial conditions

$$\eta(x,0) = \eta^0(x), \quad w(x,0) = w^0(x).$$

In (4), M_{α_j} are Fourier multiplier operators given by

$$M_{\alpha_j}\left(\sum_{k\in\mathbb{Z}} v_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

The energy associated to the model is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2)dx \tag{5}$$

and we can (formally) deduce that

$$\begin{aligned}\frac{d}{dt}E(t) = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx \\ - \int_0^{2\pi} (\eta w)_x\,\eta\,dx.\end{aligned} \tag{6}$$

Since $\beta_1, \beta_2 \geq 0$ and

$$(M_{\alpha_j}v, v)_{L^2(0,2\pi)} \geq 0, \qquad j = 1, 2,$$

the terms $M_{\alpha_1}\eta$ and $M_{\alpha_2}w$ play the role of feedback damping mechanisms, at least for the linearized system.

Assumptions on the Dissipation: $\int_{\mathbb{T}} M_{\alpha_i}\varphi(x)\varphi(x)dx \geq 0$

- Applications and study of asymptotic behavior os solutions:
  - J. L. Bona and J. Wu, M2AN Math. Model. Numer. Anal. (2000).
  - J.-P. Chehab, P. Garnier and Y. Mammeri, J. Math. Chem. (2001).
  - D. Dix, Comm. PDE (1992).
  - C. J. Amick, J. L. Bona and M. Schonbek, Jr. Diff. Eq. (1989).
  - P. Biler, Bull. Polish. Acad. Sci. Math. (1984).
  - J.-C. Saut, J. Math. Pures et Appl. (1979).
- Fractional derivative (Weyl fractional derivative operator):

$$h(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx} \Rightarrow W_x^{\alpha}(h)(x) = \sum_{k\in\mathbb{Z}} (ik)^{\alpha} a_k e^{ikx}, \quad \alpha \in (0,1).$$

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Main results

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx,$$

where

$$M_{\alpha_j} v = \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}.$$

Firstly, we analyze the linearized system:

- $\alpha_1 = \alpha_2 = 2$ and $\beta_1, \beta_2 > 0 \Longrightarrow$ the exponential decay of solutions in the H^s-setting, for any $s \in \mathbb{R}$.
- $\max\{\alpha_1,\, \alpha_2\} \in (0, 2)$, $\beta_1, \beta_2 \geq 0$ and $\beta_1^2 + \beta_2^2 > 0 \Longrightarrow$ polynomial decay rate of solutions in the H^s-setting, by considering more regular initial data.

Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

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Exponential decay estimate and contraction mapping argument $\Longrightarrow$ global well-posedness and the exponential stability property of the nonlinear system.

For any $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the $k-$Fourier coefficient of v,

$$\widehat{v}_k = \frac{1}{2\pi} \int_0^{2\pi} v(x) e^{-ikx} dx,$$

and, for any $s \in \mathbb{R}$, we define the space

$$H_p^s(0, 2\pi) = \left\{ v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx} \in H^s(0, 2\pi) \;\middle|\; \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^s < \infty \right\},$$

which is a Hilbert space with the inner product defined by

$$(v, w)_s = \sum_{k \in \mathbb{Z}} \widehat{v}_k \overline{\widehat{w}_k} (1 + k^2)^s. \tag{7}$$

Then,

$$M_{\alpha_j} : H_p^{\alpha_j}(0, 2\pi) \to L^2(0, 2\pi).$$

$$M_{\alpha_j} v = \sum_{k \in \mathbb{Z}} (1 + k^2)^{\frac{\alpha_j}{2}} \widehat{v}_k e^{ikx}, \qquad j = 1, 2.$$

The Linearized System

Since

$$(I - b\partial_x^2)\eta_t + w_x + \beta_1 M_1 \eta = 0,$$
$$(I - d\partial_x^2)w_t + \eta_x + \beta_2 M_2 \eta = 0,$$

the linear system can be written as

$$U_t + AU = 0,$$
$$U(0) = U_0,$$

where A is given by

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_1} & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & \beta_2 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_2} \end{pmatrix}. \tag{8}$$

For $\alpha > 0$, the operator $(I - \alpha\partial_x^2)^{-1}$ is defined in the following way:

$$(I - \alpha\partial_x^2)^{-1}\varphi = v \Leftrightarrow \begin{cases} v - \alpha v_{xx} = \varphi & \text{in } (0, 2\pi), \\ v(0) = v(2\pi), & v_x(0) = v_x(2\pi). \end{cases}$$

Spectral Analysis

If we assume that

$$(\eta^0, w^0) = \sum_{k \in \mathbb{Z}} (\widehat{\eta}_k^0, \widehat{w}_k^0) e^{ikx},$$

the solution can be written as

$$(\eta, \omega)(x,t) = \sum_{k \in \mathbb{Z}} (\widehat{\eta}_k(t), \widehat{\omega}_k(t)) e^{ikx},$$

where the pair $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ fulfills

$$\begin{aligned}
&(1+bk^2)(\widehat{\eta}_k)_t + ik\widehat{w}_k + \beta_1(1+k^2)^{\frac{\alpha_1}{2}}\widehat{\eta}_k = 0,\\
&(1+dk^2)(\widehat{w}_k)_t + ik\widehat{\eta}_k + \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\widehat{w}_k = 0,\\
&\widehat{\eta}_k(0) = \widehat{\eta}_k^0, \qquad \widehat{w}_k(0) = \widehat{w}_k^0,
\end{aligned} \tag{9}$$

where $t \in (0,T)$.

We set

$$A(k) = \begin{pmatrix} \frac{\beta_1 (1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2} & \frac{ik}{1+bk^2} \\ & \\ \frac{ik}{1+dk^2} & \frac{\beta_2 (1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2} \end{pmatrix}.$$

Then system (9) is equivalent to

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix}_t (t) + A(k) \begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix} (t) = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \widehat{\eta}_k \\ \\ \widehat{w}_k \end{pmatrix} (0) = \begin{pmatrix} \widehat{\eta}_k^0 \\ \\ \widehat{w}_k^0 \end{pmatrix}.$$

Lemma

The eigenvalues of the matrix A are given by

$$\lambda_k^{\pm} = \frac{1}{2}\left(\frac{\beta_1(1+k^2)^{\frac{\alpha_1}{2}}}{1+bk^2} + \frac{\beta_2(1+k^2)^{\frac{\alpha_2}{2}}}{1+dk^2} \pm \frac{2|k|\sqrt{e_k^2-1}}{\sqrt{(1+bk^2)(1+dk^2)}}\right),$$

where

$$e_k = \frac{1}{2k}\left(\beta_1(1+k^2)^{\frac{\alpha_1}{2}}\sqrt{\frac{1+dk^2}{1+bk^2}} - \beta_2(1+k^2)^{\frac{\alpha_2}{2}}\sqrt{\frac{1+bk^2}{1+dk^2}}\right),$$

and $k \in \mathbb{Z}^*$.

Observe that

- $\lambda_k^{\pm} = \lambda_{-k}^{\pm}$.
- If $e_k < 1$, the eigenvalues $\lambda_k^{\pm} \in \mathbb{C}$.
- If $e_k \geq 1$, the eigenvalues $\lambda_k^{\pm} \in \mathbb{R}$.

Lemma

The solution $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ *of* (9) *is given by*

$$\widehat{\eta}_k(t) = \tfrac{1}{1-\zeta_k^2}\left[\left(\widehat{\eta}_k^0 + i\alpha_k\zeta_k\widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(\zeta_k^2\widehat{\eta}_k^0 + i\alpha_k\zeta_k\widehat{w}_k^0\right)e^{-\lambda_k^- t}\right],$$

$$\widehat{w}_k(t) = \tfrac{1}{1-\zeta_k^2}\left[\left(i\theta_k\zeta_k\widehat{\eta}_k^0 - \zeta_k^2\widehat{w}_k^0\right)e^{-\lambda_k^+ t} - \left(i\theta_k\zeta_k\widehat{\eta}_k^0 - \widehat{w}_k^0\right)e^{-\lambda_k^- t}\right],$$

if $|e_k| \neq 1$ *and* $k \neq 0$,

$$\widehat{\eta}_k(t) = \left[\left(1 - \tfrac{k\zeta_k}{\sqrt{(1+bk^2)(1+dk^2)}}t\right)\widehat{\eta}_k^0 - \tfrac{ikt}{1+bk^2}\widehat{w}_k^0\right]e^{-\lambda_k^+ t},$$

$$\widehat{w}_k(t) = \left[-\tfrac{ikt}{1+dk^2}\widehat{\eta}_k^0 + \left(1 + \tfrac{k\zeta_k}{\sqrt{(1+bk^2)(1+dk^2)}}t\right)\widehat{w}_k^0\right]e^{-\lambda_k^+ t},$$

if $|e_k| = 1$ *and* $k \neq 0$, *and finally,*

$$\widehat{\eta}_0(t) = \widehat{\eta}_0^0 e^{-\beta_1 t}, \qquad \widehat{w}_0(t) = \widehat{w}_0^0 e^{-\beta_2 t}.$$

Here, $\alpha_k = \sqrt{\frac{1+dk^2}{1+bk^2}}$, $\theta_k = \sqrt{\frac{1+bk^2}{1+dk^2}}$ *and* $\zeta_k = e_k - \sqrt{e_k^2 - 1}$.

The case $s = 0$

For any $t \geq 0$ and $k \in \mathbb{Z}$, we have that

$$b|\widehat{\eta}_k(t)|^2 + d|\widehat{w}_k(t)|^2 \leq M \left(b|\widehat{\eta}_k^0|^2 + d|\widehat{w}_k^0|^2\right) e^{-2t \min\left\{|\Re(\lambda_k^+)|, |\Re(\lambda_k^-)|\right\}},$$

where

$$\min\{|\Re(\lambda_k^+)|, |\Re(\lambda_k^-)|\} \geq D > 0,$$

and D is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b and d.

Moreover,

- If $\beta_1\beta_2 = 0$, then $\Re(\lambda_k^{\pm}) \to 0$, as $|k| \to \infty$, and we cannot expect uniform exponential decay of the solutions.
- The fact that the decay of the solutions is not exponential is equivalent to the non uniform decay rate: given any non increasing positive function Θ, there is an initial data $\left(\eta^0, w^0\right)$ such that the $H_p^s \times H_p^s$−norm of the corresponding solution decays slower that Θ.

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and D is a positive number, depending on the parameters β_1, β_2, α_1, α_2, b and d.

Moreover,

- If $\beta_1\beta_2 = 0$, then $\Re(\lambda_k^{\pm}) \to 0$, as $|k| \to \infty$, and we cannot expect uniform exponential decay of the solutions.
- The fact that the decay of the solutions is not exponential is equivalent to the non uniform decay rate: given any non increasing positive function Θ, there is an initial data $\left(\eta^0, w^0\right)$ such that the $H_p^s \times H_p^s-$norm of the corresponding solution decays slower that Θ.

Let us introduce the space

$$V^s = H_p^s(0,2\pi) \times H_p^s(0,2\pi).$$

Then, the following holds:

Theorem (Micu, P., Preprint, 2016)

The family of linear operators $\{S(t)\}_{t\geq 0}$ *defined by*

$$S(t)(\eta^0, w^0) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}, \qquad (\eta^0, w^0) \in V^s, \tag{10}$$

is an analytic semigroup in V^s *and verifies the following estimate*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq C\|(\eta^0, w^0)\|_{V^s}, \tag{11}$$

where C *is a positive constant. Moreover, its infinitesimal generator is the compact operator* $(D(A), A)$, *where* $D(A) = V^s$ *and* A *is given by*

$$A = \begin{pmatrix} \beta_1 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_1} & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & \beta_2 \left(I - b\partial_x^2\right)^{-1} M_{\alpha_2} \end{pmatrix}. \tag{12}$$

Definition

The solutions *decay exponentially in* V^s if there exist two positive constants M and μ, such that

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq M e^{-\mu t} \|(\eta^0, w^0)\|_{V^s}, \tag{13}$$

$\forall t \geq 0$ and $(\eta^0, w^0) \in V^s$.

We have the following result:

Theorem (Micu, P., Preprint, 2016)

The solutions decay exponentially in V^s *if and only if* $\alpha_1 = \alpha_2 = 2$ *and* $\beta_1,\ \beta_2 > 0$. *Moreover,* μ *from* (13) *is given by*

$$\mu = \inf_{k \in \mathbb{Z}} \left\{ \left|\Re(\lambda_k^+)\right|,\ \left|\Re(\lambda_k^-)\right| \right\}, \tag{14}$$

where $\lambda_k^\pm$ *are the eigenvalues of the operator* A.

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Theorem (Micu, P., Preprint, 2016)

Suppose that $\beta_1, \beta_2 \geq 0$, $\beta_1^2 + \beta_2^2 > 0$ *and* $\min\{\alpha_1, \alpha_2\} \in [0, 2)$. *Then, there exists* δ *and* $M > 0$, *such that*

$$\|S(t)(\eta^0, w^0)\|_{V^s} \leq \frac{M}{(1+t)^{\frac{1}{\delta}(q-\frac{1}{2})}} ||(\eta^0, w^0)||_{V^{s+q}}, \, \forall t > 0,$$

where $s \in \mathbb{R}$ *and* $q > \frac{1}{2}$. *Moreover,* $\delta > 0$ *is defined by*

$$\delta = \begin{cases} 2 - \max\{\alpha_1, \alpha_2\} & \text{if } \alpha_1 + \alpha_2 \leq 2, \quad \max\{\alpha_1, \alpha_2\} \leq 1, \\ \max\{\alpha_1, \alpha_2\} & \text{if } \alpha_1 + \alpha_2 \leq 2, \quad \max\{\alpha_1, \alpha_2\} > 1, \\ 2 - \min\{\alpha_1, \alpha_2\} & \text{if } \alpha_1 + \alpha_2 > 2. \end{cases}$$

Remark: If $\alpha_1 = \alpha_2 = 2$ and $\beta_1 = 0$ or $\beta_2 = 0$, then $\delta = 2$.

The nonlinear problem

Theorem (Micu, P., Preprint, 2016)

Let $s \geq 0$ and suppose that $\beta_1, \beta_2 > 0$ and $\alpha_1 = \alpha_2 = 2$. There exist $r > 0$, $C > 0$ and $\mu > 0$, such that, for any $(\eta^0, w^0) \in V^s$, satisfying

$$||(\eta^0, w^0)||_{V^s} \leq r,$$

the system admits a unique solution $(\eta, w) \in C([0, \infty); V^s)$ which verifies

$$\|(\eta(t), w(t))\|_{V^s} \leq Ce^{-\mu t}\|(\eta^0, w^0)\|_{V^s}, \quad t \geq 0.$$

Moreover, μ may be taken as in the linearized problem.

The energy $E(t)$ satisfies

$$\frac{dE}{dt} = -\beta_1 \int_0^{2\pi} (M_{\alpha_1}\eta)\,\eta\,dx - \beta_2 \int_0^{2\pi} (M_{\alpha_2}w)\,w\,dx - \int_0^{2\pi} (\eta w)_x\,\eta\,dx.$$

We define the space

$$Y_{s,\mu} = \{(\eta, w) \in C_b(\mathbb{R}^+; V^s) : e^{\mu t}(\eta, w) \in C_b(\mathbb{R}^+; V^s)\},$$

with the norm

$$||(\eta, w)||_{Y_{s,\mu}} := \sup_{0 \le t < \infty} ||e^{\mu t}(\eta, w)(t)||_{V^s},$$

and the function $\Gamma : Y_{s,\mu} \to Y_{s,\mu}$ by

$$\Gamma(\eta, w)(t) = S(t)(\eta^0, w^0) - \int_0^t S(t-\tau) N(\eta, w)(\tau) \, \mathrm{d}\tau,$$

where $N(\eta, w) = (-(I - b\partial_x^2)^{-1}(\eta w)_x, -(I - d\partial_x^2)^{-1} w w_x)$ and $\{S(t)\}_{t \ge 0}$ is the semigroup associated to the linearized system.

Combining the estimates obtained for the linearized system we have

$$||\Gamma(\eta, w)(t)||_{V^s} \leq Me^{-\mu t}||(\eta^0, w^0)||_{V^s} + MCe^{-\mu t} \sup_{0\leq\tau\leq t} ||e^{\mu\tau}(\eta, w)||_{V^s},$$

for any $t \geq 0$ and some positive constants M and C.

- If we take $(\eta, w) \in B_R(0) \subset Y_{s,\mu}$, the following estimate holds

  $$||\Gamma(\eta, w)||_{Y_{s,\mu}} \leq M||(\eta^0, w^0)||_{V^s} + MC||(\eta, w)||^2_{Y_{s,\mu}} \leq MR + MCR^2.$$

- A similar calculations shows that,

  $$||\Gamma(\eta_1, w_1) - \Gamma(\eta_2, w_2)||_{Y_{s,\mu}} \leq 2RMC||(\eta_1, w_1) - (\eta_2, w_2)||_{Y_{s,\mu}},$$

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Dirichlet boundary conditions

We consider the BBM-BBM system

$$\begin{aligned} &\eta_t + w_x - b\eta_{txx} + \varepsilon a(x)\eta = 0, \qquad x \in (0, 2\pi),\ t > 0, \\ &w_t + \eta_x - dw_{txx} = 0, \qquad x \in (0, 2\pi),\ t > 0, \end{aligned}$$

with boundary conditions

$$\eta(t, 0) = \eta(t, 2\pi) = w(t, 0) = w(t, 2\pi) = 0, \quad t > 0,$$

and initial conditions

$$\eta(0, x) = \eta^0(x), \quad w(0, x) = w^0(x), \qquad x \in (0, 2\pi).$$

We assume that

- $b, d > 0$ and $\varepsilon > 0$ are parameters.
- $a = a(x)$ is a nonnegative real-valued function satisfying

$$a(x) \geq a_0 > 0, \qquad \text{in } \Omega \subset (0, 2\pi).$$

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$$a(x) \geq a_0 > 0, \qquad \text{in } \Omega \subset (0, 2\pi).$$

The energy associated to the model is given by

$$E(t) = \frac{1}{2}\int_0^{2\pi} (\eta^2 + b\eta_x^2 + w^2 + dw_x^2)dx \tag{15}$$

and we can (formally) deduce that

$$\frac{d}{dt}E(t) = -\varepsilon \int_0^{2\pi} a(x)\eta^2(t,x)dx. \tag{16}$$

Does $E(t)$ converges to zero as $t \to \infty$?

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Lack of Compactness

There exist $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \Big[\int_0^{2\pi} \varepsilon a(x) \eta^2(x,t) dx \Big] dt, \qquad (17)$$

for every finite energy solution. Indeed, from (17) and the energy dissipation law, we have that

$$E(T) \leq \frac{C}{C+1} E(0). \qquad (18)$$

Since $E(t) \leq E(kT) \leq \gamma^k E(0)$, for $0 < \gamma < 1$ and $k > 0$,

$$E(t) \leq \frac{1}{\gamma} E(0) e^{\frac{\ln \gamma}{T} t}, \text{ where } \gamma = \frac{C}{C+1}. \qquad (19)$$

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Main results

We assume that $a = a(x)$ is nonnegative and

$$a(x) \geq a_0 > 0, \text{ in } \Omega \subset (0, 2\pi), \\ a \in W^{2,\infty}(0, 2\pi), \text{ with } a(0) = a'(0) = 0. \tag{20}$$

Theorem (Micu, P., Journal d'Analyse Mathématique)

There exits ε_0, such that, for any $\varepsilon \in (0, \varepsilon_0)$ and (η^0, w^0) in $(H_0^1(0, 2\pi))^2$, the solution (η, w) of the system verifies

$$\lim_{t\to\infty} \|(\eta(t), w(t))\|_{(H_0^1(0,2\pi))^2} = 0.$$

Moreover, the decay of the energy is not exponential, i. e., there exists no positive constants M and ω, such that

$$\|(\eta(t), w(t))\|_{(H_0^1(0,2\pi))^2} \leq Me^{-\omega t}, \qquad t \geq 0.$$

Spectral analysis and eigenvectors expansion of solutions

Since

$$\begin{array}{ll} (I - b\partial_x^2)\eta_t + w_x + \varepsilon a(x)\eta = 0, & x \in (0, 2\pi),\ t > 0, \\ (I - d\partial_x^2)w_t + \eta_x = 0, & x \in (0, 2\pi),\ t > 0, \end{array}$$

the system can be written as

$$\begin{array}{l} U_t + \mathcal{A}_\varepsilon U = 0, \\ U(0) = U_0, \end{array}$$

where $\mathcal{A}_\varepsilon : (H_0^1(0, 2\pi))^2 \to (H_0^1(0, 2\pi))^2$ is given by

$$\mathcal{A}_\varepsilon = \begin{pmatrix} \varepsilon \left(I - b\partial_x^2\right)^{-1} a(\cdot)\, I & \left(I - b\partial_x^2\right)^{-1} \partial_x \\ \left(I - d\partial_x^2\right)^{-1} \partial_x & 0 \end{pmatrix}. \qquad (21)$$

We have that

$\mathcal{A}_\varepsilon \in \mathcal{L}((H_0^1(0, 2\pi))^2)$ and $\mathcal{A}_\varepsilon$ is a compact operator.

The operator $\mathcal{A}_\varepsilon$ has a family of eigenvalues $(\lambda_n)_{n\geq 1}$, such that:

1. $|\Re(\lambda_n)| \leq \dfrac{c}{|n|^2}, \ \forall\, n \geq n_0,$ and $\Re(\lambda_n) < 0, \ \ \forall\, n.$
2. The corresponding eigenfunctions $(\Phi_n)_{n\geq 1}$ form a Riesz basis in $(H_0^1(0,2\pi))^2$.

Then,

$$(\eta(t), w(t)) = \sum_{n\geq 1} a_n e^{\lambda_n t} \Phi_n$$

and

$$c_1 \sum_{n\geq n_0} |a_n|^2 e^{2\Re(\lambda_n)t} \leq \|(\eta(t), w(t))\|^2_{(H_0^1(0,2\pi))^2} \leq c_2 \sum_{n\geq 1} |a_n|^2 e^{2\Re(\lambda_n)t}.$$

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Theorem (Micu, P., Journal d'Analyse Mathématique)

Let (η, w) *be a finite energy solution of the system with* $a \equiv 0$*. If there exist* $T > 0$ *and an open set* $\Omega \subset (0, 2\pi)$*, such that*

$$\eta(x,t) = 0 \quad , \forall \ (t,x) \in (0,T) \times \Omega, \tag{22}$$

then

$$\eta = w \equiv 0 \quad \textit{in} \quad \mathbb{R} \times (0, 2\pi).$$

Idea of the proof:

- $\mathcal{A}_\varepsilon$ is a compact operator in $(H_0^1(0,2\pi))^2 \Rightarrow$ analyticity in time of solutions $\Rightarrow$ property (22) holds for $t \in \mathbb{R}$.
- Fourier decomposition of solutions.
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Main steps of the proof

1. The spectrum of the differential operator corresponding $\mathcal{A}_\varepsilon$ is located in the left open half-plane of the complex plane. We also obtain the asymptotic behavior (of the spectrum) .

2. There exists a Riesz basis $(\Phi_m)_{m\geq 1} \subset (H_0^1(0, 2\pi))^2$ consisting of generalized eigenvectors of the differential operator $\mathcal{A}_\varepsilon$.

We obtain the asymptotic behavior of the high eigenfunctions and prove that they are quadratically close to a Riesz basis $(\Psi_m)_{m\geq 1}$ formed by eigenvectors of a well chosen dissipative differential operator with constant coefficients:

$$\sum_{m\geq N+1} ||\Phi_m - \Psi_m||^2_{(H_0^1(0,2\pi))^2} \sim \frac{1}{m^2}.$$

This is done by using less common two dimensional versions of the Shooting Method and Rouché's Theorem.

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To control the low frequencies we use a result originally proved for a unbounded operator:

- B. Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass, SIAM J. Control Optim. 39 (2001), 1736–1747.
- B. Z. Guo and R. Yu, The Riesz basis property of discrete operators and application to a Euler-Bernoulli beam equation with boundary linear feedback control, IMA J. Math. Control Inform. 18 (2001), 241–251.

It was extended to the bounded case:

- X. Zhang and E. Zuazua, Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential, Math. Ann. 325 (2003), 543-582.

We consider the spectral problem

$$\mathcal{A}_\varepsilon \begin{pmatrix} \eta \\ w \end{pmatrix} = \mu \begin{pmatrix} \eta \\ w \end{pmatrix},$$

where

$$\mathcal{A}_\varepsilon : (H_0^1(0,2\pi))^2 \to (H_0^1(0,2\pi))^2,$$

which is equivalent to the BVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0,2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0,2\pi) \\ \eta(0) = \eta(2\pi) = 0 \\ w(0) = w(2\pi) = 0. \end{cases} \tag{23}$$

From (23) we obtain a family of eigenvalues and eigenfunctions.......

A two dimensional "shooting method"

For each $(\mu, \gamma) \in \mathbb{C}^2$, consider the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0, 2\pi) \\ w - d w_{xx} + \mu \eta_x = 0 & \text{for } x \in (0, 2\pi) \\ \eta(0) = 0, \ \eta_x(0) = 1 \\ w(0) = 0, \ w_x(0) = \gamma. \end{cases} \tag{24}$$

and the map $F : \mathbb{C}^2 \to \mathbb{C}^2$, given by $F(\mu, \gamma) = \begin{pmatrix} \eta(\mu, \gamma, 2\pi) \\ w(\mu, \gamma, 2\pi) \end{pmatrix}$.

Then, μ is an eigenvalue of (23) with corresponding eigenfunction $\begin{pmatrix} \eta \\ w \end{pmatrix}$, if and only if,

$$F(\mu, \gamma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The spectrum of the differential operator with variable potential is given by the zeros of the map F.

A two dimensional "shooting method"

For each $(\mu, \gamma) \in \mathbb{C}^2$, consider the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0, 2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0, 2\pi) \\ \eta(0) = 0, \ \eta_x(0) = 1 & \\ w(0) = 0, \ w_x(0) = \gamma. & \end{cases} \tag{24}$$

and the map $F : \mathbb{C}^2 \to \mathbb{C}^2$, given by $F(\mu, \gamma) = \begin{pmatrix} \eta(\mu, \gamma, 2\pi) \\ w(\mu, \gamma, 2\pi) \end{pmatrix}$.

Then, μ is an eigenvalue of (23) with corresponding eigenfunction $\begin{pmatrix} \eta \\ w \end{pmatrix}$, if and only if,

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Next, we define the map $G = G(\sigma, \beta)$, associated to a spectral problem with constant coefficients, for which the eigenfunctions form a Riesz basis:

$$\begin{cases} -b\psi_{xx} + \sigma u_x + \varepsilon a_0 \sigma \psi = 0 & \text{for } x \in (0, 2\pi) \\ -du_{xx} + \sigma \psi_x = 0 & \text{for } x \in (0, 2\pi) \\ \psi(0) = 0, \ \psi_x(0) = 1 \\ u(0) = 0, \ u_x(0) = \beta. \end{cases} \tag{25}$$

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Theorem (N. G. Lloyd, J. London Math. Soc. 2 (1979))

Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^N$ and h, G holomorphic maps of $\overline{\mathcal{D}}$ into $\mathbb{C}^N$ such that $\|h(z)\| < \|G(z)\|$ for $z \in \partial\mathcal{D}$. Then G has finitely many zeros in $\mathcal{D}$, and G and $h + G$ have the same number of zeros in $\mathcal{D}$, counting multiplicity.

Given a zero (σ_n^j, β_n^j) of the map G, we define the domain

$$D_n^j(\delta) = \left\{ (\mu, \gamma) \in \mathbb{C}^2 \, : \, \sqrt{|\mu - \sigma_n^j|^2 + |\gamma - \beta_n^j|^2} \leq \frac{\delta}{|n|} \right\}.$$

If $h = F - G$ we obtain

- $\|G(\mu, \gamma)\| \geq \dfrac{C}{n^2}$,
- $\|F(\mu, \gamma) - G(\mu, \gamma)\| \leq \dfrac{C}{n^2}, \quad \forall\, (\mu, \gamma) \in \partial D_n^j(\delta)$.

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Finally, we obtain an ansatz $\begin{pmatrix} \varphi(\mu,\gamma,x) \\ z(\mu,\gamma,x) \end{pmatrix}$ for the solutions of the IVP

$$\begin{cases} \eta - b\eta_{xx} + \mu w_x + \varepsilon a(x)\mu\eta = 0 & \text{for } x \in (0,2\pi) \\ w - dw_{xx} + \mu\eta_x = 0 & \text{for } x \in (0,2\pi) \\ \eta(0) = 0,\ \eta_x(0) = 1 \\ w(0) = 0,\ w_x(0) = \gamma. \end{cases}$$

More precisely, $\begin{pmatrix} \eta(\mu,\gamma,x) \\ w(\mu,\gamma,x) \end{pmatrix} = \begin{pmatrix} \varphi(\mu,\gamma,x) \\ z(\mu,\gamma,x) \end{pmatrix} + \mathcal{O}\left(\frac{1}{\mu^2}\right)$, where

$$\begin{cases} \varphi(\mu,\gamma,x) = & \dfrac{\sqrt{bd}}{\mu}\sinh(\alpha(x)) + \dfrac{\gamma d}{\mu}\cosh(\alpha(x)) - \dfrac{\gamma d}{\mu + da(x)} \\ z(\mu,\gamma,x) = & \dfrac{b}{\mu}(\cosh(\alpha(x)) - 1) + \dfrac{\gamma\sqrt{bd}}{\mu}\sinh(\alpha(x)) + \dfrac{\gamma d}{\mu}\displaystyle\int_0^x a(s)ds, \end{cases}$$

and $\alpha(x) = \frac{\mu x}{\sqrt{bd}} + \frac{1}{2}\sqrt{\frac{d}{b}}\int_0^x a(s)ds.$

$$\|F(\mu,\gamma) - G(\mu,\gamma)\|$$

$$\leq \left\| F(\mu,\gamma) - \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varphi(\mu\gamma, 2\pi) \\ z(\mu,\gamma,2\pi) \end{pmatrix} - G(\mu,\gamma) \right\|$$

$\leq \frac{C_1}{|\mu|^2}$ (from the ansatz property)

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Similar strategies have been successfully used by

- S. Cox and E. Zuazua, The rate at which energy decays in a damped string, Comm. Partial Differential Equations 19 (1994), 213–243.
- A. Benaddi and B. Rao, Energy decay rate of wave equations with indefinite damping, J. Differential Equations 161 (2000), 337–357.
- X. Zhang and E. Zuazua, Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential, Math. Ann. 325 (2003), 543–582.

Remarks and open problems

- **Dirichlet boundary conditions:**
- Less regularity for the potential a.
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