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# Long time behavior of solutions of local and nonlocal nondegenerate Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator

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# Long time behavior of solutions of local and nonlocal nondegenerate Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator

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# Outline of presentation

   I. Introduction.

  II. A *priori* Lipschitz regularity of solutions.

 III. Well-posedness of solutions.

 IV. Application to ergodic problem and long time behavior.

## Introduction

We study long time behavior of solution of Cauchy problem

$$\begin{cases} u_t - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H(x, Du) = f(x) \text{ in } Q, \\ u(\cdot, 0) = u_0(\cdot) \text{ in } \mathbb{R}^N, \end{cases} \quad \text{(HJt)}$$

where $Q = \mathbb{R}^N \times (0, \infty)$, $\mathcal{F}$ can be either *local* or *nonlocal*:

$$\begin{cases} \mathcal{F}(x, [u]) = \operatorname{tr}(\sigma(x)\sigma^T(x)D^2 u) \text{ (local case)}, \\ \mathcal{F}(x, [u]) = \mathcal{I}(x, u, Du) \text{ (nonlocal case)}, \end{cases}$$

the term $\mathcal{F} - \langle b, D \rangle$ is called the **Ornstein-Uhlenbeck operator** (O-U).
**The convergence that we want to obtain:**

$$u(\cdot, t) - (ct + v) \to 0 \text{ locally uniformly in } \mathbb{R}^N \text{ as } t \to \infty$$

where $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ is a solution of the associated ergodic problem:

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv \rangle + H(x, Dv) = f(x) \quad \text{in } \mathbb{R}^N. \quad \text{(HJErg)}$$

# Main steps to prove the convergence

**Goal:** $u(\cdot, t) - (ct + v) \to 0$ locally uniformly in $\mathbb{R}^N$ as $t \to \infty$.

- **Step 1:** Solving the ergodic problem

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv \rangle + H(x, Dv) = f(x),$$

where $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$.

- **Step 2:** Getting compactness property of $\{u(\cdot, t) - ct, t \geq 0\}$.
- **Step 3:** Applying strong maximum principle.

All three above main steps need Lipschitz regularity of solutions.

⇨ Establishing the Lipschitz regularity of solutions is one of the **crucial steps** to study long time behavior of solutions.

# Our improvements from Fujita, Ishii & Loreti (2006)'s work

$(FIL's \ eq): \qquad u_t - \Delta u + \alpha \langle x, Du \rangle + H(Du) = f(x).$

- Using more general diffusion terms

$$\Delta u \ \Rightarrow \ \text{tr}(\sigma(x)\sigma^T(x)D^2 u), \ \text{with } \sigma \text{ is Lipschitz bounded,}$$

$$\Rightarrow \ \text{Nonlocal integro-differential operator of fractional}$$

$$\text{Laplacian type.}$$

- Using more general class of Hamiltonians, $H$ is sublinear and depends on $x$, i.e., $|H(x,p)| \le C(1+|p|), \ \forall x, p \in \mathbb{R}^N$ instead of H is Lipschitz, i.e., $|H(p) - H(q)| \le C|p-q|, \ \forall p, q \in \mathbb{R}^N$.

# Assumptions on datas (1)

Prescribed growth of solutions and datas

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \quad \mu > 0, \qquad \text{(growth)}$$

$$\mathcal{E}_\mu(\bar{Q}) = \{v : \bar{Q} \to \mathbb{R} : \lim_{|x| \to +\infty} \sup_{0 \le t < T} \frac{v(x,t)}{\phi_\mu(x)} = 0\}, \qquad (\mathcal{E}_\mu)$$

where $Q = \mathbb{R}^N \times (0, \infty)$, $T > 0$. We look for solutions which belong to $\mathcal{E}_\mu$ with datas $f$, $u_0 \in \mathcal{E}_\mu$.

• **Local diffusion:** $\sigma$ is a bounded Lipschitz continuous in $\mathbb{R}^N$.

• **Nonlocal diffusion:** Let $x \in \mathbb{R}^N$, $\psi : \mathbb{R}^N \to \mathbb{R}$ be a $C^2$ function,

$$\mathcal{I}(x, \psi, D\psi) = \int_{\mathbb{R}^N} \{\psi(x+z) - \psi(x) - \langle D\psi(x), z \rangle \mathbf{1}_B(z)\} \nu(dz),$$

$\nu$ : Lévy measure satisfying $\int_B |z|^2 \nu(dz), \int_{B^c} \phi_\mu(z)\nu(dz) \le C_\nu$, $C_\nu > 0$,

$$\text{typical case:} \qquad \nu(dz) = \frac{e^{-\mu|z|}dz}{|z|^{N+\beta}}, \quad \mu > 0, \ \beta \in (0,2).$$

# Assumptions on datas (2)

• The **Ornstein-Uhlenbeck drift** term:

There exists $\alpha > 0$ such that $\langle b(x) - b(y), x - y \rangle \geq \alpha |x - y|^2$,   (O-U)

where $\alpha$ is called *size* of the Ornstein-Uhlenbeck operator which has a crutial role in the equation. It gives a kind of supersolution of the equation for large $x$, that is

$$- \mathcal{F}(x, [\phi_\mu]) + \langle b(x), D\phi_\mu \rangle - C|D\phi_\mu| \geq \phi_\mu - K, \qquad \text{(super pro)}$$

where $K > 0$, and $\phi_\mu$ is the growth function.

• The **Hamiltonian**:

$$|H(x, p)| \leq C_H(1 + |p|), \quad x, p \in \mathbb{R}^N. \qquad \text{(sublinear)}$$

# Lipschitz regularity results for nondegenerate elliptic equations

Let $\lambda > 0$, we consider the approximate stationary problem

$$\lambda u^\lambda - \mathcal{F}(x, [u^\lambda]) + \langle b(x), Du^\lambda \rangle + H(x, Du^\lambda) = f(x) \qquad \text{(HJ}\lambda\text{)}$$

in $\mathbb{R}^N$. This equation is called nondegenerate elliptic if
- **Local case**, $\mathcal{F}(x, [u^\lambda]) = \text{tr}(A(x)D^2 u^\lambda)$, $A$ satisfies $A(x) \geq \rho Id$, $\rho > 0$.
- **Nonlocal case**, $\mathcal{F}(x, [u^\lambda]) = \mathcal{I}(x, u^\lambda, Du^\lambda)$, with measures satisfy

<br>

Let $\beta \in (1, 2)$, $a \in \mathbb{R}^N$.

There exist $0 < \eta < 1$ and $C_\nu > 0$ such that

(M) $\qquad \forall \gamma > 0 \quad \displaystyle\int_{\mathcal{C}_{\eta,\gamma}(a)} |z|^2 \nu(dz) \geq C_\nu \eta^{\frac{N-1}{2}} \gamma^{2-\beta},$

where $\mathcal{C}_{\eta,\gamma}(a) := \{z \in B_\gamma ; (1-\eta)|z||a| \leq |a.z|\}$

here $\beta$ is called the order of nonlocal operator, see [BCCI-12].

# Statement of results

Recall:

$$\phi_\mu(x) = e^{\mu\sqrt{|x|^2+1}}, \quad \mu > 0,$$

$$\mathcal{E}_\mu(\mathbb{R}^N) = \{v : \mathbb{R}^N \to \mathbb{R} : \lim_{|x| \to +\infty} \frac{v(x)}{\phi_\mu(x)} = 0\}.$$

**Theorem 1** (Gradient bound for (HJλ) in the **nondegenerate** case)

*Let $u^\lambda \in \mathcal{E}_\mu$ with $\phi_\mu$-growth be a continuous solution of (HJλ). Suppose that (HJλ) is **nondegenerate elliptic** (in the both local and nonlocal case), O-U holds with any size $\alpha > 0$, H is sublinear, $f \in \mathcal{E}_\mu$. Then*

$$|u^\lambda(x) - u^\lambda(y)| \le C(\phi_\mu(x) + \phi_\mu(y))|x - y|, \ C \text{ \bf independent } of \lambda.$$

⇨ Same results for parabolic equations (HJt) with $C > 0$ **independent** of $t$.

# Ideas of proof (1). Viscosity inequalities.

Arguing by contradiction, using [Ishii-Lions-90] method's ideas. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a $C^2$ concave, increasing function with $\psi(0) = 0$, $\phi_\mu$ be the growth function and set $\Phi(x, y) = \phi_\mu(x) + \phi_\mu(y)$. Suppose that

$$\sup_{x, y \in \mathbb{R}^N} \{u^\lambda(x) - u^\lambda(y) - \psi(|x - y|)\Phi(x, y)\} > 0, \text{ achieved at } (\bar{x}, \bar{y}), \ \bar{x} \neq \bar{y}.$$

Writing the viscosity inequalities with $\varphi(\bar{x}, \bar{y}) = \psi(|\bar{x} - \bar{y}|)\Phi(\bar{x}, \bar{y})$ we have

$$\begin{aligned}
\lambda(u^\lambda(\bar{x}) - u^\lambda(\bar{y})) &- (\mathcal{F}(\bar{x}, [u^\lambda]) - \mathcal{F}(\bar{y}, [u^\lambda])) \\
&+ \langle b(\bar{x}), D_x\varphi \rangle - \langle b(\bar{y}), -D_y\varphi \rangle \\
\leq \quad & H(\bar{y}, -D_y\varphi) - H(\bar{x}, D_x\varphi) + f(\bar{x}) - f(\bar{y}), \quad (1)
\end{aligned}$$

**Goal:** Reach a contradiction in (1).

# Ideas of proof (2). Estimate the different terms in (1) (local case)

• **Local terms:** $\mathcal{F}(\bar{x}, [u^\lambda]) - \mathcal{F}(\bar{y}, [u^\lambda]) = \operatorname{tr}(A(\bar{x})X - A(\bar{y})Y)$, using [Ishii-Lions-90]'s method.

▷ Using the assumptions on the local diffusion matrix, (O-U), $H, f$ to estimate the different terms in (1) we get

$$\Phi(\bar{x}, \bar{y})(\overbrace{-4\rho\psi''(|\bar{x} - \bar{y}|)}^{>0 \text{ if } \psi \text{ strict.concave}} + \overbrace{\alpha\psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|}^{\text{coming from (O-U)}})$$

$$\leq \quad \Phi(\bar{x}, \bar{y})C\psi'(|\bar{x} - \bar{y}|) + C + C(\phi_\mu(\bar{x}) + \phi_\mu(\bar{y}))|\bar{x} - \bar{y}|$$
$$+ (\psi(|\bar{x} - \bar{y}|)) \left(\operatorname{tr}(A(\bar{x})D^2\phi_\mu(\bar{x})) - \langle b(\bar{x}), D\phi_\mu(\bar{x})\rangle + C|D\phi_\mu(\bar{x})|\right)$$
$$+ (\psi(|\bar{x} - \bar{y}|)) \left(\operatorname{tr}(A(\bar{y})D^2\phi_\mu(\bar{y})) - \langle b(\bar{y}), D\phi_\mu(\bar{y})\rangle + C|D\phi_\mu(\bar{y})|\right).$$

# Ideas of proof (3). Reach a contradiction (local case).
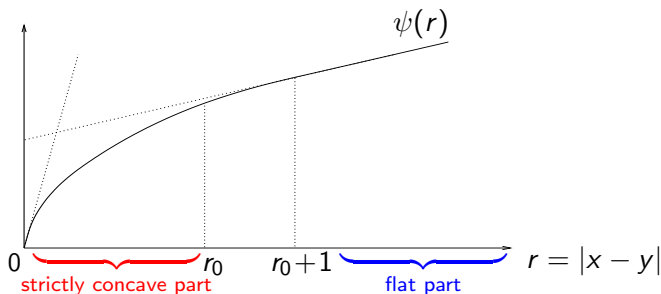


- $r := |\bar{x} - \bar{y}| \leq r_0$. We take profit of the **ellipticity** of the equation which is related with the **strictly concave** property of $\psi$ to control bad terms which come form the Hamiltonian the other terms.

- $r := |\bar{x} - \bar{y}| \geq r_0$. Using the effect of the Ornstein-Uhlenbeck operator to control everything.

▷ All of the parameters in two cases are chosen **independently** of $\lambda$.

# Ideas of proof (4). Reach a contradiction (nonlocal case).

- The ideas follow the ones in the local case in order that we can apply [Ishii-Lions-90] method's ideas.

- **Nonlocal terms:**
  $\mathcal{F}(\bar{x}, [u^\lambda]) - \mathcal{F}(\bar{y}, [u^\lambda]) = \mathcal{I}(\bar{x}, u^\lambda, D_x\varphi) - \mathcal{I}(\bar{y}, u^\lambda, -D_y\varphi)$, inspired by [BCCI-12].

- The estimates for the nonlocal terms are **more technical and complicated** than those of the local ones because of **nonlocal property of the operator and unbounded settings of solutions**.

- We need to construct concave test functions which are different with the one in the local case.

- We can not obtain the result directly. We first establish $\tau$-Hölder continuity for solutions, $\forall \tau \in (0, 1)$, then we improve that Hölder regularity to Lipschitz regularity.

# Ideas of proof (5). Concave test functions.
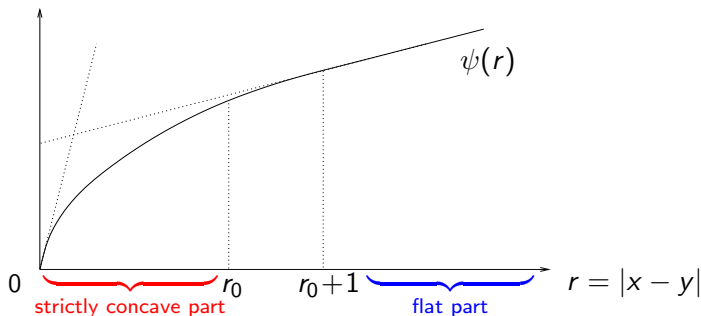


The concave test functions for $r \leq r_0$ :

- **Local case:** $\psi_{local}(r) = 1 - e^{-C_1 r}$.

- **Nonlocal case:** $\begin{cases} \psi_{nonlocal1}(r) = 1 - e^{-C_1 r^\tau}, \ \tau \in (0,1), \\ \psi_{nonlocal2}(r) = r - \varrho r^{1+\theta}, \ \theta \in (0,1), \varrho > 0. \end{cases}$

# Well-posedness of solutions (1)

∗ **Existence:**

⇨ Using truncation arguments (inspired by [Barles-Souganidis-01]) to build a continuous solution for problems. Let $m, n \geq 1$,

$$f_m(x) = \min\{f(x) + \frac{1}{m}\phi_\mu(x), m\}, \ \phi_\mu \text{ is the growth function,}$$

$$H_m(x, p) = \begin{cases} H(x, p) & |x| \leq m \\ H(m\frac{x}{|x|}, p) & |x| \geq m, \end{cases} \ H_{mn}(x, p) = \begin{cases} H_m(x, p) & |p| \leq n \\ H_m(x, n\frac{p}{|p|}) & |p| \geq n. \end{cases}$$

▷ $f_m \in BUC(\mathbb{R}^N)$ and $H_{mn} \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$ with modulus of continuity depending on $m, n \Rightarrow$ existence of $BUC$ solution of

$$\lambda u - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H_{mn}(x, Du) = f_m \ \text{ in } \mathbb{R}^N.$$

# Well-posedness of solutions (2)

⇨ Using the local gradient bounds and applying Ascoli Theorem to get the convergence of the bounded solution to the truncation equation to the initial one when $n, m \to \infty$.

∗ **Uniqueness:**

- It is a direct consequence of the comparison principle.
- In our case, the comparison principle only holds if either sub or supersolution is locally Lipschitz continuous.

# Application to ergodic problem

**Problem:** Finding $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ which is the solution (HJErg)

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv(x) \rangle + H(x, Dv(x)) = f(x).$$

Let $u^\lambda \in C(\mathbb{R}^N)$, $\lambda \in (0, 1)$ be a solution of (HJ$\lambda$). Then

$$w^\lambda := u^\lambda(x) - u^\lambda(0) \qquad \text{is a solution of}$$

$$\lambda w^\lambda - \mathcal{F}(x, [w^\lambda]) + \langle b(x), Dw^\lambda(x) \rangle + H(x, Dw^\lambda(x)) = f(x) - \lambda u^\lambda(0).$$

**Main tools:** using local uniform Lipschitz regularity of $u^\lambda$, Ascoli Theorem we get convergences, up to some subsequence,

$$w^\lambda \to v, \text{ locally uniformly in } \mathbb{R}^N$$
$$\lambda u^\lambda(0) \to c, \text{ as } \lambda \to 0.$$

Thanks to Stability results ([Crandall et al.- 92,Barles-Imbert-08]) we get a solution for (HJErg).

# Application to long time behavior

Let $\mu > 0$, fix $\theta \in (0, \mu)$. Recall that

$$\phi_\theta(x) = e^{\theta\sqrt{|x|^2+1}},$$

$$\mathcal{E}_\theta(\bar{Q}) = \{v : \bar{Q} \to \mathbb{R} : \lim_{|x|\to+\infty} \sup_{0\le t < T} \frac{v(x,t)}{\phi_\theta(x)} = 0\},$$

where $Q = \mathbb{R}^N \times [0, \infty)$.

## Theorem 2 (Long time behavior of solution)

*Let $u \in \mathcal{E}_\theta(Q)$ be the unique solution of* (HJt) *and* $(c, v) \in \mathbb{R} \times \mathcal{E}_\gamma(\mathbb{R}^N)$, $\gamma > \theta$, *be a solution of* (HJErg). *Suppose that the equations are* **nondegenerate** *(in the both local and nonlocal case). Then there is a constant $a \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \max_{B(0,R)} |u(x,t) - (ct + v(x) + a)| = 0 \quad \text{for all } R > 0.$$

# Main ideas of proof (1) (inspired by [FIL-06]'s work).

• By the Lipschitz regularity of solutions of the (HJt) we have the compactness property of the set $\{u(\cdot, t) - ct, t \geq 0\}$.

• The functions

$$
\begin{aligned}
v^+(x) &= \limsup_{t \to \infty}(u(x, t) - ct), \\
v^-(x) &= \liminf_{t \to \infty}(u(x, t) - ct)
\end{aligned}
$$

are respectively a sub and supersolution of (HJErg).

By strong maximum principle ([Bardi-Da Lio-99, Ciomaga-12]), there are $a, b \in \mathbb{R}$ such that

$$v^+(x) = v(x) + a, \quad v^-(x) = v(x) + b, \quad \Rightarrow b \leq a.$$

$\Rightarrow$ We only need to prove $a \leq b$.

# Main ideas of proof (2)

**We prove** $a \leq b$ : Let $T, \epsilon > 0$, $\gamma > \theta > 0$, $\phi_\gamma$ be the growth function, $K > e^\gamma$ be fixed so that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma \rangle - C|D\phi_\gamma| \geq \phi_\gamma(x) - K, \quad \forall x \in \mathbb{R}^N.$$

Hence, the functions

$$\eta^T(x, t) = u(x, t+T) - ct - v(x) - b, \ g_\epsilon(x, t) = \epsilon(\phi_\gamma(x) - K)e^{-t} + \epsilon(1 + K)$$

are respectively a sub and supersolution of

$$\eta_t - \mathcal{F}(x, [\eta]) + \langle b(x), D\eta \rangle - L_H|D\eta| = 0 \text{ in } \mathbb{R}^N \times (0, \infty).$$

By comparison principle we get $\eta^T \leq g_\epsilon$. Taking $x = 0$ we have

$$\begin{aligned}
u(0, t+T) &\leq v(0) + b + \epsilon(\phi_\gamma(0) - K)e^{-t} + \epsilon(1 + K) \\
&\leq v(0) + b + \epsilon(1 + K).
\end{aligned}$$

Sending $t \to \infty$ along a sequence yields $v^+(0) \leq v(0) + b + \epsilon(1 + K)$ which implies $a \leq b$, as $\epsilon > 0$ is arbitrary.

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# Thank you for your attention!