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ANNEXE 3.7

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The Schrödinger equation on a star-shaped graph under general coupling conditions.

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THE SCHRÖDINGER EQUATION ON A STAR-SHAPED GRAPH UNDER GENERAL COUPLING CONDITIONS

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Abstract. We investigate dispersive and Strichartz estimates for the Schrödinger time evolution propagator $\mathrm{e}^{-\mathrm{i}tH}$ on a star-shaped metric graph. The linear operator, H, taken into consideration is the self-adjoint extension of the Laplacian, subject to a wide class of coupling conditions. The study relies on an explicit spectral representation of the solution in terms of the resolvent kernel which is further analyzed using results from oscillatory integrals. As an application, we obtain the global well-posedness for a class of semilinear Schrödinger equations.

Keywords: Quantum graphs, Schrödinger operator, dispersion, Strichartz estimates, nonlinear Schrödinger equation, spectral theory, star-shaped network.

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1. Introduction

It is well-known that in the case of the linear Schrödinger equation on $\mathbb{R}$

$$\begin{cases} \mathrm{i}u_t(t,x) + \Delta u(t,x) = 0, & t \neq 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & \qquad\qquad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

the unitary group, $(\mathrm{e}^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$, possesses two important properties which can be derived for instance via Fourier transform (e.g. [22, Theorem IX.30]): the conservation of the L^2-norm

$$\left\|\mathrm{e}^{\mathrm{i}t\Delta}u_0\right\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R},$$

and the dispersion property

$$\left\|\mathrm{e}^{\mathrm{i}t\Delta}u_0\right\|_{L^\infty(\mathbb{R})} \lesssim \frac{1}{\sqrt{|t|}}\|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0.$$

Once the previous inequalities are obtained, the following holds by interpolation for all $p \in [1,2]$:

(i) Dispersive estimates:

$$\left\|\mathrm{e}^{\mathrm{i}t\Delta}u_0\right\|_{L^{p'}(\mathbb{R})} \lesssim |t|^{\frac{1}{2}-\frac{1}{p}}\|u_0\|_{L^p(\mathbb{R})}, \ t \neq 0.$$

Moreover, by a classical result of Keel and Tao in [14] more general space-time estimates, known as Strichartz estimates, also hold:

(ii) Strichartz estimates (homogeneous, dual homogenous and inhomogeneous):

$$\|\mathrm{e}^{\mathrm{i}t\Delta}u_0\|_{L^q_t(\mathbb{R},L^r_x(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})},$$

$$\Big\| \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}s\Delta} F(s,\cdot)ds \Big\|_{L^2(\mathbb{R})} \lesssim \|F\|_{L^{q'}_t(\mathbb{R},L^{r'}_x(\mathbb{R}))},$$

$$\Big\| \int_{s<t} \mathrm{e}^{\mathrm{i}(t-s)\Delta} F(s,\cdot)ds \Big\|_{L^q_t(\mathbb{R},L^r_x(\mathbb{R}))} \lesssim ||F||_{L^{\tilde{q}'}_t(\mathbb{R},L^{\tilde{r}'}_x(\mathbb{R}))},$$

where (q,r) and $(\tilde{q},\tilde{r})$ are sharp $1/2$–admissible exponent pairs. We recall that (q,r) is $1/2$–admissible if $2 \leq q,r \leq \infty$ and satisfy the relation

$$\frac{1}{q} = \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{r}\Big). \tag{1.2}$$

These estimates are key ingredients in order to obtain well-posedness of the nonlinear Schrödinger equation for a class of power nonlinearities (e.g. [27, 7, 18]).

In this paper we derive analogous dispersive and Strichartz estimates for the Schrödinger time evolution propagator $\mathrm{e}^{\mathrm{i}t\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))$ in the case of a star-shaped metric graph, with general boundary conditions at the common vertex which induce a self-adjoint extension of the Laplacian $\Delta(A,B)$, where $\mathcal{P}_{ac}(-\Delta(A,B))$ denotes the projection onto the absolutely continuous spectral subspace of $L^2(\mathcal{G})$ associated to $-\Delta(A,B)$. The considered star-graph consists of a finite number of edges of infinite length attached to a common vertex, each of them being identified with a copy of the positive semi-axis. We point out that in this context we cannot use Fourier analysis to obtain the dispersive properties, as in the case of the real line. In order to overcome this impediment we adopt a spectral theoretical approach and we use some tools specific to oscillatory integrals.

It is worth to mention that in the case of a star-shaped metric graph these results were established in [1] only for particular cases of boundary conditions, namely Kirchhoff, δ and δ' coupling conditions. So, this makes our result, to the best of our knowledge, new. Also for star-graphs, in the case of Kirchhoff boundary conditions, similar results were obtained in [19] where the Schrödinger group $\mathrm{e}^{\mathrm{i}t(-\Delta+V)}$ is considered for real valued potentials V satisfying some regularity and decay assumptions. We mention here that the latter is still in open problem in the case of more general boundary conditions. Bănică and Ignat proved the same results in the case of trees with the last generation of edges of infinite length, with Kirchhoff coupling condition at the vertices, in the joint work [3] and in [12]. With the same conditions, dispersive estimates were obtained in the case of the tadpole graph in [20]. The extension of the results in this paper to general trees as in [3, 2] remains to be investigated, but new ideas have to be used.

The paper is organized as follows: in Section 2, we give some preliminaries on Laplacians on metric star-graphs. We start by fixing some notations, we define the function spaces, we introduce the Laplace operator on star-graph, moreover, we recall a general result of Kostrykin and Schrader [17] that provides necessary and sufficient conditions for the self-adjointness of the Laplacian. Then we introduce the Schrödinger equation on a star-shaped metric graph and we state the main results of this paper, Theorem A concerning the dispersive and Strichartz estimates, and Theorem B, a well-posedness result of the nonlinear equation.

In Section 3 we give the proofs of the main results following several steps. First we localize the absolutely continuous spectrum of the operator and then we give an explicit description of the solution of system (2.2) using spectral theory tools. The dispersion properties are obtained based on this explicit form, relying on tools specific to oscillatory integrals, and the Strichartz space-time estimates follow as soon as the previously mentioned properties are proven. Finally, we prove the well-posedness in $L^2(\mathcal{G})$ in mild form of the semi-linear Schrödinger equation for a class of nonlinearities in the sub-critical case (the terminology in this framework is described in detail in [6]).

2. Preliminaries and main results

In the following we collect a few results necessary for a self-contained presentation as possible. For further details we refer to [5, 21] and references therein.

Definition 2.1. *A discrete graph $\mathcal{G} := (V, E, \partial)$ consists of a finite or countably infinite set of vertices $V = \{v_i\}$, a set of adjacent edges at the vertices $E = \{e_j\}$ of positive length $l_j \in (0, \infty]$ (namely internal edge if $l_j < \infty$ and external edge if $l_j = \infty$), and an orientation map $\partial : E \to V \times V$ which associates to each internal e_j edge the pair $(\partial_- e_j, \partial_+ e_j)$ of its initial and terminal vertex, and to an external edge its initial vertex only.*

Each edge $e_j \in E$ of the graph can be identified with a finite or infinite segment $I_j = [0, l_j]$ of the real line. This defines a natural topology on $\mathcal{G}$ (the space of union of all edges).

Definition 2.2. *A metric graph is a discrete graph equipped with a natural metric, with the distance of two points to be the length of the shortest path in $\mathcal{G}$ linking the points.*

In the sequel, we consider a metric graph $\mathcal{G}$ as below, given by a finite number $n \in \mathbb{N}^*, n \geq 3$ of infinite length edges attached to a common vertex (a so called star-shaped metric graph), having each edge identified with a copy of the half-line $[0, \infty)$. A function u defined on $\mathcal{G}$ is a vector $u = (u_1, \dots, u_n)^T$, each u_j being defined on the interval $I_j = [0, \infty)$, $j = 1, \dots, n$.

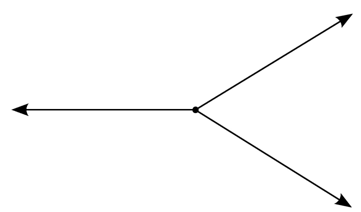

Figure 1: The star-graph $\mathcal{G}$ (one vertex, n edges)

The Lebesgue measures on the intervals I_j induce a Lebesgue measure on the space $\mathcal{G}$. We introduce the Hilbert space $L^2(\mathcal{G})$ as the space of measurable and square-integrable functions on each edge of $\mathcal{G}$, i.e.

$$L^2(\mathcal{G}) = \bigoplus_{j=1}^{n} L^2(I_j), \quad \|f\|^2_{L^2(\mathcal{G})} = \sum_{j=1}^{n} \int_{I_j} |f_j(s)|^2 ds,$$

with $f = (f_j)^T_{j=\overline{1,n}}$, where $f_j \in L^2(I_j)$ is a complex valued function. The inner product $\langle \cdot, \cdot \rangle$ is the one induced by the usual inner products in $L^2([0,\infty))$, i.e.

$$\langle f, g \rangle = \sum_{j=1}^{n} \int_{I_j} f_j(s)\overline{g_j}(s)\, ds.$$

Analogously, given $1 \le p \le \infty$ one can define $L^p(\mathcal{G})$ as the set of functions whose components are elements of $L^p(I_j)$, and the corresponding norm defined by

$$\|f\|^p_{L^p(\mathcal{G})} = \sum_{j=1}^{n} \|f\|^p_{L^p(I_j)} \quad for \quad 1 \le p < \infty, \quad and \quad \|f\|_{L^\infty(\mathcal{G})} = \sup_{1\le j\le n} \|f\|_{L^\infty(I_j)}.$$

In order to define Laplacian operators, we consider the Sobolev space

$$H^2(\mathcal{G}) = \bigoplus_{j=1}^{n} H^2(I_j), \quad \|f\|^2_{H^2(\mathcal{G})} = \sum_{j=1}^{n} \|f\|^2_{H^2(I_j)},$$

where $H^2(I_j)$ is the classical Sobolev space on I_j, $j = 1, \dots, n$.

In the sequel we introduce the Laplacian $\Delta(A, B)$ being the operator with the domain

$$\mathcal{D}(\Delta(A,B)) = \{u \in H^2(\mathcal{G}) : A\underline{u} + B\underline{u'} = 0\}, \tag{2.1}$$

acting as the second derivative along the edges,

$$\Delta(A,B)u = (u_1'', u_2'', \dots, u_n'')^T.$$

In the definition of the domain, A and B are $\mathbb{C}^{n\times n}$ matrices which express the coupling condition at the common vertex, and $\underline{u} = (u_1(0), \dots, u_n(0))^T$, $\underline{u'} = (u_1'(0_+), \dots, u_n'(0_+))^T$, respectively.

A crucial result concerning the parametrization of all self-adjoint extensions of the Laplace operator in $L^2(\mathcal{G})$ in terms of the boundary conditions, due to [17], is the following: for any two $n \times n$ matrices A and B, the next two assertions are equivalent:

- (i) The operator $\Delta(A, B)$ is self-adjoint;
- (ii) A and B satisfy:
  - (H1) The horizontally concatenated matrix (A, B) has maximal rank;
  - (H2) $AB^\dagger$ is self-adjoint, i.e. $AB^\dagger = BA^\dagger$.

Matrices A and B satisfying (H1) and (H2) may also appear in the literature under the name of *Nevanlinna* pair (e.g. [4]). The most used type of couplings that satisfy the above hypotheses (H1) and (H2) are the following ones:

(1) *Kirchhoff–coupling:*

$$\psi_i(0) = \psi_j(0), \quad i, j = 1, \dots, n, \qquad \sum_{j=1}^{n} \psi_j'(0_+) = 0;$$

(2) *δ–coupling:*

$$\psi_i(0) = \psi_j(0), \quad i, j = 1, \dots, n, \qquad \sum_{j=1}^{n} \psi_j'(0_+) = \alpha \psi_k(0), \ \alpha \in \mathbb{R};$$

(3) *δ'–coupling:*

$$\psi_i'(0_+) = \psi_j'(0_+), \quad i, j = 1, \dots, n, \quad \sum_{j=1}^{n} \psi_j(0) = \beta \psi_k'(0_+), \ \beta \in \mathbb{R}.$$

The subject of self-adjoint Laplacians on metric graphs has become popular under the name of "quantum of graphs", known for its wide applications in quantum mechanics. Since the dynamics of a quantum system is described by unitary operators, most of the literature has been concerned with self-adjoint operators. However, in [11], non-self-adjoint Laplacians on graphs were considered, since there are cases where a system is described by non-conservative equations of motion. Also, in [15], contraction semigroups which are generated by Laplace operators which are not necessarily self-adjoint are studied. They provide a criteria for such semigroups to be continuity and positivity preserving and a characterization of generators of Feller semigroups on metric graphs.

The model we will consider in this paper is given by the following linear Schrödinger equation

$$\begin{cases} \mathrm{i}u_t(t,x) + \Delta(A,B)u(t,x) = 0, & t \neq 0, \quad x \in \mathcal{G}, \\ u(0,x) = u_0(x), & \qquad\quad\;\; x \in \mathcal{G}, \end{cases} \tag{2.2}$$

where matrices A and B are assumed to satisfy (H1) and (H2). We will give now the statements of the main results of our paper.

In the following, $\mathrm{e}^{\mathrm{i}t\Delta(A,B)}$ is the unitary group generated by $\Delta(A,B)$ with domain (2.1), $\mathcal{P}_{ac}(-\Delta(A,B))$ denotes the projection onto the absolutely continuous spectral subspace of $L^2(\mathcal{G})$ associated to $-\Delta(A,B)$. Along the paper (q,r) and $(\tilde{q},\tilde{r})$ are sharp $1/2$–admissible exponent pairs, i.e. (1.2) holds true, unless we specify otherwise and p' stands for the dual exponent of p. Also in order to simplify the presentation, by writing $f \lesssim g$ we understand that the inequality holds up to some positive multiplicative constants that depends on the matrices A and B and on the involved exponents.

Theorem A. *The time-evolution propagator* $\mathrm{e}^{\mathrm{i}t\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))$ *satisfies*

(i) Dispersive estimates: for all $p \in [1,2]$,

$$\left\| \mathrm{e}^{\mathrm{i}t\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))u_0 \right\|_{L^{p'}(\mathcal{G})} \leq C(A,B,p)|t|^{\frac{1}{2}-\frac{1}{p}} \|u_0\|_{L^p(\mathcal{G})}, \forall\ t \neq 0; \tag{2.3}$$

(ii) Strichartz estimates (homogeneous, dual homogenous and inhomogeneous):

$$\|e^{it\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))u_0\|_{L^q(\mathbb{R},L^r(\mathcal{G}))} \lesssim \|u_0\|_{L^2(\mathcal{G})},$$

$$\Big\|\int_{\mathbb{R}} e^{-is\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))F(s,\cdot)ds\Big\|_{L^2(\mathcal{G})} \lesssim \|F\|_{L^{q'}(\mathbb{R},L^{r'}(\mathcal{G}))},$$

$$\Big\|\int_{s<t} e^{i(t-s)\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))F(s,\cdot)ds\Big\|_{L^q(\mathbb{R},L^r(\mathcal{G}))} \lesssim ||F||_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathcal{G}))}.$$

In all the estimates above one requires the projection onto the absolutely continous spectrum as the Laplace operator $-\Delta(A,B)$ on $L^2(\mathcal{G})$ may posses non-empty point spectrum, for which the time decay in (2.3) of the unitary group $e^{it\Delta(A,B)}$ cannot occur and then the global estimates in the second part do not hold. If we need only local in time estimates, as in the case of the proof of the local in time existence of solutions for nonlinear problems, the restriction on $\mathcal{P}_{ac}(-\Delta(A,B))$ is not required and the following holds true.

Corollary A 1. *For every $\alpha \geq 1$,*

$$\|e^{it\Delta(A,B)}u_0\|_{L^q((-T,T),L^r(\mathcal{G}))} \lesssim \|u_0\|_{L^2(\mathcal{G})} + C(\alpha,r)(2T)^{1/q}\|u_0\|_{L^\alpha(\mathcal{G})}$$

and

$$\Big\|\int_0^t e^{i(t-s)\Delta(A,B)}F(s,\cdot)ds\Big\|_{L^q((-T,T),L^r(\mathcal{G}))} \lesssim ||F||_{L^{\tilde{q}'}((-T,T)),L^{\tilde{r}'}(\mathcal{G}))} + C(\alpha,r)(2T)^{1/q}||F||_{L^1((-T,T)),L^\alpha(\mathcal{G}))}.$$

We emphasize that choosing $T<1$, $\alpha=2$ and $\alpha=\tilde{r}'$ in the first respectively second estimate we obtain local Strichartz estimates similar to the ones in the classical case.

As an application of Corollary A1, we give the following well-posedness result.

Theorem B. *For any $p\in(1,5)$ and for every $u_0\in L^2(\mathcal{G})$, there exists a unique mild solution $u\in C(\mathbb{R},L^2(\mathcal{G}))\bigcap_{(q,r)-adm} L^q_{loc}(\mathbb{R},L^r(\mathcal{G}))$ of the nonlinear Schrödinger equation*

$$\begin{cases} iu_t(t,x)+\Delta(A,B)u(t,x)+|u|^{p-1}u=0, & t\neq 0, \quad x\in\mathcal{G},\\ u(0,x)=u_0(x), & x\in\mathcal{G}, \end{cases} \tag{2.4}$$

where the intersection $\bigcap_{(q,r)-adm}$ is taken over all sharp $1/2-$admissible exponent pairs. Moreover, the $L^2(\mathcal{G})$-norm of u is preserved along time:

$$\|u(t)\|_{L^2(\mathcal{G})} = \|u_0\|_{L^2(\mathcal{G})}.$$

We recall that the nonlinearity in (2.4) is understood component-wise, i.e. $|u|^{p-1}u = (|u_1|^{p-1}u_1,\dots,|u_n|^{p-1}u_n)^T$.

The study of nonlinear propagation in ramified structures is of relevance in several branches of pure and applied science, modeling phenomena such as nonlinear electromagnetic pulse propagation in optical fibers, the hydrodynamic flow, electrical signal propagation in the nervous system, etc. For a large classification of applications and references, we refer to [5, Chapter 7]. We also point out here that the modelization depends on the real network, where each edge has a thickness, but usually idealizations of these graphs are considered. The convergence of these so-called graph-like spaces to metric graphs (with 0-thickness limit) is analyzed in [21, 10, 28].

3. Proofs of the main results

3.1. The solution via the resolvent. In the sequel we make use of the explicit representation of the integral resolvent kernel:

Lemma 3.1. [17, Lemma 4.2] *The resolvent* $(-\Delta(A,B)-\mathrm{k}^2)^{-1}$*, for* $\mathrm{k}^2 \in \mathbb{C}\setminus\sigma(-\Delta(A,B))$*,* $\Im(\mathrm{k}) > 0$*, is the integral operator with the* $n \times n$ *matrix-valued integral kernel* $\mathrm{r}(x,y,\mathrm{k})$*, admitting the representation*

$$\mathrm{r}(x,y,\mathrm{k}) = \mathrm{r}^{(0)}(x,y,\mathrm{k}) + \frac{\mathrm{i}}{2\mathrm{k}}\phi(x,\mathrm{k})G(\mathrm{k},A,B)\phi(y,\mathrm{k}), \tag{3.1}$$

with $[\mathrm{r}^{(0)}(x,y,\mathrm{k})]_{j,j'} = \dfrac{\mathrm{i}}{2\mathrm{k}}\delta_{j,j'}\mathrm{e}^{\mathrm{i}\mathrm{k}|x_j-y_{j'}|}$, $x_j \in I_j, y_{j'} \in I_{j'}$, $\phi(x,\mathrm{k}) = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}\mathrm{k}x_j}\}_{j=\overline{1,n}}$, $\phi(y,\mathrm{k}) = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}\mathrm{k}y_j}\}_{j=\overline{1,n}}$ *and* $G(\mathrm{k},A,B) = -(A+\mathrm{i}\mathrm{k}B)^{-1}(A-\mathrm{i}\mathrm{k}B)$.

We start by localizing the absolutely continuous spectrum of the self-adjoint Laplacian on the star-graph by using some classical results of Weyl and information concerning the eigenvalues, provided by Kostrykin and Schrader.

Proposition 3.2. *Assume* $n \times n$ *matrices* A *and* B *satisfy hypotheses (H1) and (H2). Then the absolutely continuous spectrum of the corresponding Hamiltonian* $-\Delta(A,B)$ *with domain given in* (2.1) *is the interval* $[0,\infty)$.

We include below the proof of this proposition. However, we would like to mention that we recently encountered a similar result in [23].

Proof. The operator not being positive definite, the proof is thus not immediate, so we perform a different approach. Since the self-adjoint operator $-\Delta(A,B)$ may have at most a finite collection of negative eigenvalues [17, Theorem 3.7], $\sigma_{ac}(-\Delta(A,B)) = \sigma_{ess}(-\Delta(A,B))$. Now, the idea is to show first that $\sigma_{ess}(-\Delta(A,B)) = \sigma_{ess}(-\Delta(A' = \mathbb{I}_n, B' = \mathbb{O}_n))$, where $-\Delta(A' = \mathbb{I}_n, B' = \mathbb{O}_n)$ is the Hamiltonian with homogeneous Dirichlet boundary conditions, and secondly, that $\sigma_{ess}(-\Delta(A' = \mathbb{I}_n, B' = \mathbb{O}_n)) = [0,\infty)$.

Since both Laplacians are self-adjoint (they fulfill (H1) and (H2)), in order to prove the first part, it is enough to show that the resolvent difference $R_{-\Delta(A,B)} - R_{-\Delta(A'=\mathbb{I}_n,B'=\mathbb{O}_n)}$ is a compact operator for some element (and hence, for all) in both resolvent sets, due to Theorem of Weyl in [26, Theorem 6.19, p. 146]. Using each resolvent's representation in terms of its explicit integral kernel expressed in (3.1), one can check that the difference of resolvents is a Hilbert Schmidt integral operator for some element in both resolvent sets, and hence, compact (take for instance the spectral parameter k^2 to be any negative real number less than the smallest negative eigenvalue, given in [17, Remark 3.11].

For the second part, we will proceed by double inclusion. Since $-\Delta(A' = \mathbb{I}_n, B' = \mathbb{O}_n)$ posses no point spectrum and it is a positive definite self-adjoint operator, $\sigma_{ess}(-\Delta(A' = \mathbb{I}_n, B' = \mathbb{O}_n)) = \sigma(-\Delta(A' = \mathbb{I}_n, B' = \mathbb{O}_n))$ and the direct inclusion follows immediately. For the reverse inclusion, we rely on the criteria of Weyl [26, Lemma 2.16]. □

Description of the solution. By Stone's Theorem in [25], which ensures the existence and uniqueness of an unitary group $(\mathrm{e}^{\mathrm{i}t\Delta(A,B)})_{t\in\mathbb{R}}$, the linear Schrödinger equation (2.2) is well-posed and the unique global solution is given by $\mathrm{e}^{\mathrm{i}t\Delta(A,B)}u_0$, for all $t \in \mathbb{R}$, and thus, the study of its properties is consistent. In the sequel, we provide an explicit representation of the solution, more precisely, of a sequence of functions converging in $L^2(\mathcal{G})$ to the solution, and hence, almost everywhere along a sub-sequence.

A careful analysis of the integral resolvent kernel (3.1) leads to the following lemma.

Lemma 3.3. *There exists $\delta_0 = \delta_0(A, B)$ such that*

$$\sup_{x,y\in\mathcal{G},k\in\mathbb{R}\setminus\{0\},0\leq\delta\leq\delta_0} |\mathrm{kr}(x, y, \sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta})| < \infty \tag{3.2}$$

Moreover, for any $x, y \in \mathcal{G}$

$$\lim_{\delta\to 0} \mathrm{kr}(x, y, \sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta}) = \mathrm{kr}(x, y, \pm|\mathrm{k}|),\ \forall\, \mathrm{k} \in \mathbb{R} \setminus \{0\}.$$

Proof. In view of [16, Proposition 3.11], $\det(A + \mathrm{ik}B)$ has zeros only on the imaginary axis. Denote by ρ its smallest non-zero root in absolute value. We will chose latter δ_0 such that $\delta_0 \ll \rho$.

By Lemma 3.1, for $\mathrm{k} \in \mathbb{R}$ and $j, j' = 1, \dots, n$,

$$[\,\mathrm{kr}(x, y, \sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta})\,]_{j,j'} = \frac{\mathrm{ik}}{2\sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta}} \Big[\delta_{j,j'}\mathrm{e}^{\mathrm{i}\sqrt{\mathrm{k}^2\pm i\delta}|x_j-y_{j'}|}+\mathrm{e}^{\mathrm{i}\sqrt{\mathrm{k}^2\pm i\delta}(x_j+y_{j'})}G_{j,j'}(\sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta}, A, B)\Big], \tag{3.3}$$

where $[\cdot]_{j,j'}$ denotes the j, j' entry of the corresponding matrix. Recall that here the complex square root is chosen in such a way that $\sqrt{r\mathrm{e}^{\mathrm{i}\theta}} = \sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}$ with $r > 0$ and $\theta \in [0, 2\pi)$.

We analyze the elements of the matrix $G(\sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta}, A, B)$, making use of the properties of $G(\mathrm{k}, A, B) = (A + \mathrm{ik}B)^{-1}(A - \mathrm{ik}B)$, for $\mathrm{k} \in \mathbb{R} \setminus \{0\}$. Using the matrix inverse formula, each element of $(A + \mathrm{ik}B)^{-1}$ is the quotient of two polynomials, except for a finite number of complex values k, i.e. the complex zeros of $\det(A + \mathrm{ik}B)$. It follows that for any $1 \leq j, j' \leq n$, there exist two complex polynomials $p_{jj'}, q_{jj'} \in \mathbb{C}[X]$ such that, excepting a finite number o values of k, $G_{j,j'}(\mathrm{k}, A, B)$ is the quotient of the two polynomials, $p_{jj'}(\mathrm{k}), q_{jj'}(\mathrm{k})$. Since $\det(A + \mathrm{ik}B)$ is a polynomial in k with zeros on the imaginary axis it follows that the following holds in a strip around the real axis

$$G_{j,j'}(z, A, B) = \frac{p_{jj'}(z)}{q_{jj'}(z)}, \quad |\Im z| < \rho, z \neq 0.$$

Moreover $G(\mathrm{k}, A, B)$ is a unitary matrix for any $\mathrm{k} \in \mathbb{R} \setminus \{0\}$ (due to [16, Proposition 3.7]). It follows that the above polynomials satisfy for any $\mathrm{k} \in \mathbb{R} \setminus \{0\}$

$$\sum_{l=1}^{n} \frac{p_{jl}(\mathrm{k})}{q_{jl}(\mathrm{k})} \overline{\frac{p_{j'l}(\mathrm{k})}{q_{j'l}(\mathrm{k})}} = \delta_{jj'}, \qquad j, j' = 1, \dots, n.$$

Consequently, all the involved quotients satisfy

$$\left| \frac{p_{jj'}(\mathrm{k})}{q_{jj'}(\mathrm{k})} \right| \leq 1, \quad j, j' = 1, \dots, n, \ k \in \mathbb{R} \setminus \{0\}. \tag{3.4}$$

This implies in particular that the degrees of the involved polynomials satisfy $\deg(p_{jj'}) \leq \deg(q_{jj'})$ for any $1 \leq j, j' \leq n$. In order to control the behaviour of $G_{j,j'}(z)$ near $z = 0$ we decompose the above polynomials as

$$p_{jj'}(\mathrm{k}) = \tilde{p}_{jj'}(\mathrm{k}) p^*_{jj'}(\mathrm{k}), \quad q_{jj'}(\mathrm{k}) = \tilde{q}_{jj'}(\mathrm{k}) q^*_{jj'}(\mathrm{k}),$$

where $\tilde{p}_{jj'}$ and $\tilde{q}_{jj'}$ denote the monomials with the largest degree such that $p^*_{jj'}$ and $q^*_{jj'}$ are polynomials with non-zero constant term. Using again property (3.4) for $\mathrm{k} \sim 0$ we obtain that the degrees of the new polynomials can also be compared, more precisely $\deg(\tilde{p}_{jj'}) \leq \deg(\tilde{q}_{jj'})$. The above properties of the involved polynomials imply that for any $1 \leq j, j' \leq n$ the following holds

$$\sup_{|\Im z| \leq \rho/2} |G_{j,j'}(z, A, B)| \leq C(A, B) < \infty. \tag{3.5}$$

We now express the above estimate in terms of k and δ by choosing $z = \sqrt{\mathrm{k}^2 \pm i\delta}$. For all real numbers k and $\delta \geq 0$, the following hold

$$0 \leq \Im \sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta} \leq (\delta/2)^{1/2}, \tag{3.6}$$

$$|\mathrm{k}| \leq \Re \sqrt{\mathrm{k}^2 + i\delta} \leq \sqrt{\mathrm{k}^2 + \frac{\delta}{2}},$$

$$-\sqrt{\mathrm{k}^2 + \frac{\delta}{2}} \leq \Re \sqrt{\mathrm{k}^2 - i\delta} \leq -|\mathrm{k}|.$$

We choose $\delta_0 = 2(|\rho|/2)^2$ such that $(\delta_0/2)^{1/2} = \rho/2$. Let us fix $\delta \leq \delta_0$. Using (3.6) and (3.5) we obtain that $G_{jj'}(\sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta}, A, B)$ is uniformly bounded for $0 \leq \delta \leq \delta_0$ and $\mathrm{k} \in \mathbb{R}$. In view of representation (3.3) we obtain the desired estimate (3.2).

Let us fix two points on $\mathcal{G}$, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, $x_i \geq 0$, $y_i \geq 0$ for all $i = \{1, \dots, n\}$, and choose $k \in \mathbb{R} \setminus \{0\}$. Notice that the matrix $G(|\mathrm{k}|, A, B)$ is well defined. Since $\sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta} \to \pm|k|$ as $\delta > 0$ tends to zero and the composing terms are continuous as function of k it is immediate that

$$\lim_{\delta \to 0} \mathrm{kr}(x, y, \sqrt{\mathrm{k}^2 \pm \mathrm{i}\delta}) = \mathrm{kr}(x, y, |\mathrm{k}|), \ \forall \mathrm{k} \neq 0.$$

The proof is now finished. □

We denote by $C_0^\infty(\mathcal{G})$ the set of all functions $f = (f_j)_{j=\overline{1,n}}^T$ such that each f_j belongs to the space of compactly supported infinitely differentiable functions on the interior of the edge $\mathring{I}_j = (0, \infty)$.

Proposition 3.4. *For every test function $u_0 \in C_0^\infty(\mathcal{G})$, the solution for the linear Schrödinger equation (2.2) can be written as*

$$\mathrm{e}^{\mathrm{i}t\Delta(A,B)}\mathcal{P}_{ac}(-\Delta(A,B))u_0(x) = \lim_{\epsilon\searrow 0} \frac{1}{\pi \mathrm{i}} \int_{\mathcal{G}} u_0(y) \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2} \mathrm{k}\mathrm{r}(x,y,\mathrm{k})\ d\mathrm{k}\ dy, \tag{3.7}$$

where the limit is taken in $L^2(\mathcal{G})$.

Proof. In the sequel, for the ease of notation, we may refer to $-\Delta(A,B)$ also as H. Let $u_0 \in C_0^\infty(\mathcal{G})$. By [26, Theorem 3.1], it holds that

$$\mathrm{e}^{-\mathrm{i}tH}u_0 = \lim_{\epsilon\searrow 0} \mathrm{e}^{-(\mathrm{i}t+\epsilon)H}u_0 \quad in\ L^2(\mathcal{G}).$$

Since the absolutely continuous spectrum of H is the interval $[0,\infty)$, by the spectral theorem of representation via the resolvent for bounded functions of unbounded self-adjoint operators [9, Theorem XII.2.11], we have

$$\mathrm{e}^{-(\mathrm{i}t+\epsilon)H}\mathcal{P}_{ac}(H)u_0(x) = \lim_{\delta\to 0} \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-(\mathrm{i}t+\epsilon)\lambda} \left[R_{\lambda+\mathrm{i}\delta} - R_{\lambda-\mathrm{i}\delta}\right] u_0(x)\ d\lambda,$$

where $R_z = (H-z)^{-1}$. Hence, using the resolvent representation in Lemma 3.1 together with with *the dominated convergence theorem*,

$$\begin{aligned}
&\mathrm{e}^{-(\mathrm{i}t+\epsilon)H}\mathcal{P}_{ac}(H)u_0(x) = \\
&\quad = \lim_{\delta\to 0} \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-(\mathrm{i}t+\epsilon)\lambda} \int_{\mathcal{G}} \left[\mathrm{r}(x,y,\sqrt{\lambda+\mathrm{i}\delta}) - \mathrm{r}(x,y,\sqrt{\lambda-\mathrm{i}\delta})\ \right] u_0(y)\ dy\ d\lambda \\
&\quad = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-(\mathrm{i}t+\epsilon)\lambda} \lim_{\delta\to 0} \int_{\mathcal{G}} \left[\mathrm{r}(x,y,\sqrt{\lambda+\mathrm{i}\delta}) - \mathrm{r}(x,y,\sqrt{\lambda-\mathrm{i}\delta})\ \right] u_0(y)\ dy\ d\lambda.
\end{aligned}$$

Making now the change of variables $\mathrm{k} = \sqrt{\lambda}$ in the integral containing $\mathrm{r}(x,y,\sqrt{\lambda+\mathrm{i}\delta})$ and $\mathrm{k} = -\sqrt{\lambda}$ in the integral containing $\mathrm{r}(x,y,\sqrt{\lambda-\mathrm{i}\delta})$ leads to

$$\begin{aligned}
\mathrm{e}^{-(\mathrm{i}t+\epsilon)H}\mathcal{P}_{ac}(H)u_0(x) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^0 \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2} \lim_{\delta\searrow 0} \int_{\mathcal{G}} \mathrm{k}\mathrm{r}(x,y,\sqrt{\mathrm{k}^2-\mathrm{i}\delta})\ u_0(y)\ dy\ d\mathrm{k} \\
+ \frac{1}{\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2} \lim_{\delta\searrow 0} \int_{\mathcal{G}} \mathrm{k}\mathrm{r}(x,y,\sqrt{\mathrm{k}^2+\mathrm{i}\delta})\ u_0(y)\ dy\ d\mathrm{k}.
\end{aligned}$$

Using again *the dominated convergence* and *Fubini's* theorems, together with Lemma 3.3 we finally obtain

$$\mathrm{e}^{-(\mathrm{i}t+\epsilon)H}\mathcal{P}_{ac}(H)u_0(x) =$$
$$= \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{0} \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2}\int_{\mathcal{G}} \mathrm{kr}(x,y,-|\mathrm{k}|)u_0(y)dyd\mathrm{k} + \frac{1}{\pi \mathrm{i}}\int_{0}^{\infty} \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2}\int_{\mathcal{G}} \mathrm{kr}(x,y,|\mathrm{k}|)u_0(y)dyd\mathrm{k}$$
$$= \frac{1}{\pi \mathrm{i}}\int_{\mathcal{G}} u_0(y)\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\epsilon)k^2}\mathrm{kr}(x,y,\mathrm{k})\ d\mathrm{k}\ dy.$$

Passing now to the limit $\epsilon \to 0$ in $L^2(\mathcal{G})$, we obtain the desired representation

$$\mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_{ac}(H)u_0(x) = \lim_{\epsilon\searrow 0}\frac{1}{\pi \mathrm{i}}\int_{\mathcal{G}} u_0(y)\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2}\mathrm{kr}(x,y,\mathrm{k})\ d\mathrm{k}\ dy,$$

which completes the proof. □

3.2. **Dispersive and Strichartz estimates.** Having the explicit form of the solution expressed in Proposition 3.4, we obtain first the $L^\infty-$time decay of the solution, using a classical result for oscillatory integrals and, as a consequence, $L^p \to L^{p'}$ dispersive properties. Together with the properties of the unitary group and the projection operator, the Strichartz estimates follow using a more general result of Keel and Tao.

Proof of Theorem A. We first establish the $L^p - L^{p'}$ decay and then the Strichartz estimates.

Step I. $L^p \to L^{p'}$ **estimates.** Let us first establish the $L^1 - L^\infty$ estimate. By density, it is sufficient to consider $u_0 \in C_0^\infty(\mathcal{G})$. In view of Proposition 3.7, each component $j = 1,\dots,n$ of the solution (3.7) can be rewritten as

$$\left(\mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_{ac}(H)u_0(x)\right)_j = \lim_{\epsilon\searrow 0}\frac{1}{\pi \mathrm{i}}\sum_{j'=1}^{n}\int_{I_{j'}}\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\epsilon)\mathrm{k}^2}(\mathrm{kr}(x,y,\mathrm{k}))_{j,j'}\ d\mathrm{k}\ u_{0j'}(y)dy, \qquad (3.8)$$

with

$$\mathrm{kr}(x,y,\mathrm{k}) = \frac{\mathrm{i}}{2}\mathrm{diag}(\mathrm{e}^{\mathrm{ik}|x_l-y_l|})_{l=\overline{1,n}} + \frac{\mathrm{i}}{2}\ \mathrm{diag}(\mathrm{e}^{\mathrm{ik}x_l})_{l=\overline{1,n}}\ G(\mathrm{k},A,B)\ \mathrm{diag}(\mathrm{e}^{\mathrm{ik}y_l})_{l=\overline{1,n}}.$$

Estimate (3.4) and the properties on the degrees of the polynomials we obtained in the proof of Lemma 3.3 show that there exists a positive constant $C(A,B)$ such that for all $i,j = 1,\dots,n$:

$$\sup_{\mathrm{k}\in\mathbb{R}}|G_{i,j}(\mathrm{k},A,B)| \le 1, \qquad \int_{\mathbb{R}}|G'_{i,j}(\mathrm{k},A,B)|\ d\mathrm{k} \le C(A,B) < \infty.$$

Since each component (3.8) is the limit of a sum of n integrals with respect to $y \in \mathcal{G}$, with oscillatory integrals with respect to $\mathrm{k} \in \mathbb{R}$ as integrands, applying Van Der Corput Lemma (e.g. [24, p. 334]), for each $j = 1,\dots,n$ follows that

$$\begin{aligned}\left\|\left(\mathrm{e}^{-\mathrm{it}H}\mathcal{P}_{ac}(H)u_0(x)\right)_j\right\|_{L^\infty(I_j)} &\le \frac{1}{\sqrt{t}}\sum_{j'=1}^{n}\left(||\mathrm{e}^{-\epsilon\mathrm{k}^2}G_{j,j'}||_{L^\infty(\mathbb{R})}+||(\mathrm{e}^{-\epsilon\mathrm{k}^2}G_{j,j'})'||_{L^1(\mathbb{R})}\right)||u_{0j'}||_{L^1(I_{j'})}\\ &\le \frac{1}{\sqrt{t}}\sum_{j'=1}^{n}\left(||G_{j,j'}||_{L^\infty(\mathbb{R})}+||G'_{j,j'}||_{L^1(\mathbb{R})}\right)||u_{0j'}||_{L^1(I_{j'})}\\ &\le C(A,B)\frac{1}{\sqrt{t}}\sum_{j'=1}^{n}||u_{0j'}||_{L^1(I_{j'})} = C(A,B)\frac{1}{\sqrt{t}}\|u_0\|_{L^1(\mathcal{G})}.\end{aligned}$$

Since $(\mathrm{e}^{-\mathrm{it}H})_{t\in\mathbb{R}}$ is a family of unitary operators preserving the L^2-norm we obtain

$$\left\|\mathrm{e}^{-\mathrm{it}H}\mathcal{P}_{ac}(H)u_0\right\|_{L^2(\mathcal{G})} \le ||u_0||_{L^2(\mathcal{G})}.$$

By *Riesz-Thorin's interpolation theorem*, one has that for all $p \in [1,2]$ and $t \neq 0$,

$$\left\|\mathrm{e}^{-\mathrm{it}H}\mathcal{P}_{ac}(H)u_0\right\|_{L^{p'}(\mathcal{G})} \le C(A,B,p)|t|^{-\frac{1}{p}+\frac{1}{2}}\ ||u_0||_{L^p(\mathcal{G})}.$$

Step II. Strichartz estimates. We employ here a result of Keel and Tao in [14], concerning Strichartz estimates in the following abstract setting: let (X, dx) be a measure space and $\mathcal{H}$ a Hilbert space. Suppose that for each time $t \in \mathbb{R}$, we have an operator $U(t) : \mathcal{H} \to L^2(X)$ which obeys estimates:

(i) For all t and $f \in \mathcal{H}$,
$$||U(t)f||_{L^2(X)} \lesssim ||f||_{\mathcal{H}};$$
(ii) For some $\sigma > 0$, for all $t \neq s$ and all $g \in L^1(X)$
$$||U(t)U^*(s)g||_{L^\infty(X)} \lesssim |t-s|^{-\sigma}\ ||g||_{L^1(X)}.$$

We recall that an exponent pair (q, r) is called sharp $\sigma-$admissible if $q, r \ge 2$, $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q}+\frac{\sigma}{r}=\frac{\sigma}{2}. \tag{3.9}$$

Theorem 3.5. [14] *If $U(t)$ obeys (i) and (ii), then the estimates*

(I)
$$\left\|U(t)f\right\|_{L^q_tL^r_x} \lesssim ||f||_{\mathcal{H}}$$

(II)
$$\left\|\int_{\mathbb{R}} U(t)U^*(s)F(s,\cdot)ds\right\|_{\mathcal{H}} \lesssim ||F||_{L^{q'}_tL^{r'}_x}$$

(III)
$$\left\|\int_{s<t} U(t)U^*(s)F(s,\cdot)ds\right\|_{L^q_tL^r_x} \lesssim ||F||_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x}$$

hold for all sharp σ-admissible exponent pairs (q, r), $(\tilde{q}, \tilde{r})$.

From the properties of the unitary group $\mathrm{e}^{-\mathrm{it}H}$, the projection operator $\mathcal{P}_{ac}(H)$ (mass conservation, self-adjointness and the comutativity property in [9, Corollary XII.2.7] and the dispersive estimate 2.3 in Theorem A, conditions (i) and (ii), are satisfied for $X = \mathcal{G}$,

$\mathcal{H} = L^2(\mathcal{G})$, $U(t) = \mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_{ac}(H)$ and $\sigma = \dfrac{1}{2}$, for $t \in \mathbb{R}$. Thus, by Theorem 3.5, the Strichartz estimates in Theorem A hold.

Hence, the proof of Theorem A is now complete. □

Proof of Corollary A1. We know by [17, Theorem 3.7] that in the case of a star-shaped metric graph the Hamiltonian $H = -\Delta(A, B)$ has at most a finite number of negative eigenvalues. More precisely, it is equal with $n_+(AB^\dagger)$, i.e. the number of positive eigenvalues of the matrix $AB^\dagger$. For every $j \in \{1, \ldots, n_+(AB^\dagger)\}$ the eigenfunction φ_j corresponding to the eigenvalue $\lambda_j = -k_j^2$, $k_j > 0$, is of type $c_j e^{-k_j x}$ on each edge parametrized by $x \in (0, \infty)$. Hence,

$$\varphi_j \in L^r(\mathcal{G}), \qquad \forall\ 1 \le r \le \infty.$$

For every $f \in L^2(\mathcal{G})$ the projection $\mathcal{P}_p(H)$ on the closed subspace spanned by the corresponding eigenfunctions $\{\varphi_j\}_{j=1}^{n_+(AB^\dagger)}$ is given by

$$\mathcal{P}_p(H)f = \sum_{j=1}^{n_+(AB^\dagger)} < f, \varphi_j > \varphi_j.$$

and

$$\mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_p(H)f = \sum_{j=1}^{n_+(AB^\dagger)} \mathrm{e}^{-\mathrm{i}\lambda_j t} < f, \varphi_j > \varphi_j.$$

Using that $\lambda_j \in \mathbb{R}$ and *Hölder's inequality*, we obtain that for every $\alpha \ge 1$,

$$\begin{aligned}\|\mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_p(H)f\|_{L^r(\mathcal{G})} &\le \sum_{j=1}^{n_+(AB^\dagger)} | < f, \varphi_j > | \|\varphi_j\|_{L^r(\mathcal{G})} \\ &\le \|f\|_{L^\alpha(\mathcal{G})} \sum_{j=1}^{n_+(AB^\dagger)} \|\varphi_j\|_{L^{\alpha'}(\mathcal{G})}\|\varphi_j\|_{L^r(\mathcal{G})} \\ &\le C(\mathcal{G}, A, B, \alpha, r)\|f\|_{L^\alpha(\mathcal{G})}.\end{aligned}$$

Taking now the L^q–norm on the time interval $(-T, T)$, we get that for every $\alpha \ge 1$,

$$\|\mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_p(H)f\|_{L^q((-T,T),L^r(\mathcal{G}))} \le C(\mathcal{G}, A, B, \alpha, r)(2T)^{1/q}\|f\|_{L^\alpha(\mathcal{G})}. \tag{3.10}$$

Hence, we consequently obtain that

$$\begin{aligned}\left\| \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H}\mathcal{P}_p(H)F(s)\, ds\right\|_{L^q((-T,T),L^r(\mathcal{G}))} &\le \left\| \int_0^t \|\mathrm{e}^{-\mathrm{i}(t-s)H}\mathcal{P}_p(H)F(s)\|_{L^r(\mathcal{G})}\, ds\right\|_{L^q(-T,T)} \\ &\le C(\mathcal{G}, A, B, \alpha, r)(2T)^{1/q}\int_{-T}^{T} \|F(s)\|_{L^\alpha(\mathcal{G})}\, ds.\end{aligned} \tag{3.11}$$

Taking now into account that

$$\mathrm{e}^{-\mathrm{i}tH}u_0 = \mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_{ac}(H)u_0 + \mathrm{e}^{-\mathrm{i}tH}\mathcal{P}_p(H)u_0,$$

the desired result follows after corroborating the estimates (3.10) and (3.11) with Theorem A. ☐

3.3. **Well-posedness of the nonlinear Schrödinger equation.** In this subsection, making use of the Strichartz estimates, we show that the problem (2.4) is globally well-posed in the mild form. First we establish local in time existence and uniqueness, then, once the conservation of the $L^2(\mathcal{G})-$norm is obtained, we prove that the solution is global.

Proof of Theorem B. We divide the proof in few steps.

Step I. Local well-posedness. We consider the integral equation:

$$u(t) = \underbrace{\mathrm{e}^{\mathrm{i}tH}u_0 + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-s)H}(|u|^{p-1}u)(s)ds}_{\Phi(t)} \tag{3.12}$$

Let us set the admissible pair $(q_0, r_0) = (4(p+1)/(p-1), p+1)$. The proof follows the same steps as the ones in [18, Theorem 5.2], more precisely, using the Strichartz estimates in Corollary A1, one shows that Φ is a strict-contraction on some balls of the space $C([-T,T] : L^2(\mathcal{G})) \cap L^{q_0}([-T,T], L^{r_0}(\mathcal{G}))$, for some convenient $T < 1$ depending on $\|u_0\|_{L^2(\mathcal{G})}$ and p. In particular, we use that the nonlinearity $f(u) = |u|^{p-1}u$ satisfies the following estimate when $p \in (1,5)$ (see [18, Theorem 5.2])

$$\|f(u)\|_{L^{q_0'}([-T,T],L^{r_0'}(\mathcal{G}))} \le (2T)^{\frac{5-p}{4}}\|u\|^p_{L^{q_0}([-T,T],L^{r_0}(\mathcal{G}))}.$$

In fact when $p \le 5$ for any admissible pair (q,r) we can find another admissible pair (q^*, r^*) such that $pr' = r^*$, $pq' \le q^*$ and then

$$\|f(u)\|_{L^{q'}([-T,T],L^{r'}(\mathcal{G}))} = \|u\|^p_{L^{pq'}([-T,T],L^{pr'}(\mathcal{G}))} \le C(T)\|u\|^p_{L^{q^*}([-T,T],L^{r^*}(\mathcal{G}))}. \tag{3.13}$$

Step II. Extra-integrability. We now prove that the solution u obtained above belongs to $L^q([-T,T], L^r(\mathcal{G}))$ for all sharp $1/2-$admissible exponent pairs (q,r). Due to the Strichartz estimates in Corollary A1, we have that

$$\begin{aligned}
\|u\|_{L^q([-T,T],L^r(\mathcal{G}))} &\lesssim \|\mathrm{e}^{-\mathrm{i}tH}\|_{L^q([-T,T],L^r(\mathcal{G}))} + \Big\|\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H}(|u|^{p-1}u)(s)ds\Big\|_{L^q([-T,T],L^r(\mathcal{G}))}\\
&\lesssim (1+T^{1/q})\|u_0\|_{L^2(\mathcal{G})} + \|f(u)\|_{L^{q_0'}([-T,T],L^{r_0'}(\mathcal{G}))} + (2T)^{1/q_0}||f(u)||_{L^1((0,T)),L^{r_0'}(\mathcal{G}))},\\
&\lesssim (1+T^{1/q})\|u_0\|_{L^2(\mathcal{G})} + [1+(2T)^{1/q+1/q_0}]\|f(u)\|_{L^{q_0'}([-T,T],L^{r_0'}(\mathcal{G}))}\\
&\lesssim (1+T^{1/q})\|u_0\|_{L^2(\mathcal{G})} + C(T)\|u\|_{L^{q_0}([-T,T],L^{r_0}(\mathcal{G}))}.
\end{aligned}$$

Step III. Conservation of the L^2-norm and the global well-posedness. In order to prove the conservation of the L^2-norm we will regularize the solution obtained above and analyze the L^2-norm of the regularized solution. Once the conservation of the L^2-norm is obtained, we can repeat the local argument in Step I as many times as we wish, preserving the length of time interval to get a global solution. The following lemma will play an important role in our approach. Its proof will be given latter.

Lemma 3.6. *The operator* $(I+\varepsilon^2 H)^{-1}$ *satisfies the following:*

(i) There exist two constants $\varepsilon_0=\varepsilon_0(A,B)$ *and* $C(A,B)$ *such that for any* $1\le p\le\infty$

$$\left\|(I+\varepsilon^2 H)^{-1}\right\|_{\mathcal{L}(L^p(\mathcal{G}),L^p(\mathcal{G}))}\le C(A,B),\quad \forall\,|\varepsilon|<\varepsilon_0,$$

(ii) For any $1\le p<\infty$ *and* $\varphi\in L^p(\mathcal{G})$,

$$\lim_{\varepsilon\to 0}(I+\varepsilon^2 H)^{-1}\varphi=\varphi\qquad \textit{in } L^p(\mathcal{G});$$

(iii) For any (q,r) *such that* $1\le q,r<\infty$ *and* $\psi\in L^q((0,T),L^r(\mathcal{G}))$,

$$\lim_{\varepsilon\to 0}(I+\varepsilon^2 H)^{-1}\psi=\psi\qquad \textit{in } \in L^q((0,T),L^r(\mathcal{G})).$$

For any initial data $u_0\in L^2(\mathcal{G})$ we consider the local solution obtained at Step I, $u\in C([0,T],L^2(\mathcal{G}))\bigcap_{(q,r)-adm}L^q((0,T),L^r(\mathcal{G}))$ satisfying

$$u(t)=\mathrm{e}^{-\mathrm{i}tH}u_0+\mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H}(|u|^{p-1}u)(s)\,ds.$$

Let us consider u_ε the regularization of u defined as

$$u_\varepsilon(t):=(I+\varepsilon^2 H)^{-1}u(t).$$

Since the operator $(I+\varepsilon^2 H)^{-1}$ commutes with the unitary group $\mathrm{e}^{-\mathrm{i}tH}$, we obtain that

$$\begin{cases} u_\varepsilon\in C([-T,T],\mathcal{D}(H)),\\ u_\varepsilon(t)=\mathrm{e}^{-\mathrm{i}tH}u_{0\varepsilon}+\mathrm{i}\displaystyle\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H}(I+\varepsilon^2 H)^{-1}(|u|^{p-1}u)(s)\,ds,\end{cases}$$

where $u_{0\varepsilon}=(I+\varepsilon^2 H)^{-1}u_0$, with $u_0\in L^2(\mathcal{G})$. In view of (3.13) we have that $f(u)\in L^1((0,T),L^2(\mathcal{G}))$ and hence $(I+\varepsilon^2 H)^{-1}f(u)\in L^1((0,T),\mathcal{D}(H))$. Applying now [8, Proposition 4.1.9], we obtain that $u_\varepsilon\in C([0,T],\mathcal{D}(H))\cap W^{1,1}((0,T),L^2(\mathcal{G}))$, and, moreover,

$$\begin{cases} \mathrm{i}\dfrac{\partial u_\varepsilon}{\partial t}(t)=Hu_\varepsilon(t)+(I+\varepsilon^2 H)^{-1}f(u),\qquad for\,a.e.\ t\in[0,T],\\ u_\varepsilon(0)=u_{0\varepsilon}.\end{cases}$$

This shows that the map $[0,T]\ni t\longmapsto\|u_\varepsilon(t)\|^2_{L^2(\mathcal{G})}$ is absolutely continuous and

$$\|u_\varepsilon(t)\|^2_{L^2(\mathcal{G})}-\|u_{0\varepsilon}\|^2_{L^2(\mathcal{G})}=2\Re\,\mathrm{i}\int_0^t\int_{\mathcal{G}}(I+\varepsilon^2 H)^{-1}(f(u))(s,x)\,\overline{u}_\varepsilon(s,x)\,dx\,ds.$$

In view of Lemma 3.6, as ε goes to zero,

$$u_\varepsilon(t)\to u(t)\qquad in\ L^2(\mathcal{G}),\ \forall\ t\ge 0,$$

and for every (q,r) sharp 1/2–admissible the following holds

$$u_\varepsilon\to u\qquad in\ L^q((0,T),L^r(\mathcal{G})).$$

Furthermore, taking into account (3.13), we also have that $f(u) \in L^{q'}((0,T), L^{r'}(\mathcal{G})$ and as ε goes to zero

$$(I+\varepsilon^2 H)^{-1} f(u) \to f(u) \qquad in\ L^{q'}((0,T), L^{r'}(\mathcal{G}))$$

This leads to

$$\int_0^t \int_{\mathcal{G}} (I+\varepsilon^2 H)^{-1}(f(u))(s,x)\overline{u}_\varepsilon(s,x)\ dx\ ds \to \int_0^t \int_{\mathcal{G}} f(u(s,x))\overline{u}(s,x)\ dx\ ds,$$

as ε tends to zero. Finally, since $\Re(\mathrm{i} f(u), u) = 0$, we obtain the conservation of the $L^2(\mathcal{G})$–norm of $u(t)$.

The proof of Theorem B is now complete. □

Proof of Lemma 3.6 . Using the representation formula (3.1) of the integral resolvent for any $\varphi = (\varphi_j)_{j=1}^n$ we obtain the following representation

$$(I+\varepsilon^2 H)^{-1}\varphi = \frac{1}{2\varepsilon}\int_{\mathcal{G}} \mathrm{diag}(e^{-\frac{|x-y|}{\varepsilon}})\varphi(y)\ dy + \frac{1}{2\varepsilon}\int_{\mathcal{G}} \mathrm{diag}(e^{-\frac{x}{\varepsilon}}) G\Big(\frac{\mathrm{i}}{\varepsilon}, A, B\Big)\mathrm{diag}(e^{-\frac{y}{\varepsilon}})\varphi(y)\ dy. \tag{3.14}$$

Using the representation of matrix G obtained in the proof of Lemma 3.3 as quotient of polynomials we obtain that all the components of the matrix $G(\mathrm{i}/\varepsilon, A, B)$ are uniformly bounded if ε is small enough (see also [16, Proposition 3.11] and the proof of [16, Theorem 3.12]), i.e.

$$\Big\| G\Big(\frac{\mathrm{i}}{\varepsilon}, A, B\Big)\Big\| \leq \mathcal{C}(A,B) < \infty,$$

for all $1/\varepsilon > \rho > \rho_0 \geq 0$, where ρ_0 is the radius of the disk containing all the zeros of $\det(A + ikB)$, $k \in \mathbb{C}$, which lie on the imaginary axis. Using Young's inequality and the fact that the map $K_\varepsilon(x) = e^{-\varepsilon|x|}/\varepsilon$ belongs to $L^1(\mathbb{R})$ with a norm independent of ε we obtain the first estimate.

The third estimate follows by applying Lebesgue dominated convergence theorem in the time variable and the first two properties in this lemma. It remains to concentrate on proving the second property.

We recall that in the case of an even function $K \in L^1(\mathbb{R}, 1+|x|^2)$ it has been proved in [13, Lemma 2.2] that for any $1 \leq p \leq \infty$

$$\|K_\varepsilon * \phi - \phi\|_{L^p(\mathbb{R})} \leq C(K)\varepsilon^2 \|\phi''\|_{L^p(\mathbb{R})}.$$

By an approximation argument we have that for any $\phi \in L^p(\mathbb{R})$, $1 \leq p < \infty$, the following holds

$$K_\varepsilon * \phi \to \phi \quad \text{in } L^p(\mathbb{R}), \text{ as } \varepsilon \to 0. \tag{3.15}$$

In order to prove the convergence in $L^p(\mathcal{G})$ with $1 \leq p < \infty$, we will treat each integral separately and we show that

$$T^1_\varepsilon(\varphi) = \frac{1}{2\varepsilon}\int_{\mathcal{G}} \mathrm{diag}(e^{-\frac{|x-y|}{\varepsilon}})\varphi(y)\ dy \to \varphi \quad \text{in } L^p(\mathcal{G})$$

and

$$\frac{1}{2\varepsilon}\int_{\mathcal{G}} \mathrm{diag}(e^{-\frac{x}{\varepsilon}})G\Big(\frac{\mathrm{i}}{\varepsilon}, A, B\Big)\mathrm{diag}(e^{-\frac{y}{\varepsilon}})\varphi(y)\,dy \to 0 \quad \text{in } L^p(\mathcal{G}). \tag{3.16}$$

Note that each component of $T^1_\varepsilon(\varphi)$ is explicitly given by

$$[T^1_\varepsilon(\varphi)]_j(x_j) = \frac{1}{2\varepsilon}\int_0^\infty e^{-\frac{|x_j-y_j|}{\varepsilon}}\varphi_j(y)\,dy,\ x_j\in(0,\infty),\ j\in\{1,\dots,n\}.$$

We will prove that $[T^1_\varepsilon(\varphi)]_j \to \varphi_j$ in $L^p(0,\infty)$. To do that we observe that

$$[T^1_\varepsilon(\varphi)]_j(x) = (K_\varepsilon * \tilde{\varphi}_j)(x),\ x\in(0,\infty),$$

where

$$\widetilde{\varphi}_j(y) = \begin{cases} \varphi_j(y), & \text{if } y\ge 0,\\ 0, & \text{otherwise.}\end{cases}$$

Using (3.15) we obtain that $K_\varepsilon * \tilde{\varphi}_j \to \widetilde{\varphi}_j$ in $L^p(\mathbb{R})$ as $\varepsilon\to 0$. Thus,

$$[T^1_\varepsilon(\varphi)]_j = (K_\varepsilon * \tilde{\varphi})|_{x>0} \to \tilde{\varphi}_j|_{x>0} = \varphi_j \quad \text{in } L^p(0,\infty).$$

Let us now prove (3.16). Since the elements of the matrix $G(i/\varepsilon, A, B)$ are uniformly bounded for small ε it is sufficient to prove that for any $\phi\in L^p(0,\infty)$ the following holds

$$T^2_\varepsilon(\phi)(x) := \frac{1}{2\varepsilon}\int_0^\infty e^{-\frac{x+y}{\varepsilon}}\phi(y)\,dy \to 0 \quad \textit{in } L^p(0,\infty).$$

Observe again that for any $x\in(0,\infty)$ we have $T^2_\varepsilon(\phi)(x) = (K_\varepsilon * \tilde{\phi})(x)$, where this time

$$\tilde{\phi}(y) = \begin{cases} 0, & \text{if } y>0,\\ \phi(-y), & \text{otherwise.}\end{cases}$$

Using again (3.15) we obtain that $K_\varepsilon * \tilde{\phi} \to \tilde{\phi}$ in $L^p(\mathbb{R})$ as $\varepsilon\to 0$. Thus,

$$T^2_\varepsilon(\phi) = (K_\varepsilon * \tilde{\phi})_{x>0} \to \tilde{\phi}|_{x>0} = 0 \quad \text{in } L^p(0,\infty).$$

The proof is now completed. □

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