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ANNEXE 3.7

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Null Controllability of the Kuramoto-Sivashinsky Equation on star-shaped trees

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NULL-CONTROLLABILITY OF THE LINEAR KURAMOTO-SIVASHINSKY EQUATION ON STAR-SHAPED TREES

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Abstract. In this paper we treat null-controllability properties for the linear Kuramoto-Sivashinsky equation on a network with two types of boundary conditions. More precisely, the equation is considered on a star-shaped tree with Dirichlet and Neumann boundary conditions. By using the moment theory we can derive null-controllability properties with boundary controls acting on the external vertices of the tree. In particular, the controllability holds if the *anti-diffusion* parameter of the involved equation does not belong to a critical countable set of real numbers. We point out that the critical set for which the null-controllability fails differs from the first model to the second one.

Key words: Kuramoto-Sivashinsky equation, null-controllability, star-shaped trees, method of moments.

Mathematics Subject Classification 2010: 35K25, 35R02, 93B05, 93B60.

1. Introduction

In this paper, we consider two control problems on the same simple network formed by the edges of a tree. The problem we address here enter in the framework of quantum graphs. The name quantum graph is used for a graph considered as a one-dimensional singular variety and equipped with a differential operator. Those quantum graphs are metric spaces which can be written as the union of finitely many intervals, which are compact or $[0,\infty)$ and any two of these intervals are either disjoint or intersect only at one of their endpoints.

Our main goal is to study boundary null-controllability properties for the Kuramoto-Sivashinsky (KS) equation

$$y_t + \lambda y_{xx} + y_{xxxx} = 0, \tag{1}$$

on a star-shaped tree denoted Γ. More precisely, Γ is a simplified topological graph with $N \geq 2$ edges e_i, $i \in \{1, \dots, N\}$, of the same given length $L > 0$ and $N + 1$ vertices. Besides, all edges intersect at a unique endpoint which is the interior vertex of the graph. The mathematical formulation of the control problems that we address on Γ stands for a system of N-KS equations on the interval $(0, L)$ coupled

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through the left endpoint $x = 0$ as follows

$$
\text{(2)} \qquad \begin{cases}
y_t^k + \lambda y_{xx}^k + y_{xxxx}^k = 0, & (t,x) \in (0,T) \times (0,L) \\
y^i(t,0) = y^j(t,0), & t \in (0,T), \quad i,j \in \{1,\dots,N\} \\
\sum_{k=1}^N y_x^k(t,0) = 0, & t \in (0,T) \\
y_{xx}^i(t,0) = y_{xx}^j(t,0), & t \in (0,T), \quad i,j \in \{1,\dots,N\} \\
\sum_{k=1}^N y_{xxx}^k(t,0) = 0, & t \in (0,T) \\
y^k(0,x) = y_0^k(x), & x \in (0,L).
\end{cases}
$$

For system (2) we study two types of boundary control conditions:

$$
\text{(I)} : \quad \begin{cases}
y^k(t,L) = 0, \\
y_x^k(t,L) = u^k(t), \qquad k \in \{1,\dots,N\},
\end{cases}
$$

respectively

$$
\text{(II)} : \quad \begin{cases}
y_x^k(t,L) = a^k(t), \\
y_{xxx}^k(t,L) = b^k(t), \qquad k \in \{1,\dots,N\}.
\end{cases}
$$

Next in the paper we will refer to (2)-(I) for system (2) subject to the boundary conditions (I) and to (2)-(II) for system (2) with boundary conditions (II).

In system (2), λ is a positive constant, the functions $y^k = y^k(t,x)$ are real-valued for any $k \in \{1,\dots,N\}$, t denotes the time variable, x denotes the space variable and the subscripts for both t and x indicate partial differentiation with respect to each one. The boundary functions u^k, a^k and b^k are considered as control inputs acting on the external nodes. In model (II) we impose two controls to act on the same vertex whereas in model (I) we only require one control. Our main aims are to see whether we can force the solutions of system (2) to have certain properties by choosing appropriate control inputs. The focus here is on the following null-controllability issue:

Given any finite time $T > 0$ and any initial state $y_0 = (y_0^k)_{k=1,N}$, can we find proper control inputs in (I) or (II) ($u = (u^k)_{k=1,N}$ and $a = (a^k)_{k=1,N}$, $b = (b^k)_{k=1,N}$, respectively) to lead the solution of system (2) to the zero state, i.e.,

$$
\text{(3)} \qquad y^k(T,x) = 0, \quad \textit{for any } x \in (0,L), \quad k \in \{1,\dots,N\}?
$$

For parabolic control problems, in general, it is not possible to steer the system to an arbitrary prescribed state. Thus, we do not expect the exact controllability to be true neither for the KS control system. Our motivation for studying such control systems goes back to the quasilinear KS equation

$$
\text{(4)} \qquad y_t + \lambda y_{xx} + y_{xxxx} + y y_x = 0,
$$

which was derived independently by Kuramoto and Tsuzuki in [17, 18] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky in [19, 20] as a model for plane flame propagation. The real positive number λ in (4) is called the anti-diffusion parameter. This nonlinear partial differential equation also describes incipient instabilities in a variety of physical and chemical systems (see, for instance, [9] and [16]).

The linear control problem on the interval $(0,1)$ has been first studied in [4] considering Dirichlet boundary conditions. By using the moment theory developed by Fattorini and Russell in [13], it was proved that this system is null controllable if two controls act only on the left endpoint of the interval, or more general, at one endpoint of the interval. If one control is removed the system is controllable if and only if the anti-diffusion parameter λ does not belong to the following countable set of critical values:

(5)
$$\mathcal{N}_0 := \left\{\pi^2(m^2+n^2) : (m,n) \in \mathbb{N}^2, 0 \le m < n,\ m \text{ and } n \text{ have the same parity}\right\}.$$

More precisely, there exists a finite-dimensional space of initial conditions that cannot be driven to zero with only one control. We point out that the results in [4] could be extended to intervals of any length $L > 0$ by re-scaling the set $\mathcal{N}_0$ in terms of L accordingly.

Later on, the boundary controllability to the trajectories of (4) was proved in [6] when two controls act on the left Dirichlet boundary conditions. We also refer to [2, 5, 7, 8, 21, 10, 14, 15] for related problems and results on the subject. Particularly, the strategy used in [21] can be regarded as a possibility to study the nonlinear problem for the systems under consideration in this paper.

To our knowledge, the study of the controllability properties of KS systems in the context of quantum graphs has not been yet addressed in the literature neither for the linear equation (1). At this respect, the program of this work was carried out for a choice of classical boundary conditions and aims to establish as a fact that the models under consideration inherit the interesting controllability properties initially observed for the KS equation posed on a bounded interval.

In order to present our main results, we introduce the following countable sets

$$\mathcal{N}_1 := \left\{\frac{\pi^2(m^2+n^2)}{L^2} : (m,n) \in \mathbb{N}^2,\ 1 \le m < n\right\},$$

$$\mathcal{N}_2 := \left\{\frac{\pi^2 m^2}{L^2} : m \in \mathbb{N},\ 1 \le m\right\},$$

$$\mathcal{N}_3 := \left\{\frac{\pi^2}{L^2}\left(n+\frac{1}{2}\right)^2 : n \in \mathbb{N},\ n \ge 0\right\},$$

$$\mathcal{N}_4 := \left\{\frac{\pi^2}{L^2}\left(m^2+\left(n+\frac{1}{2}\right)^2\right) : (m,n) \in \mathbb{N}^2,\ 1 \le m, 0 \le n\right\},$$

$$\mathcal{N}_{odd} := \left\{\frac{\pi^2}{4L^2}\left((2m+1)^2+(2n+1)^2\right) : (m,n) \in \mathbb{N}^2,\ 0 \le m < n\right\}.$$

To simplify the presentation of our main results let us also introduce the notations

$$\mathcal{N}_{even} := \mathcal{N}_1 \cup \mathcal{N}_2,$$
$$\mathcal{N}_{mixt} := \mathcal{N}_3 \cup \mathcal{N}_4.$$

Observe that

$$\mathcal{N}_{even} \cup \mathcal{N}_{odd} = \frac{\mathcal{N}_0}{4L^2}. \tag{6}$$

A priori, for the models we study here each control possesses N components but some of the inputs might not be necessary and could vanish completely. In fact, in the case when the system is null-controllable the goal is to intend the controls to act on a minimal number of components. Our main results will be stated accurately in the following. Roughly speaking, for problem (2)-(I) we prove that the null controllability property holds with $N-1$ control inputs, which is the optimal number of controls in this case, whereas for the model (2)-(II) the optimal number of control inputs is $2N-2$ or even less. In both cases the control properties are obtained under some restrictions on the parameter λ. For our purposes we need to work in a rigorous functional framework in which Sobolev spaces play a crucial role, namely $L^2(\Gamma)$ and $H^m(\Gamma)$ for $m \in \mathbb{N}^\star$. By $L^2(\Gamma)$ and $H^m(\Gamma)$ we understand the Hilbert spaces

$$L^2(\Gamma) := \prod_{i=1}^{N} L^2(0,L), \quad H^m(\Gamma) := \prod_{i=1}^{N} H^m(0,L), \quad m \geq 1,$$

endowed with their natural norms.

Thus, our main results are as follows.

Theorem 1.1 (Null-controllability for model (2)-(I))**.** *Let $T>0$ be fixed and $\lambda \notin \mathcal{N}_{even} \cup \mathcal{N}_{odd}$. Then*

(1) For any edge e_i with $i \in \{1,\dots,N\}$ and any initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$, there exist controls $u = (u^k)_{k=1,N} \in (H^1(0,T))^N$ with $u^i \equiv 0$ such that the solution of system (2)-(I) satisfies

$$y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}. \tag{7}$$

(2) For any given two edges e_{i_0} and e_{j_0}, with $i_0, j_0 \in \{1,\dots,N\}$, there exist initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$ such that for any control $u = (u^k)_{k=1,N} \in (H^1(0,T))^N$ with $u^{i_0}(t) = u^{j_0}(t) = 0$ for all $t \in (0,T)$, the solution of system (2)-(I) satisfies

$$y^{k_0}(T,\cdot) \not\equiv 0,$$

for some $k_0 \in \{1,\dots,N\}$.

Theorem 1.2 (Null-controllability for model (2)-(II))**.** *Let $T>0$ be fixed and $\lambda \notin \mathcal{N}_{mixt}$. Then*

(1) For any edge e_i with $i \in \{1,\dots,N\}$ and any initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$, there exist controls $a = (a^k)_{k=1,N}, b = (b^k)_{k=1,N} \in (H^1(0,T))^N$ such that $a^i(t) = b^i(t) = 0$ for all $t \in (0,T)$ and the solution of system (2)-(II) satisfies

$$y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}. \tag{8}$$

(2) If moreover $\lambda \in \mathcal{N}_{odd}$ then for any given two edges e_{i_0} and e_{j_0}, with $i_0, j_0 \in \{1,\dots,N\}$, there exist initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$ such that for any controls $a = (a^k)_{k=1,N}, b = (b^k)_{k=1,N} \in (H^1(0,T))^N$ with either $a^{i_0}(t) = a^{j_0}(t) = b^{i_0}(t) = 0$ or $b^{i_0}(t) = b^{j_0}(t) = a^{i_0}(t) = 0$ for all $t \in (0,T)$, the solution of system (2)-(II) satisfies

$$y^{k_0}(T,\cdot) \not\equiv 0,$$

for some $k_0 \in \{1,\dots,N\}$.

(3) Otherwise if $\lambda \notin \mathcal{N}_{odd}$ and $N \geq 3$ for any given two edges e_{i_0} and e_{j_0}, with $i_0, j_0 \in \{1,\dots,N\}$ and any initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$, there exist controls $a = (a^k)_{k=1,N}, b = (b^k)_{k=1,N} \in (H^1(0,T))^N$ with either $a^{i_0}(t) = a^{j_0}(t) = b^{i_0}(t) = 0$ or $b^{i_0}(t) = b^{j_0}(t) = a^{i_0}(t) = 0$ for all $t \in (0,T)$, such that the solution of system (2)-(II) satisfies

$$y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}.$$

Remark 1.3. *The first statement in Theorem 1.1 asserts that, roughly speaking, for any $\lambda \notin \mathcal{N}_{even} \cup \mathcal{N}_{odd}$ system (2)-(I) can be driven to the zero state acting with controls just on $N-1$ components within the N edges of the tree. The inactive control input could be taken on any edge of the tree independently on the choice of* *the initial data y_0. The second statement in Theorem 1.1 ensures that the minimal number of control inputs needed to control system (2)-(I) is exactly $N-1$.*

Similarly, Theorem 1.2 says that it is necessary to act only on $2N-2$ components of the system within the all $2N$ components to ensure the null-controllability of the problem. The second part of Theorem 1.2 represents the optimality of acting on $2N-2$ components to control any initial data of system (2)-(II) when $\lambda \in \mathcal{N}_{odd}$. When $\lambda \notin \mathcal{N}_{odd}$ the third statement of Theorem 1.2 asserts that we can obtain null-controllability with $2N-3$ active controls.

In order to prove Theorems 1.1 and 1.2 we make a careful spectral analysis of the corresponding elliptic differential operator which allows us to transform the controllability problem into an equivalent moment problem. The later will be solved combining an adaptation of the general moment theory developed by Fattorini and Russell [13] and the asymptotic analysis of the eigenvalues and the corresponding eigenfunctions. One of the main difficulty in applying directly the method in [13] appears in both systems (2)-(I) and (2)-(II) and is represented by the presence of multiple eigenvalues for the corresponding eigenvalue problems as shown with precision in Lemmas 2.5 and (3.6). At that point it is important to note that, for making possible the existence of the controls, the choice of the parameter λ plays an important role. It is also worth mentioning that the general theory in [13] allows to build solutions to the moment problem in L^2. However, as the authors in [13] assert later on one can obtain the existence of smoother controls not only in L^2 but in any space H^s, $s \in \mathbb{R}$ and in consequence in $C^\infty([0,T])$.

The extension of the above results to the case when the edges of the star shaped tree have different lengths needs a different approach. Following [12] other coupling conditions may be imposed at the internal node. The analysis of the controllability properties for the KS system with other coupling conditions at $x=0$ remains to be considered elsewhere.

The analysis described above is organized in two sections: in section 2 we study problem (2)-(I) and prove Theorem 1.1. Section 3 is devoted to problem (2)-(II) and the proof of Theorem 1.2.

The main effort we put in this paper concerns the proofs of Theorems 1.1 and 1.2 where we focus to obtain control results with minimal number of active control inputs. On the other hand we may also wonder in which conditions we can obtain control properties with no restrictions on the number of the control components. We answer to these questions in section 4.1.

Besides, in view of the spectral results in sections 3.1 and 3.2 we are able to obtain new control results for the linear KS on an interval which were not analyzed in [4]. These aspects will be detailed in section 4.2.

2. The KS equation of type (I)

The controllability problem (2)-(I) will be studied by using the method of moments due to Fattorini and Russell [13]. Therefore, a careful spectral analysis of the involved elliptic operator is necessary.

2.1. **Spectral analysis.** For any $\lambda > 0$ let us consider the following spectral problem on Γ:

$$
\begin{cases}
\lambda \phi^k_{xx} + \phi^k_{xxxx} = \sigma \phi^k, \quad x \in (0, L) \\
\phi^i(0) = \phi^j(0), \qquad\qquad i, j \in \{1, \dots, N\} \\
\phi^k(L) = \phi^k_x(L) = 0, \quad k \in \{1, \dots, N\} \\
\sum_{k=1}^N \phi^k_x(0) = 0, \\
\phi^i_{xx}(0) = \phi^j_{xx}(0), \qquad i, j \in \{1, \dots, N\} \\
\sum_{k=1}^N \phi^k_{xxx}(0) = 0.
\end{cases} \tag{9}
$$

To begin with, we introduce the fourth order operator

$$A : D(A) \subset L^2(\Gamma) \to L^2(\Gamma)$$

given by
(10)

$$
\begin{cases}
A\phi^k = \lambda \phi^k_{xx} + \phi^k_{xxxx}, \quad k \in \{1, \dots, N\} \\
D(A) = \left\{ \begin{array}{l} \phi = (\phi^k)_{k=1,N} \in H^4(\Gamma) \mid \phi^k(L) = \phi^k_x(L) = 0, \quad k \in \{1, \dots, N\}, \\ \phi^i(0) = \phi^j(0), \ \phi^i_{xx}(0) = \phi^j_{xx}(0), \quad i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi^k_x(0) = 0, \ \sum_{k=1}^N \phi^k_{xxx}(0) = 0 \end{array} \right\}.
\end{cases}
$$

Remark that spectral problem (9) is equivalent to

$$
\begin{cases}
A\phi = \sigma \phi, \\
\phi \in D(A).
\end{cases} \tag{11}
$$

To study this eigenvalue problem we firstly claim that

Proposition 2.1. *For any $\mu > \lambda^2/4$ the operator*

$$A + \mu I : D(A) \subset L^2(\Gamma) \to L^2(\Gamma),$$

is a non-negative self-adjoint operator with compact inverse. In particular, it has a pure discrete spectrum consisted by a sequence of nonnegative eigenvalues $\{\sigma_{\mu,n}\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \sigma_{\mu,n} = \infty$. Moreover, up to a normalization, the corresponding eigenfunctions $\{\phi_{\mu,n}\}_{n\in\mathbb{N}}$ form an orthonormal basis of $L^2(\Gamma)$.

Proof. Firstly, it is easy to observe that $D(A)$ is dense in $L^2(\Gamma)$. Next the proof will be done in several steps.
Step 1: A is a symmetric operator. Indeed, after integrations by parts, we get

$$
\begin{aligned}
(Au, v)_{L^2(\Gamma)} &= (u, Av)_{L^2(\Gamma)} \\
&= \sum_{k=1}^N \int_0^L (u^k_{xx} v^k_{xx} - \lambda u^k_x v^k_x)\, \mathrm{d}x, \quad \forall u, v \in D(A).
\end{aligned}
$$

Step 2: $A + \mu I$ is maximal monotone for any $\mu > \lambda^2/4$. Firstly, let us show the monotonicity property

$$((A + \mu I)u, u)_{L^2(\Gamma)} > 0, \quad \forall u \in D(A).$$

Indeed,

$$((A + \mu I)u, u)_{L^2(\Gamma)} = \sum_{k=1}^N \int_0^L (|u^k_{xx}|^2 + \mu |u^k|^2 - \lambda |u^k_x|^2)\, \mathrm{d}x. \tag{12}$$

On the other hand we have

$$\sum_{k=1}^{N}\int_0^L u_{xx}^k u^k \,\mathrm{d}x = \sum_{k=1}^{N} u_x^k u^k \Big|_{x=0}^{x=L} - \sum_{k=1}^{N}\int_0^L |u_x^k|^2 \,\mathrm{d}x$$
$$= -\sum_{k=1}^{N}\int_0^L |u_x^k|^2 \,\mathrm{d}x.$$

Therefore, from the inequality of arithmetic and geometric means it holds

$$\sum_{k=1}^{N}\int_0^L |u_x^k|^2 \,\mathrm{d}x \leq \sum_{k=1}^{N}\int_0^L \left(\frac{1}{\lambda}|u_{xx}^k|^2 + \frac{\lambda}{4}|u^k|^2\right) \mathrm{d}x. \tag{13}$$

Combining (12) and (13) we obtain

$$((A+\mu I)u, u)_{L^2(\Gamma)} \geq \left(\mu - \frac{\lambda^2}{4}\right)\sum_{k=1}^{N}\int_0^L |u^k|^2 \,\mathrm{d}x,$$

which is positive provided $\mu > \lambda^2/4$.

Next we emphasize that $A+\mu I$ is maximal, i.e., for any $f \in L^2(\Gamma)$ there exists a unique $u \in D(A)$ such that $(A+\mu I)u = f$. To do that, first we consider the variational formulation

$$\begin{cases} a(u,v) = (f,v)_{L^2(\Gamma)}, \quad \forall v \in V \\ u \in V, \end{cases} \tag{14}$$

where V denotes the Hilbert space

$$V = \left\{ \begin{array}{l} \phi = (\phi^k)_{k=1,N} \in H^2(\Gamma) \mid \phi^k(L) = \phi_x^k(L) = 0, \quad k \in \{1,\ldots,N\}, \\ \phi^i(0) = \phi^j(0), \ i,j \in \{1,\ldots,N\}, \ \sum_{k=1}^N \phi_x^k(0) = 0 \end{array} \right\}$$

endowed with the $H^2(\Gamma)$-norm and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ denotes the bilinear form

$$a(u,v) = \sum_{k=1}^{N}\int_0^L \left(u_{xx}^k v_{xx}^k - \lambda u_x^k v_x^k + \mu u^k v^k\right) \mathrm{d}x.$$

It is easy to see that $a(\cdot,\cdot)$ is symmetric and continuous. In addition, it is also coercive. Indeed, let $\delta > 0$ small enough such that $\mu > \lambda^2/(4-4\delta)$. Then, arguing as in (13) we obtain

$$\sum_{k=1}^{N}\int_0^L |u_x^k|^2 \,\mathrm{d}x \leq \sum_{k=1}^{N}\int_0^L \left(\frac{1-\delta}{\lambda}|u_{xx}^k|^2 + \frac{\lambda}{4(1-\delta)}|u^k|^2\right) \mathrm{d}x,$$

which together with (12) leads to

$$a(u,u) \geq \delta \sum_{k=1}^{N}\int_0^L |u_{xx}^k|^2 \,\mathrm{d}x + \left(\mu - \frac{\lambda^2}{4-4\delta}\right)\sum_{k=1}^{N}\int_0^L |u^k|^2 \,\mathrm{d}x,$$

and so a is coercive. Applying Lax-Milgram lemma we ensure the existence of a unique $u \in V$ satisfying the variational problem (14). In order to justify the maximality of $A+\mu I$ it is sufficient to show that the solution u of (14) belongs actually to $D(A)$. For that, we refer to the classical regularity arguments for elliptic operators (see, for instance, [1]).

Step 3: A is a self-adjoint operator with compact inverse. Since $A+\mu I : D(A) \subset L^2(\Gamma) \to L^2(\Gamma)$ is a symmetric operator and maximal monotone, it is a self-adjoint operator (see, e.g., [1]).

Moreover, we have that the linear operator $(A+\mu I)^{-1} : L^2(\Gamma) \to L^2(\Gamma)$, given by

$$(A+\mu I)^{-1} f = u \in D(A) \subset L^2(\Gamma),$$

satisfies

$$\|(A+\mu I)^{-1}f\|_{H^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)},$$

for some positive constant $C > 0$. Since the embedding $H^2(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact it follows that $(A+\mu I)^{-1}$ is a compact operator. Then applying the classical spectral results for compact self-adjoint operators we conclude the proof of Proposition 2.1. □

Remark 2.2. *The spectrum $\{\sigma_n\}_{n\in\mathbb{N}}$ of problem (11) is obtained by shifting the spectrum of $A+\mu I$ in Proposition 2.1, i.e.,*

$$\sigma_n := \sigma_{\mu,n} - \mu, \quad \forall n \in \mathbb{N},$$

with the corresponding eigenfunctions

$$\phi_n := \phi_{\mu,n}, \quad \|\phi_n\|_{L^2(\Gamma)} = 1, \quad \forall n \in \mathbb{N}.$$

In particular, it holds that

$$-\frac{\lambda^2}{4} \leq \sigma_n, \ \forall n \in \mathbb{N}, \qquad \sigma_n \to \infty, \ \text{as } n \to \infty. \tag{15}$$

The following proposition will be also very useful in our analysis.

Proposition 2.3. *The lower bound $-\lambda^2/4$ is not an eigenvalue for the operator A.*

Proof. If $(-\lambda^2/4, \phi)$ were an eigenpair for A, then $A\phi = \lambda^2/4\phi$ for some nontrivial $\phi = (\phi^k)_{k=1,N} \in D(A)$. Then, integration by parts leads to

$$0 = \left(\left(A + \frac{\lambda^2}{4}\right)\phi, \phi\right)_{L^2(\Gamma)} = \sum_{k=1}^{N}\int_0^L \left(\phi^k_{xx} + \frac{\lambda}{2}\phi^k\right)^2 dx.$$

Therefore, for any $k \in \{1,\dots,N\}$ the component ϕ^k must satisfy the equation $\phi^k_{xx} + \frac{\lambda}{2}\phi^k = 0$ in $(0,L)$ subject to the boundary conditions $\phi^k(L) = \phi^k_x(L) = 0$. These allow us to obtain $\phi^k \equiv 0$, for all $k \in \{1,\dots,N\}$ which is in contradiction with $\phi \not\equiv 0$. □

2.2. **Qualitative properties of the eigenvalues.** The main goal of this section is to provide an asymptotic formula for the behavior of eigenvalues of system (9). Particularly, this result will play an important role to prove the null-controllability of problem (2)-(I).

For any fixed $\lambda > 0$ let us firstly consider the following two eigenvalue problems on the interval $(0, L)$:

$$\begin{cases} \lambda\Psi_{xx} + \Psi_{xxxx} = \sigma\Psi, & x \in (0,L), \\ \Psi_x(0) = 0, \ \Psi_{xxx}(0) = 0, & \\ \Psi(L) = \Psi_x(L) = 0 & \end{cases} \tag{16}$$

and

$$\begin{cases} \lambda\Phi_{xx} + \Phi_{xxxx} = \sigma\Phi, & x \in (0,L), \\ \Phi(0) = 0, \ \Phi_{xx}(0) = 0, & \\ \Phi(L) = \Phi_x(L) = 0, & \end{cases} \tag{17}$$

respectively. As in Section 2.1 we can easily show that systems (16)-(17) possess a sequence of eigenvalues $\{\sigma_n\}_n$ which tends to infinity and is strictly bounded from below by $-\lambda^2/4$. Before going through let us fix some notations which will be useful in the forthcoming sections. For any $\lambda > 0$ we denote

$$
\begin{cases} \alpha := \sqrt{\frac{-\lambda+\sqrt{\lambda^2+4\sigma}}{2}}, \quad \beta := \sqrt{\frac{\lambda+\sqrt{\lambda^2+4\sigma}}{2}}, & \text{if } \sigma \geq 0 \\ \gamma := \sqrt{\frac{\lambda-\sqrt{\lambda^2+4\sigma}}{2}} \quad \beta := \sqrt{\frac{\lambda+\sqrt{\lambda^2+4\sigma}}{2}}, & \text{if } -\frac{\lambda^2}{4} < \sigma < 0 \end{cases} \tag{18}
$$

for which we have the relations

$$
\begin{cases} \beta^2 - \alpha^2 = \lambda, & \text{if } \sigma \geq 0 \\ \beta^2 + \gamma^2 = \lambda, & \text{if } -\frac{\lambda^2}{4} < \sigma < 0 \end{cases} \tag{19}
$$

and

$$
\sigma = \begin{cases} \alpha^2\beta^2, & \text{if } \sigma \geq 0 \\ -\beta^2\gamma^2, & \text{if } -\frac{\lambda^2}{4} < \sigma \leq 0. \end{cases} \tag{20}
$$

Coming back to our spectral problem (9), we introduce the functions

$$
S := \sum_{k=1}^{N} \phi^k \tag{21}
$$

and

$$
D^k := \phi^k - \frac{S}{N}, \quad k \in \{1, \dots, N\}. \tag{22}
$$

The motivation for analyzing systems (16) and (17) is due to the fact that S verifies (16) whereas D^k satisfies (17) for all $k \in \{1, \dots, N\}$.

Next we state and prove some preliminary results.

Lemma 2.4. *For any $\lambda > 0$ the eigenvalue problems (16) and (17) have no any common eigenvalue σ. In addition, any eigenvalue of either (16) or (17) is simple.*

Moreover, if $\lambda(2L)^2 \notin \mathcal{N}_0$ then any eigenfunction ϕ of either (16) or (17) satisfies $\phi_{xx}(L) \neq 0$.

Proof. First we show that problems (16) and (17) have no common eigenvalues. Let us assume that there exists σ and two functions Ψ and Φ not identically vanishing, satisfying (16) and (17), respectively.

The boundary conditions at $x = 0$ in (16) and (17) allow to introduce the even and odd extensions of Ψ, respectively Φ, with respect to $x = 0$. More precisely, we consider

$$
\overline{\Psi}(x) := \begin{cases} \Psi(x), & x \in [0, L] \\ \Psi(-x), & x \in [-L, 0] \end{cases}
$$

and

$$
\overline{\Phi}(x) := \begin{cases} \Phi(x), & x \in [0, L] \\ -\Phi(-x), & x \in [-L, 0]. \end{cases}
$$

Then $\overline{\Psi}$ and $\overline{\Phi}$ verify

$$
\begin{cases} \lambda\overline{\Psi}_{xx} + \overline{\Psi}_{xxxx} = \sigma\overline{\Psi}, & x \in (-L, L) \\ \overline{\Psi}(-L) = \overline{\Psi}(L) = \overline{\Psi}_x(-L) = \overline{\Psi}_x(L) = 0 \end{cases} \tag{23}
$$

and

$$
\begin{cases} \lambda\overline{\Phi}_{xx} + \overline{\Phi}_{xxxx} = \sigma\overline{\Phi}, & x \in (-L, L) \\ \overline{\Phi}(-L) = \overline{\Phi}(L) = \overline{\Phi}_x(-L) = \overline{\Phi}_x(L) = 0, \end{cases} \tag{24}
$$

respectively. Finally, let us denote

$$\hat{\Psi}(y) := \overline{\Psi}(2Ly - L), \quad \hat{\Phi}(y) := \overline{\Phi}(2Ly - L), \quad y \in (0,1).$$

In view of (23) and (24) it follows that $\hat{\Psi}$ and $\hat{\Phi}$ satisfy the same eigenvalue problem

$$\begin{cases} \lambda(2L)^2\phi_{xx} + \phi_{yyyy} = \sigma(2L)^4\phi, & y \in (0,1) \\ \phi(0) = \phi(1) = \phi_y(0) = \phi_y(1) = 0. \end{cases} \tag{25}$$

The arguments in [4] show that problem (25) admits simple eigenvalues. This means that $\hat{\Phi} = \alpha\hat{\Psi}$ for some constant $\alpha \neq 0$ and, equivalently, we have $\overline{\Phi} = \alpha\overline{\Psi}$. Since $\overline{\Psi}$ is an even function and $\overline{\Phi}$ is an odd function (both of them vanishing on the boundary) we necessarily have $\overline{\Psi} = \overline{\Phi} \equiv 0$, which contradicts our assumption. Then the first part of lemma is proved.

The fact that the eigenvalues of both (16) and (17) are simple is a consequence of the construction above. Indeed, if (σ, Ψ_1) and (σ, Ψ_2) are eigenpairs for (16) we get that $(\sigma, \hat{\Psi}_1)$ and $(\sigma, \hat{\Psi}_2)$ are eigenpairs of (25). Since any eigenvalue of (25) is simple there exists a constant $c \neq 0$ such that $\hat{\Psi}_1(y) = c\hat{\Psi}_1(y)$ for all $y \in [0,1]$. This is equivalent to $\overline{\Psi_1}(x) = c\overline{\Psi_1}(x)$ for all $x \in [-L, L]$ and therefore $\Psi_1(x) = c\Psi_1(x)$ for any $x \in [0, L]$. Hence σ is a simple eigenvalue for (16). With a similar argument we get that each eigenvalue of (17) is also simple.

For the last part of the proof let us assume that (σ, Ψ) is an eigenpair of (16). With the notations above, applying [4, Lemma 2.1] for (25) under the assumption $\lambda(2L)^2 \notin \mathcal{N}_0$ it follows $\hat{\Psi}_{xx}(0) \neq 0$. Due to the symmetry of the boundary conditions we notice that the function $x \mapsto \hat{\Psi}(1 - x)$ is also an eigenfunction of problem (25) and therefore we also have $\hat{S}_{xx}(1) \neq 0$. This gives us the desired property for Ψ, $\Psi_{xx}(L) \neq 0$. Analogously, it follows that $\Phi_{xx}(L) \neq 0$ for any eigenfunction of (17). The proof of Lemma 2.4 is finished.

□

In consequence we have the following partition for the eigenvalues of system (9).

Lemma 2.5. *For any given $\lambda > 0$, σ is an eigenvalue for system (9) if and only if σ is an eigenvalue for either (16) or (17). More precisely*

(i). If $(\sigma, \phi = (\phi^k)_{k=1,N})$ is an eigenpair of (9) we have the alternative:
  (a) either $\phi^k = \phi^1$ for all $k \in \{1, \dots, N\}$, (σ, ϕ^1) is an eigenpair of (16) and
  $$\phi = (\phi^1, \dots, \phi^1),$$
  or
  (b) there exists $k_0 \in \{1, \dots, N\}$ such that (σ, ϕ^{k_0}) is an eigenpair of (17) and there exist the constants $c_1, \dots, c_N$ (not all vanishing) with $\sum_{k=1}^N c_k = 0$ such that
  $$\phi = (c_1\phi^{k_0}, \dots, c_N\phi^{k_0}).$$

(ii). Conversely, if (σ, Ψ) is an eigenpair of (16) then $(\sigma, \phi = (\Psi, \dots, \Psi, \Psi))$ is an eigenpair of (9) whereas if (σ, Φ) is an eigenpair of (17) then, for any constants c_k (not all vanishing) such that $\sum_{k=1}^N c_k = 0$, it holds that $(\sigma, \phi = (c_1\Phi, \dots, c_N\Phi))$ is also eigenpair for (9).

Proof. The "if" implication is a consequence of (ii) whose proof is trivial. Let us prove the "only if" implication. For that, let us assume that $(\sigma, \phi = (\phi^k)_{k=1,N})$ is an eigenpair of (9). Then σ verifies (16) for $\Psi = S$ in (21). In addition, σ verifies (17) for any $\Phi = D^k$ in (22). If $S \neq 0$ then (σ, S) is an eigenpair for (16). Otherwise, if $S = 0$ then $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. From the initial assumption there exists $k_0 \in \{1, \dots, N\}$ such that $\phi^{k_0} \neq 0$, and therefore, (σ, ϕ^{k_0}) is an eigenpair for (17).

Moreover, according to Lemma 2.4 we distinguish two cases: $S \equiv 0$ or $S \not\equiv 0$ and $D^k \equiv 0$ for all $k \in \{1, \dots, N\}$.

In the second case we obtain $\phi = (\phi^1, \dots, \phi^1)$ where $\phi^1 = S/N$ is an eigenfunction for problem (16) and (1a) is proved.

Let us now consider the case $S \equiv 0$. From above we know that (σ, ϕ^{k_0}) is an eigenpair of (17) and (σ, ϕ^k) also verifies (17) for any k. In view of Lemma 2.4 the eigenspace of σ in (17) has dimension 1. Thus, there exist some constants c_k such that $\phi^k = c_k \phi^{k_0}$ for any $k \in \{1, \dots, N\}$. Then, since $S \equiv 0$ we get $\sum_{k=1}^N c_k = 0$. This proves (1b) which completes the proof of Lemma 2.5.

□

Lemma 2.6. *For any $\lambda(2L)^2 \notin \mathcal{N}_0$ any eigenfunction $\phi = (\phi^k)_{k=1,N}$ of (9) satisfies $\phi^k_{xx}(L) \neq 0$ for at least two indexes $k \in \{1, \dots, N\}$.*

Proof. This is a trivial consequence of Lemma 2.4 and Lemma (2.5). □

Lemma 2.7. *The positive eigenvalues $\{\sigma_n\}_{n \geq n_0}$ (without counting the multiplicity) of problem (9) can be partitioned into*

$$\{\sigma_n \mid n \geq n_0\} := \{\sigma_{1,n} \mid n \in \mathbb{N}\} \cup \{\sigma_{2,n} \mid n \in \mathbb{N}\},$$

where $\{\sigma_{1,n}\}_{n\geq 0}$ are the different (simple) eigenvalues of (16) whereas $\{\sigma_{2,n}\}_{n\geq 0}$ are the different eigenvalues of (17) each of them having multiplicity $N-1$. Moreover, they satisfy the following asymptotic properties:

$$\sigma_{1,n} = \left(\frac{\pi}{L}\right)^4 \left(n - \frac{1}{4}\right)^4 + o(n^3), \quad n \to \infty$$

and

$$\sigma_{2,n} = \left(\frac{\pi}{L}\right)^4 \left(n + \frac{1}{4}\right)^4 + o(n^3), \quad n \to \infty.$$

Threfore,

$$\sigma_n = \left(\frac{\pi}{2L}\right)^4 \left(n - n_0 - \frac{1}{2}\right)^4 + o(n^3), \quad n \to \infty.$$

Proof. Since there exists a finite number of non-positive eigenvalues we just concentrate on the positive eigenvalues σ.

Let us consider $(\sigma, \phi = (\phi^1, \dots \phi^N))$ an eigenpair of (9) with $\sigma > 0$. In view of Lemma 2.5 we distinguish two cases.

Case 1. We have $\phi = (\phi^1, \dots, \phi^1)$ where (σ, ϕ^1) is an eigenpair for (16). Since $\sigma > 0$, with the notations in (18) then

$$\phi^1(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x), \tag{26}$$

where C_1, C_2, C_3, C_4 are such that to satisfy the boundary conditions in (16) and $\phi^1 \not\equiv 0$. From the conditions at $x = 0$ we get $C_2 = C_4 = 0$. From the conditions at $x = L$ we obtain $\phi^1 \not\equiv 0$ when the compatibility conditions

$$\beta \cosh(\alpha L) \sin(\beta L) + \alpha \sinh(\alpha L) \cos(\beta L) = 0, \quad \alpha, \beta > 0, \tag{27}$$

are satisfied. This is equivalent to

$$\beta \tan(\beta L) = -\alpha \tanh(\alpha L), \quad \cos(\beta L) \neq 0$$

or equivalently (in view of (18))

$$\sqrt{\lambda + \alpha^2} \tan(\sqrt{\lambda + \alpha^2} L) = -\alpha \tanh(\alpha L), \quad \cos(\sqrt{\lambda + \alpha^2} L) \neq 0. \tag{28}$$

It is not difficult to note that the function $(0, \infty) \ni \alpha \mapsto -\alpha \tanh(\alpha L)$ is strictly decreasing whereas the function $(0, \infty) \setminus \{\alpha \mid \sqrt{\lambda + \alpha^2} L = n\pi + \pi/2 \ , n \in \mathbb{N}\} \ni$

$\alpha \mapsto \sqrt{\lambda+\alpha^2}\tan(\sqrt{\lambda+\alpha^2}L)$ is increasing on each interval of the domain (for that to be proved we just have to look at the sign of the corresponding derivatives, see also [11] for similar arguments). So, there exists two strictly increasing sequences $\{\alpha_{1,n}\}_{n\geq 0}$ and $\{\beta_{1,n}\}_{n\geq 0}$ with $\beta_{1,n}^2 = \lambda + \alpha_{1,n}^2$, solutions for (27), where $\beta_{1,n}L \in (n\pi - \pi/2, n\pi + \pi/2)$. Then the sequence of positive eigenvalues $\{\sigma_{1,n}\}_{n\geq 0}$ is given by

$$\sigma_{1,n} = \beta_{1,n}^2 \alpha_{1,n}^2 \tag{29}$$

Coming back to (28) we have

$$-\frac{\alpha_{1,n}}{\sqrt{\lambda+\alpha_{1,n}^2}} = \frac{\tan\left(\sqrt{\lambda+\alpha_{1,n}^2}L\right)}{\tanh(\alpha_{1,n}L)}.$$

Passing to the limit we obtain

$$\tan(\beta_{1,n}L) = \tan\left(\sqrt{\lambda+\alpha_{1,n}^2}L\right) \to -1, \quad n\to\infty.$$

Then we get

$$\beta_{1,n} = \frac{n\pi}{L} - \frac{\pi}{4L} + o(1), \qquad \text{as } n\to\infty,$$

which combined with (29) gives the asymptotic behavior of $\{\sigma_{1,n}\}_{n\geq 0}$ in the present lemma.

Case 2. We have $\phi = (c_1\phi^{k_0}, \dots, c_k\phi^{k_0})$ for some constants c_k with $\sum_{k=1}^N c_k = 0$, where (σ, ϕ^{k_0}) is an eigenpair for (17). The requirement for σ to be an eigenvalue of (17) is equivalent to

$$\beta\sinh(\alpha L)\cos(\beta L) - \alpha\cosh(\alpha L)\sin(\beta L) = 0 \tag{30}$$

or equivalently

$$\frac{1}{\alpha}\tanh(\alpha L) = \frac{1}{\beta}\tan(\beta L), \qquad \text{with } \beta = \sqrt{\lambda+\alpha^2}, \quad \cos(\beta L)\neq 0. \tag{31}$$

In the same way as in the previous case it is not difficult to prove that the function $(0,\infty)\ni\alpha\mapsto\tanh(\alpha L)/\alpha$ is decreasing whereas the function $(0,\infty)\setminus\{\alpha \mid \sqrt{\lambda+\alpha^2}L = n\pi+\pi/2, n\in\mathbb{N}\}\ni\alpha\mapsto\tan(\sqrt{\lambda+\alpha^2}L)/(\sqrt{\lambda+\alpha^2}L)$ is strictly increasing on each open interval of the domain. Thus we obtain two sequences of solutions $\{\alpha_{2,n}\}_{n\geq 0}$ $\{\beta_{2,n}\}_{n\geq 0}$, to (31) with $\beta_{2,n}L\in(n\pi-\pi/2, n\pi+\pi/2)$. Rewriting (30) as

$$\frac{\sqrt{\lambda+\alpha_{2,n}^2}}{\alpha_{2,n}} = \frac{\tan\left(\sqrt{\lambda+\alpha_{2,n}^2}L\right)}{\tanh(\alpha_{2,n}L)}$$

and passing to the limit as n tends to infinity we obtain similarly as in Case 1 that

$$\beta_{2,n} = \frac{n\pi}{L} + \frac{\pi}{4L} + o(1), \qquad \text{as } n\to\infty.$$

Since $\sigma_{2,n} = \beta_{2,n}^2\alpha_{2,n}^2$ we obtain the asymptotic behaviour. The multiplicity of eigenvalues is a consequence of Lemma 2.5.

On the other hand let us observe that

$$\sigma_{1,n} < \sigma_{2,n} < \sigma_{1,n+1} < \sigma_{2,n+1}, \quad \forall n\in\mathbb{N}.$$

By concatenating the sequences $\{\sigma_{1,n}\}$ and $\{\sigma_{2,n}\}$ we obtain $\sigma_{2n} = \sigma_{1,2n-2n_0}$ and $\sigma_{2n+1} = \sigma_{2,2n+1-2n_0}$ for all $n\geq n_0$ which implies the asymptotic behavior of the sequence $\{\sigma_n\}_{n\geq 0}$ Thus the present lemma is proved.

□

As a consequence of Lemma 2.7 we easily obtain the spectral gap

Lemma 2.8. *Assume* $\{\sigma_n\}_{n\geq n_0}$ *is the increasing set of positive eigenvalues of problem* (9)*. Then there is a spectral gap at infinity. In fact, we have more than that i.e.*

$$\liminf_{n\to\infty}(\sigma_{n+1}-\sigma_n)=\infty. \tag{32}$$

Consequently, there exists a constant $c_{gap}>0$ *such that*

$$\sigma_{n+1}-\sigma_n>c_{gap},\quad \forall n\geq n_0.$$

For our purposes we also need to prove some asymptotic properties for the eigenfunctions evaluated at $x=L$ as follows.

Lemma 2.9. *Let* $(\sigma_n,\phi_n=(\phi_n^k)_{k=1,N})$ *be an eigenpair of problem* (9) *such that* $\sigma_n>0$ *and* $\|\phi_n\|_{L^2(\Gamma)}=1$ *for any* $n\in\mathbb{N}$*. Then there exist the constants* $a,b>0$ *depending only on* L *and* N *such that for all* $n\in\mathbb{N}$ *and any* $k\in\{1,\ldots,N\}$ *it holds*

$$an^2\leq|\phi_{n,xx}^k(L)|\leq bn^2, \tag{33}$$

Proof. As in the proof of Lemma 2.7 we distinguish two cases as follows.
Case 1. This case corresponds to eigenfunctions ϕ_n which have the form $\phi_n=(\phi_n^1,\ldots,\phi_n^1)$ where (σ_n,ϕ_n^1) are eigenpairs of problem (16). Let $\{\alpha_{1,n}\}_{n\geq0}$, $\{\beta_{1,n}\}_{n\geq0}$ be the sequences defined in the proof of Lemma 2.7 satisfying the compatibility condition (27). Then in view of the boundary conditions at $x=L$ we get

$$\phi_n^1(x)=C_1\cos(\beta_{1,n}x)+C_3\cosh(\alpha_{1,n}x),$$

where C_1,C_3 are as in (26). In view of (27) observe that $\cos(\beta_{1,n}L)\neq0$ and since $\phi_n^1(L)=0$ we obtain

$$\phi_n^1(x)=-\frac{C_3\cosh(\alpha_{1,n}L)}{\cos(\beta_{1,n}L)}\cos(\beta_{1,n}x)+C_3\cosh(\alpha_{1,n}x). \tag{34}$$

The constant C_3 in (34) is chosen such that $\|\phi_n\|_{L^2(\Gamma)}=1(=\sqrt{N}\|\phi_n^1\|_{L^2(0,L)})$, i.e.

$$C_3=\frac{\cos(\beta_{1,n}L)}{\sqrt{N}\sqrt{\int_0^L|\cosh(\alpha_{1,n}L)\cos(\beta_{1,n}x)+\cosh(\alpha_{1,n}x)\cos(\beta_{1,n}L)|^2\,dx}}.$$

Due to (27) after expanding the square within the integral of C_3 we notice that the cross term is vanishing, i.e.

$$\int_0^L\cos(\beta_{1,n}x)\cosh(\alpha_{1,n}x)dx=0. \tag{35}$$

Indeed, using integration by parts we have the formula

$$\int_0^L\cos(\beta x)e^{\alpha x}dx=\frac{1}{\alpha^2+\beta^2}\left(\alpha\cos(\beta L)e^{\alpha L}-\alpha+\beta\sin(\beta L)e^{\alpha L}\right),\quad\forall\alpha,\beta\neq0.$$

Applying it two times for $(\alpha,\beta)=(\alpha_{1,n},\beta_{1,n})$ and $(\alpha,\beta)=(-\alpha_{1,n},\beta_{1,n})$, respectively, we obtain the validity of (35) taking into account (27). Therefore, due to (35) and (34) we obtain

$$\phi_n^1(x)=\frac{-\cosh(\alpha_{1,n}L)\cos(\beta_{1,n}x)+\cos(\beta_{1,n}L)\cosh(\alpha_{1,n}x)}{\sqrt{N}\sqrt{\cosh^2(\alpha_{1,n}L)\int_0^L\cos^2(\beta_{1,n}x)dx+\cos^2(\beta_{1,n}L)\int_0^L\cosh^2(\alpha_{1,n}x)dx}}. \tag{36}$$

In consequence we get

$$\phi_{n,xx}^1(L)=\frac{(\alpha_{1,n}^2+\beta_{1,n}^2)\cosh(\alpha_{1,n}L)\cos(\beta_{1,n}L)}{\sqrt{N}\sqrt{\cosh^2(\alpha_{1,n}L)\int_0^L\cos^2(\beta_{1,n}x)dx+\cos^2(\beta_{1,n}L)\int_0^L\cosh^2(\alpha_{1,n}x)dx}}. \tag{37}$$

Then, in view of the behavior of $\{\alpha_{1,n}\}_n,\ \{\beta_{1,n}\}_n$ we successively have

$$\lim_{n\to\infty} \frac{\cosh(\alpha_{1,n}L)}{\sqrt{\cosh^2(\alpha_{1,n}L)\int_0^L \cos^2(\beta_{1,n}x)dx + \cos^2(\beta_{1,n}L)\int_0^L \cosh^2(\alpha_{1,n}x)dx}}$$

$$= \lim_{n\to\infty} \frac{\cosh(\alpha_{1,n}L)}{\sqrt{\cosh^2(\alpha_{1,n}L)\Big(\frac{L}{2} + \frac{\sin(2\beta_{1,n}x)}{4\beta_{1,n}}\Big) + \cos^2(\beta_{1,n}L)\Big(\frac{L}{2} + \frac{\sinh(2\alpha_{1,n}L)}{4\alpha_{1,n}}\Big)}}$$

$$= \sqrt{\frac{2}{L}}. \tag{38}$$

On the other hand, we also have

$$\inf_{n\in\mathbb{N}} |\cos(\beta_{1,n}L)| > 0, \tag{39}$$

otherwise it contradicts (27). Finally, combining (37)-(39) and the behavior of the sequences $\alpha_{1,n}, \beta_{1,n}$ (as $n \to \infty$) we are able to establish the behavior of $|\phi^1_{n,xx}(L)|$ as in (33).

Case 2. This case corresponds to eigenfunctions of the form $\phi_n = (\phi_n^1, c_2\phi_n^1, \ldots, c_N\phi_n^1)$, for each constants c_k, $k \in \{2, \ldots, N\}$, where (σ_n, ϕ_n^1) are eigenpairs for the eigenvalue problem (17). Let $\{\alpha_{2,n}\}_{n\geq 0}$, $\{\beta_{2,n}\}_{n\geq 0}$ be the sequences built in the proof of Lemma 2.7 satisfying the compatibility condition (30) to ensure $\phi_n^1 \neq 0$. Imposing the boundary conditions at the origin in (17) from (26) we get

$$\phi_n^1(x) = C_2\sin(\beta_{2,n}x) + C_4\sinh(\alpha_{2,n}x). \tag{40}$$

Taking into account the normalization of the L^2-norm for ϕ_n^1, using similar steps as in Case 1 we finally obtain

$$\phi_n^1(x) = c_0\frac{-\sinh(\alpha_{2,n}L)\sin(\beta_{2,n}x) + \sin(\beta_{2,n}L)\sinh(\alpha_{2,n}x)}{\sqrt{\sinh^2(\alpha_{2,n}L)\int_0^L \sin^2(\beta_{2,n}x)dx + \sin^2(\beta_{1,n}L)\int_0^L \sinh^2(\alpha_{2,n}x)dx}}, \tag{41}$$

where $c_0 := 1/\sqrt{1 + c_2^2 + \ldots + c_N^2}$. Consequently

$$\phi^1_{n,xx}(L) = c_0\frac{(\alpha_{2,n}^2 + \beta_{2,n}^2)\sinh(\alpha_{2,n}L)\sin(\beta_{2,n}L)}{\sqrt{\sinh^2(\alpha_{2,n}L)\int_0^L \sin^2(\beta_{2,n}x)dx + \sin^2(\beta_{1,n}L)\int_0^L \sinh^2(\alpha_{2,n}x)dx}}. \tag{42}$$

Similarly as in Case 1, one can show that

$$\lim_{n\to\infty} \frac{\sinh(\alpha_{2,n}L)}{\sqrt{\sinh^2(\alpha_{2,n}L)\int_0^L \sin^2(\beta_{2,n}x)dx + \sin^2(\beta_{1,n}L)\int_0^L \sinh^2(\alpha_{2,n}x)dx}} = \sqrt{\frac{2}{L}} \tag{43}$$

and

$$\inf_{n\in\mathbb{N}} |\sin(\beta_{2,n}L)| > 0, \tag{44}$$

which concludes the proof of the lemma. □

2.3. **Well-posedness.** In order to study the well-posedness of problem (2)-(I) we apply the semigroup theory. First, let us consider the polynomial

$$P(x) = \Big(\frac{x}{L}\Big)^4 (x - L) \tag{45}$$

and denote $z^k = y^k - P(x)u^k(t)$, $k \in \{1, \ldots, N\}$. For our purpose it is more convenient to analyze first the equation satisfied by $z = (z^k)_{k=1,N}$. Indeed, it is easy to see that z satisfies the following nonhomogeneous problem with zero boundary conditions:

(46)
$$\begin{cases} z_t^k + \lambda z_{xx}^k + z_{xxxx}^k = -Pu_t^k(t) - (\lambda P_{xx} + P_{xxxx})u^k(t), & (t,x) \in (0,T)\times(0,L) \\ z^i(t,0) = z^j(t,0), & t \in (0,T), \quad i,j \in \{1,\dots,N\} \\ z^k(t,L) = z_x^k(t,L) = 0, & k \in \{1,\dots,N\} \\ \sum_{k=1}^N z_x^k(t,0) = 0, & t \in (0,T) \\ z_{xx}^i(t,0) = z_{xx}^j(t,0), & t \in (0,T), \quad i,j \in \{1,\dots,N\} \\ \sum_{k=1}^N y_{xxx}^k(t,0) = 0, & t \in (0,T) \\ z^k(0,x) = y_0^k(x) - P(x)u^k(0), & x \in (0,L). \end{cases}$$

Problem (46) can be written as an abstract Cauchy problem. Indeed, it follows that

$$\begin{cases} z_t + Az = F(t,x,u), \quad t \in (0,T) \\ z(0) = z_0, \end{cases} \tag{47}$$

where A is the operator defined in (10), whereas $F = (F^k)_{k=1,N}$ and $z_0 = (z_0^k)_{k=1,N}$ are given by

$$F^k(t,x,u) = -P(x)u_t^k(t) - (\lambda P_{xx}(x) + P_{xxxx}(x))u^k(t),$$

respectively,

$$z_0^k(x) = y_0^k(x) - P(x)u^k(0),$$

for any $k \in \{1,\dots,N\}$. In the previous section we have proved that $(A, D(A))$ generates a semigroup in $L^2(\Gamma)$. Therefore, applying the Hille-Yosida theory for the Cauchy problem (47) (see, e.g., [3, Proposition 4.1.6 and Lemma 4.1.5]) we finally obtain

Proposition 2.10. *If $z_0 \in D(A)$ and $F \in C([0,T], L^2(\Gamma)) \cap L^1((0,T), D(A))$ there exists a function $z \in C([0,T], D(A)) \cap C^1([0,T], L^2(\Gamma))$ solution to (46). Moreover, if $z_0 \in L^2(\Gamma)$ and $F \in C([0,T], L^2(\Gamma))$ (it can be extended to $L^1((0,T), L^2(\Gamma))$) there exists a mild function $z \in C([0,T], L^2(\Gamma))$ solution to (46).*

Proposition 2.10 extends for y solution of (2) with initial data $y_0 \in D(A)$ and $y_0 \in L^2(\Gamma)$, respectively.

2.4. **Controllability problem.** Next we address the controllability problem (2)-(I) by using the method of moments [13]. In view of that, let us consider the so-called adjoint problem, that is

$$
\begin{cases}
-q^k_t + \lambda q^k_{xx} + q^k_{xxxx} = 0, & (t,x) \in (0,T)\times(0,L),\\
q^i(t,0) = q^j(t,0), & t\in(0,T), \quad i,j\in\{1,\dots,N\},\\
q^k(t,L) = q^k_x(t,L) = 0, & t\in(0,T), \quad k\in\{1,\dots,N\},\\
\sum_{k=1}^N q^k_x(t,0) = 0, & t\in(0,T),\\
q^i_{xx}(t,0) = q^j_{xx}(t,0), & t\in(0,T), \quad i,j\in\{1,\dots,N\},\\
\sum_{k=1}^N q^k_{xxx}(t,0) = 0, & t\in(0,T),\\
q^k(T,x) = q^k_T(x), & x\in(0,L),
\end{cases}
\tag{48}
$$

Then, we have the following characterization of the null-controllability property.

Lemma 2.11. *The system (2)-(I) is null-controllable in time $T>0$ if and only if, for any initial data $y_0 = (y^k_0)_{k=1,N} \in L^2(\Gamma)$ there exists a control function $u = (u^k)_{k=1,N} \in H^1(\Gamma)$ such that, for any $q_T = (q^k_T)_{k=1,N} \in L^2(\Gamma)$*

$$
(y_0, q(0))_{L^2(\Gamma)} = \sum_{k=1}^N \int_0^T u^k(t) q^k_{xx}(t,L)\,\mathrm{d}t,
\tag{49}
$$

where q is the solution of (48).

Proof. We proceed as in the classical duality approach. We first multiply the equation in (2) by $q = (q^k)_{k=1,N}$, the solution of (48) to obtain

$$
\sum_{k=1}^L \int_0^T\int_0^L \left(y^k_t + \lambda y^k_{xx} + y^k_{xxxx}\right) q^k \,\mathrm{d}x\,\mathrm{d}t = 0.
$$

Integration by parts leads to

$$
\begin{aligned}
0 = &\sum_{k=1}^N \int_0^L y^k q^k\Big|_{t=0}^{t=T} \mathrm{d}x + \sum_{k=1}^N \int_0^T\int_0^L \left(-q^k_t + \lambda q^k_{xx} + q^k_{xxxx}\right) y^k \,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^T \Big(\lambda y^k_x q^k\Big|_{x=0}^{x=L} - \lambda y^k q^k_x\Big|_{x=0}^{x=L} + y^k_{xxx} q^k\Big|_{x=0}^{x=L}\\
&\qquad - y^k_{xx} q^k_x\Big|_{x=0}^{x=L} + y^k_x q^k_{xx}\Big|_{x=0}^{x=L} - y^k q^k_{xxx}\Big|_{x=0}^{x=L}\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{50}
$$

In view of the boundary conditions satisfied by $y = (y^k)_{k=1,N}$ and $q = (q^k)_{k=1,N}$, identity (50) is equivalent to

$$
\sum_{i=1}^N \int_0^L \left(y^k(T,x) q^k(T,x) - y^k_0(x) q^k_0(x)\right) \mathrm{d}x + \int_0^T u^k(t) q^k_{xx}(t,L)\,\mathrm{d}t = 0.
\tag{51}
$$

"Only if" implication. Since (2)-(I) is null-controllable (i.e. $y(T,x) = 0$ for any $x\in\Gamma$) it follows from (51) that condition (49) holds true.

"If" implication. Let us assume the validity of (49). In this case, due to (51) we get

$$
(y(T), q_T)_{L^2(\Gamma)} = 0,
$$

for any $q_T \in L^2(\Gamma)$. This implies $y(T) = 0$. □

As a consequence of Lemma 2.11 and the spectral analysis developed in subsection 2.1 the controllability problem reduces to the following moment problem.

Lemma 2.12. *Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be the distinct eigenvalues of A. Let $m(\sigma_n)$ be the multiplicity of each eigenvalue σ_n whose eigenspace is generated by linear independent eigenvectors $\{\phi_{n,l}\}_{l=1,m(\sigma_n)}$, normalized in $L^2(\Gamma)$. Since $\{\phi_{n,l}\}_{l=1,m(\sigma_n),n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Gamma)$. Then system (2)-(I) is null-controllable if and only if for any initial data $y_0 \in L^2(\Gamma)$,*

$$y_0 = \sum_{n\in\mathbb{N}} \sum_{l=1}^{m(\sigma_n)} y_{0,n,l}\phi_{n,l}, \tag{52}$$

and any time $T > 0$, there exists a control $u = (u^k)_{k=1,N} \in H^1(\Gamma)$ such that
(53)

$$y_{0,n,l}e^{-T\sigma_n} = \sum_{k=1}^{N} \phi^k_{n,l,xx}(L) \int_0^T u^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t, \quad \forall n\in\mathbb{N}, \quad \forall l = 1, m(\sigma_n).$$

Proof. For any $q_T \in L^2(\Gamma)$ we have

$$q_T = \sum_{n\in\mathbb{N}} \sum_{l=1}^{m(\sigma_n)} q_{n,l}\phi_{n,l},$$

where $\sum_{n\in\mathbb{N}} \sum_{l=1}^{m(\sigma_n)} |q_{n,l}|^2 < \infty$. Then, seeking for solutions in separable variable

$$q(t,x) = \sum_{n\in\mathbb{N}} \sum_{l=1}^{m(\sigma_n)} \overline{q}_{n,l}(t)\phi_{n,l}(x),$$

the time coefficients $\overline{q}_{n,l}$ satisfy

$$\overline{q}^{'}_{n,l} - \sigma_n\overline{q}_{n,l} = 0, \quad \overline{q}_{n,l}(T) = q_{n,l}.$$

Then, we obtain

$$q(t,x) = \sum_{n\in\mathbb{N}} e^{(-T+t)\sigma_n} q_{n,l}\phi_{n,l}(x),$$

and therefore

$$q_{xx}(t,L) = \sum_{n\in\mathbb{N}} \sum_{l=1}^{m(\sigma_n)} e^{(-T+t)\sigma_n} q_{n,l}\phi_{n,l,xx}(L). \tag{54}$$

Plugging (54) and (52) in the controllability condition (49) we obtain that the existence of a function u satisfying the moment problem (53) suffices to prove the null-controllability property. □

2.5. **Proof of Theorem 1.1.** We show that a control acting on $N-1$ nodes is sufficient to obtain the null-controllability of system (2)-(I).

According to Lemma 2.12 we have to solve the moment problem (53) by constructing a control u. In order to do that we settle one of the components of $u = (u^k)_{k=1,N}$ to be identically zero. For simplicity, we assume $u^N = 0$. Then the problem of moments (53) becomes
(55)

$$y_{0,n,l}e^{-T\sigma_n} = \sum_{k=1}^{N-1} \phi^k_{n,l,xx}(L) \int_0^T u^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t, \quad \forall n\in\mathbb{N}, \quad \forall l = 1, m(\sigma_n).$$

In fact, in view of the analysis done in the proof of Lemma 2.7 we may have $m(\sigma_n) \in \{1, N-1\}$. More precisely, if σ_n is simple (i.e. $m(\sigma_n) = 1$) then, in view of Lemma 2.5 we may choose $\phi_{n,1} = (\Psi_n, \ldots, \Psi_n)$ where (σ_n, Ψ_n) is an eigenpair of problem (16) such that $\|\phi_{n,1}\|_{L^2(\Gamma)} = 1$. On the other hand, if $m(\sigma_n) = N-1$ we may choose for any $l \in \{1, \dots, N-1\}$

$$\phi_{n,l} = \Phi_n e_l - \Phi_n e_{l+1}, \tag{56}$$

where $\{e_i\}_{i=1,N}$ is the canonical basis of the euclidian space $\mathbb{R}^N$ and (σ_n, Φ_n) is an eigenpair of problem (17). Then $\{\phi_{n,l}\}_{l=1,N-1}$ form an orthonormal basis of the eigenspace of σ_n. If $m(\sigma_n) = 1$ then (55) becomes

$$y_{0,n,1} e^{-T\sigma_n} = \Psi_{n,xx}(L) \sum_{k=1}^{N-1} \int_0^T u^k(T-t) e^{-t\sigma_n}\, \mathrm{d}t, \quad \forall n \in \mathbb{N}. \tag{57}$$

Therefore, it suffices that the control inputs $u^k \in L^2(0,T)$, $k \in \{1, \dots, N-1\}$ satisfy the following moment problems

$$\begin{cases} \int_0^T u^1(T-t)e^{-t\sigma_n} dt = \frac{y_{0,n,1}e^{-T\sigma_n}}{\Psi_{n,xx}(L)}, & \forall n \\ \int_0^T u^k(T-t)e^{-t\sigma_n} dt = 0, & \text{if } k = 2, N-1, \quad \forall n, \end{cases} \tag{58}$$

where by convention the second line in (58) is not taken into account when $N = 2$. On the other hand, if σ_n is a multiple eigenvalue (i.e. $m(\sigma_n) = N-1$) then (55) becomes

$$\begin{cases} y_{0,n,l} e^{-T\sigma_n} = \Phi_{n,xx}(L) \left(\int_0^T u^l(T-t) e^{-t\sigma_n}\, \mathrm{d}t - \int_0^T u^{l+1}(T-t) e^{-t\sigma_n}\, \mathrm{d}t \right) \\ \qquad \forall n \in \mathbb{N},\ \forall l = 1, N-2 \\ y_{0,n,N-1} e^{-T\sigma_n} = \Phi_{n,xx}(L) \int_0^T u^{N-1}(T-t) e^{-t\sigma_n},\ \forall n \in \mathbb{N}, \end{cases} \tag{59}$$

where we make the convection that when $N = 2$ the first relation in (59) does not appear. Equivalently we can write

$$\begin{pmatrix} y_{0,n,1} \\ \vdots \\ \vdots \\ y_{0,n,N-1} \end{pmatrix} e^{-T\sigma_n} = \Phi_{n,xx}(L) A \begin{pmatrix} \int_0^T u^1(T-t) e^{-t\sigma_n}\, \mathrm{d}t \\ \vdots \\ \vdots \\ \int_0^T u^{N-1}(T-t) e^{-t\sigma_n}\, \mathrm{d}t \end{pmatrix} \tag{60}$$

where

$$A = (a_{ij})_{i,j=1,N-1} := \begin{pmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 & -1 \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Let us denote by $A^{-1} = (a^{ij})_{i,j=1,N-1}$ the inverse of matrix A. Then we get

$$\begin{pmatrix} \int_0^T u^1(T-t) e^{-t\sigma_n}\, \mathrm{d}t \\ \vdots \\ \vdots \\ \int_0^T u^{N-1}(T-t) e^{-t\sigma_n}\, \mathrm{d}t \end{pmatrix} = \frac{1}{\Phi_{n,xx}(L)} A^{-1} \begin{pmatrix} y_{0,n,1} \\ \vdots \\ \vdots \\ y_{0,n,N-1} \end{pmatrix} e^{-T\sigma_n}. \tag{61}$$

To summarize, the moment problems to be solved are

$$\int_0^T u^k(T-t) e^{-t\sigma_n}\, \mathrm{d}t = c_{n,k}, \quad n \in \mathbb{N}^\star \tag{62}$$

for each $k = 1, N-1$, where $c_{n,k}$ are given by

$$c_{n,1} := \begin{cases} \frac{y_{o,n,1} e^{-T\sigma_n}}{\Psi_{n,xx}(L)} & \text{if } m(\sigma_n) = 1 \\ \sum_{j=1}^{N-1} \frac{a^{ij} y_{0,j,1} e^{-T\sigma_n}}{\Phi_{n,xx}(L)} & \text{if } m(\sigma_n) = N-1, \end{cases}$$

respectively

$$k \geq 2: \quad c_{n,k} := \begin{cases} 0 & \text{if } m(\sigma_n) = 1 \\ \sum_{j=1}^{N-1} \frac{a^{ij} y_{0,j,k} e^{-T\sigma_n}}{\Phi_{n,xx}(L)} & \text{if } m(\sigma_n) = N-1. \end{cases}$$

Due to the asymptotic behavior of the distinct eigenvalues $\{\sigma_n\}_n$ given in Lemma 2.7 with the asymptotic properties of the corresponding eigenfunctions $\{\phi_n\}_n$ in Lemma 2.9 we are able to show that the series

$$\sum_{n \geq 1} c_{n,k} \prod_{j=1, j \neq n}^{\infty} \frac{\sigma_n + \sigma_j}{\sigma_n - \sigma_j}, \quad \forall k = 1, N-1,$$

is absolutely convergent. Indeed, this is true since $\{\sigma_n\}_n$ fulfill condition (3.10) in [13] and we can apply both Lemma 3.1 in [13] and our asymptotic results above to guarantee the convergence. More exactly, in view of (3.3) and (3.9) in [13] we can employ the method of moments to construct a so-called *bona fide* solution for each one of the moment problems (62). Thus we are able to build the controls $(u^k)_{k=1,N-1}$ verifying (58) and finalize the proof.

Optimality of $N-1$ controls. For the proof of the second statement in Theorem 4.5 let us assume without losing the generality that $i_0 = 1$ and $j_0 = 2$. Then we consider as the initial datum $y_0 = (\phi, \phi, 0, \ldots, 0)$, where ϕ is a normalized eigenfunction (i.e. $\|\phi\|_{L^2} = 1$) of system (64) corresponding to some eigenvalue σ. Then system (53) becomes

$$e^{-T\sigma} = \phi_{xx}(L) \int_0^T (u^1(T-t) - u^2(T-t)) e^{-t\sigma} \, dt. \tag{63}$$

Therefore, such y_0 cannot be driven to zero by any control $u = (u^k)_{k=1,N}$ with the first two components vanishing ($u^1 = u^2 = 0$). The proof of the theorem is finally complete.

3. The KS equation of type (II)

The results obtained in this section are based on a careful analysis of the eigenvalues for the corresponding elliptic operator of system (2)-(II). In the previous section we addressed this problem for system (2)-(I) by using specific spectral results done by Cerpa [4]. In the present case such results are not applicable. Therefore, to take advantage of the strategy implemented in the previous case an additional work has to be done by determining explicitly the spectrum and the eigenfunctions for two different eigenvalue problems as follows.

3.1. Preliminaries I.

In this subsection we analyze the following eigenvalue problem

$$\begin{cases} \lambda\phi_{xx} + \phi_{xxxx} = \sigma\phi, & x \in (0, L) \\ \phi_x(0) = \phi_{xxx}(0) = 0, \\ \phi_x(L) = \phi_{xxx}(L) = 0. \end{cases} \tag{64}$$

Again, we can employ spectral analysis tools to show that system (64) has a sequence of eigenvalues which tends to infinity and is bounded from below by $-\lambda^2/4$.

Making usage of the characteristic equation of the equation in (64), i.e.,

$$r^4 + \lambda r^2 - \sigma = 0$$

in view of the notations in (18) we distinguish several cases as follows.

Case I: $\sigma > 0$**.** In this case, the general solution of the equation in (64) is given by

$$\phi(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x),$$

where C_i are real constants, $i = 1, 4$. Imposing the boundary conditions at $x = 0$ we easily obtain that $C_2 = C_4 = 0$. The boundary conditions at $x = L$ provide a nontrivial solution ϕ if $\sinh(\alpha L)\sin(\beta L) = 0$. Since $\alpha > 0$ this is equivalent to the compatibility condition

$$\sin(\beta L) = 0.$$

Then, we get a sequence $\{\beta_n\}_{n\geq 1}$ of positive solutions, $\beta_n = n\pi/L$. In view of (18) we obtain that the sequence of positive simple eigenvalues is given by

$$\sigma_n = \left(\frac{n\pi}{L}\right)^4 - \lambda\left(\frac{n\pi}{L}\right)^2, \quad n \geq \left[\frac{L\sqrt{\lambda}}{\pi}\right] + 1,$$

where $[\cdot]$ is the floor function. The corresponding eigenfunctions are

$$\phi_n(x) = C_1 \cos(\beta_n x), \quad C_1 \neq 0.$$

Case II: $\sigma = 0$**.** The general solution for the equation in (64) is

$$\phi(x) = C_1 + C_2 x + C_3 \cos(\sqrt{\lambda}x) + C_4 \sin(\sqrt{\lambda}x).$$

From the boundary conditions at $x = 0$ we deduce that $C_2 = C_4 = 0$. According to the boundary conditions at $x = L$ we produce the following alternatives.

(1) If $\sin(\sqrt{\lambda}L) = 0$ then $\sigma = 0$ is an eigenvalue with multiplicity 2 and the eigenfunctions are

$$\phi_0(x) = C_1 + C_3 \cos(\sqrt{\lambda}x), \quad C_1^2 + C_3^2 \neq 0.$$

(2) If $\sin(\sqrt{\lambda}L) \neq 0$ then $\sigma = 0$ is a simple eigenvalue and the eigenfunctions are constant functions, i.e.,

$$\phi_0(x) = C, \quad C \neq 0.$$

Case III: $-\lambda^2/4 < \sigma < 0$**.** The general solution of the equation in (64) is

$$\phi(x) = C_1 \cos(\gamma x) + C_2 \sin(\gamma x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

The boundary conditions at $x = 0$ lead to $C_2 = C_4 = 0$. Defining the quantity $\delta := \sin(\beta L)\sin(\gamma L)$, from the boundary conditions at $x = L$ we obtain that ϕ is an eigenfunction if and only if $\delta = 0$. We distinguish the following cases for $\delta = 0$.

(1) *The case* $\sin(\beta L) = \sin(\gamma L) = 0$. We obtain two sequences of solutions $\beta_n = n\pi/L$ and $\gamma_m = m\pi/L$ with $n \neq m$ (since $\beta \neq \gamma$), $n, m \geq 1$. From (19) this is equivalent to $\lambda \in \mathcal{N}_1$ and in this case the finite set of negative eigenvalues is given by

$$\sigma_{n,m}(x) = -\beta_n^2\gamma_m^2 = -\frac{n^2m^2\pi^4}{L^4}; \quad 1 \leq m < n, \; nm < \frac{\lambda L^2}{2\pi^2}.$$

The corresponding eigenfunctions are

$$\phi_{n,m}(x) = C_1 \cos(\beta_n x) + C_3 \cos(\beta_m x), \quad C_1^2 + C_3^2 \neq 0.$$

(2) *The case* $\delta = 0$, *such that* $\sin^2(\beta L) + \sin^2(\gamma L) > 0$, *i.e.* $\lambda \notin \mathcal{N}_1$. Then we obtain the eigenvalues, i.e.

$$\sigma_n = \left(\frac{n\pi}{L}\right)^4 - \lambda\left(\frac{n\pi}{L}\right)^2, \quad 1 \le n < \frac{L\sqrt{\lambda}}{\pi},$$

with the corresponding eigenfunctions

$$\phi_n(x) = C\cos\left(\frac{n\pi x}{L}\right), \quad C \neq 0.$$

From the spectral analysis developed above it is easy to check the following lemma that will play an important role in the proof of Theorem 1.2.

Lemma 3.1. *Let* $\lambda > 0$ *and* (σ, ϕ) *be an eigenpair of system* (64). *The following holds:*

(1) If $\sigma > 0$ *then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

(2) If $\sigma = 0$ *and* $\lambda \in \mathcal{N}_2$ *then* $\phi(L)$ *and* $\lambda\phi(L) + \phi_{xx}(L)$ *cannot vanish simultaneously.*

(3) If $\sigma = 0$ *and* $\lambda \notin \mathcal{N}_2$ *then* $\phi(L) \neq 0$ *and* $\lambda\phi(L) + \phi_{xx}(L) \neq 0$.

(4) If $\sigma < 0$ *and* $\lambda \notin \mathcal{N}_1$ *then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

(5) If $\sigma < 0$ *and* $\lambda \in \mathcal{N}_1$ *then* $\phi(L)$ *and* $\lambda\phi(L) + \phi_{xx}(L)$ *cannot vanish simultaneously.*

3.2. **Preliminaries II.** Secondly we analyze the following eigenvalue problem

$$\begin{cases} \lambda\phi_{xx} + \phi_{xxxx} = \sigma\phi, & x \in (0, L), \\ \phi(0) = \phi_{xx}(0) = 0, \\ \phi_x(L) = \phi_{xxx}(L) = 0. \end{cases} \tag{65}$$

Again, the spectrum of (65) is pure discrete, bounded from below by $-\lambda^2/4$ and tends to infinity. Making use of the notations (18) we distinguish the following cases.

Case I: $\sigma > 0$. The solution of the equation in (64) is

$$\phi(x) = C_1\cosh(\alpha x) + C_2\sinh(\alpha x) + C_3\cos(\beta x) + C_4\sin(\beta x).$$

From the boundary conditions at $x = 0$ we easily obtain that $C_1 = C_3 = 0$. The conditions at $x = L$ say that ϕ is an eigenfunction under the constraint

$$\cos(\beta L) = 0.$$

We get a sequence $\{\beta_n\}_{n\ge 0}$ of positive solutions, $\beta_n = (2n+1)\pi/2L$. In view of (18)-(19) we obtain the sequence of simple eigenvalues $\sigma_n = \beta_n^2(\beta_n^2 - \lambda)$, i.e.

$$\sigma_n = \frac{(2n+1)^2\pi^2}{4L^2}\left(\frac{(2n+1)^2\pi^2}{4L^2} - \lambda\right), \quad n \ge \max\left\{0, \left[\frac{1}{2}\left(\frac{2L\sqrt{\lambda}}{\pi} - 1\right)\right] + 1\right\},$$

with the corresponding family of eigenfunctions

$$\phi_n(x) = C_2\sin(\beta_n x), \quad C_2 \neq 0.$$

Case II: $\sigma = 0$**.** The general solution for the equation in (64) is

$$\phi(x) = C_1 + C_2 x + C_3 \cos(\sqrt{\lambda} x) + C_4 \sin(\sqrt{\lambda} x).$$

From the boundary conditions at $x = 0$ we deduce that $C_1 = C_3 = 0$. Then, the boundary conditions at $x = L$ produce the following cases.

(1) If $\cos(\sqrt{\lambda} L) \neq 0$, which is equivalent to $\lambda \notin \mathcal{N}_3$, then $\sigma = 0$ is not an eigenvalue.

(2) On the contrary, if $\cos(\sqrt{\lambda} L) = 0$, which is equivalent to $\lambda \in \mathcal{N}_3$, then $\sigma = 0$ is a simple eigenvalue with the corresponding eigenfunctions

$$\phi_0(x) = C_4 \sin(\sqrt{\lambda} x), \quad C_4 \neq 0.$$

Case III: $-\lambda^2/4 < \sigma < 0$**.** The general solution of the equation in (65) is

$$\phi(x) = C_1 \cos(\gamma x) + C_2 \sin(\gamma x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

The boundary conditions at $x = 0$ give $C_1 = C_3 = 0$. Then, from the conditions at $x = L$ we obtain that ϕ is an eigenfunction if and only if

$$\cos(\beta L) \cos(\gamma L) = 0.$$

We distinguish the following cases.

(1) *The case* $\cos(\beta L) = \cos(\gamma L) = 0$. We obtain $\beta_n = (2n+1)\pi/2L$ and $\gamma_m = (2m+1)\pi/2L$, with $n \neq m$ (since $\beta \neq \gamma$). From (18)-(19) this is equivalent to $\lambda \in \mathcal{N}_{odd}$. In this case the eigenvalues have multiplicity 2 and they are given by

$$\sigma_{n,m}(x) = -\beta_n^2 \beta_m^2; \quad 0 \leq m < n, \; (2n+1)(2m+1) \leq \frac{2\lambda L^2}{\pi^2}.$$

The corresponding eigenfunctions are

$$\phi_{n,m}(x) = C_2 \sin(\beta_n x) + C_4 \sin(\beta_m x), \quad C_1^2 + C_3^2 \neq 0.$$

(2) *The case* $\cos^2(\beta L) + \cos^2(\gamma L) > 0$, *i.e.* $\lambda \notin \mathcal{N}_{odd}$. We obtain a finite number of simple eigenvalues such as

$$\sigma_n = \frac{(2n+1)^2\pi^2}{4L^2} \left(\frac{(2n+1)^2\pi^2}{4L^2} - \lambda \right), \quad 0 \leq n \leq \max \left\{ \left[\frac{1}{2} \left(\frac{2L\sqrt{\lambda}}{\pi} - 1 \right) \right], 0 \right\}.$$

The corresponding eigenfunctions are given by

$$\phi_n(x) = C \sin\left(\frac{(2n+1)\pi x}{2L} \right), \quad C \neq 0.$$

Combining the spectral results of this section we conclude

Lemma 3.2. *Let* $\lambda > 0$ *and* (σ, ϕ) *be an eigenpair of system* (65).

(1) If $\sigma > 0$ *then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

(2) If $\lambda \notin \mathcal{N}_3$ *then* $\sigma = 0$ *is not an eigenvalue.*

(3) If $\lambda \in \mathcal{N}_3$ *then* $\sigma = 0$ *is an eigenvalue and*

$$\phi(L) = \lambda\phi(L) + \phi_{xx}(L) = 0.$$

(4) If $\sigma < 0$ *and* $\lambda \in \mathcal{N}_{odd}$ *then* $\phi(L)$ *and* $\lambda\phi(L) + \phi_{xx}(L)$ *cannot vanish simultaneously.*

(5) If $\sigma < 0$ *and* $\lambda \notin \mathcal{N}_{odd}$ *then*

$$\phi(L) \neq 0 \text{ and } \lambda\phi(L) + \phi_{xx}(L) \neq 0.$$

3.3. **Spectral analysis.** In this section we aim to discuss some general properties of the following spectral problem

$$\begin{cases} \lambda\phi^k_{xx} + \phi^k_{xxxx} = \sigma\phi^k, & x \in (0, L), \\ \phi^i(0) = \phi^j(0), & i, j \in \{1, \dots, N\}, \\ \phi^k_x(L) = \phi^k_{xxx}(L) = 0, & k \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi^k_x(0) = 0, & \\ \phi^i_{xx}(0) = \phi^j_{xx}(0), & i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi^k_{xxx}(0) = 0, & \end{cases} \tag{66}$$

which governs our control system (2)-(II). This is equivalent to study the spectral properties of the fourth order operator

$$A : D(A) \subset L^2(\Gamma) \to L^2(\Gamma)$$

given by
(67)

$$\begin{cases} A\phi^k = \lambda\phi^k_{xx} + \phi^k_{xxxx}, \quad k \in \{1, \dots, N\} \\ D(A) = \left\{ \begin{array}{l} \phi = (\phi^k)_{k=1,N} \in H^4(\Gamma) \mid \phi^k_x(L) = \phi^k_{xxx}(L) = 0, \quad k \in \{1, \dots, N\}, \\ \phi^i(0) = \phi^j(0),\ \phi^i_{xx}(0) = \phi^j_{xx}(0), \quad i, j \in \{1, \dots, N\}, \\ \sum_{k=1}^N \phi^k_x(0) = 0,\ \sum_{k=1}^N \phi^k_{xxx}(0) = 0, \end{array} \right\}. \end{cases}$$

Similar to the operator induced by the model (2)-(I) we obtain

Proposition 3.3. *For any $\mu > \lambda^2/4$ the operator*

$$A + \mu I : D(A) \subset L^2(\Gamma) \to L^2(\Gamma),$$

is a non-negative self-adjoint operator with compact inverse. In particular, it has a pure discrete spectrum consisted by a sequence of nonnegative eigenvalues $\{\sigma_{\mu,n}\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \sigma_{\mu,n} = \infty$. Moreover, up to a normalization, the corresponding eigenfunctions $\{\phi_{\mu,n}\}_{n\in\mathbb{N}}$ form an orthonormal basis of $L^2(\Gamma)$.

Moreover,

Proposition 3.4. *The spectrum $\{\sigma_n\}_{n\in\mathbb{N}}$ of problem (66) verifies*

$$-\frac{\lambda^2}{4} < \sigma_n, \ \forall n \in \mathbb{N}, \qquad \sigma_n \to \infty, \ \text{as } n \to \infty. \tag{68}$$

The details of the proof of Propositions 3.3 and 3.4 are quite similar as in the model (2)-(I), therefore they will be omitted here.

3.4. **Qualitative properties of the eigenvalues.** In this section we apply the preliminary results shown in subsections 3.1 and 3.2 to obtain useful properties of the eigenvalues of (66) which will play a crucial role for proving the null-controllability of problem (2)-(II). In particular we obtain the asymptotic behavior of the eigenvalues of system (66) and useful asymptotic properties of the corresponding eigenfunctions.

For any fixed $\lambda > 0$ let us firstly consider the following two eigenvalue problems which have been already analyzed in subsections 3.1 and 3.2:

$$(69)\qquad E_1 : \begin{cases} \lambda\Psi_{xx} + \Psi_{xxxx} = \sigma\Psi, & x \in (0, L), \\ \Psi_x(0) = 0,\ \Psi_{xxx}(0) = 0, \\ \Psi_x(L) = \Psi_{xxx}(L) = 0, \end{cases}$$

and

$$(70)\qquad E_2 : \begin{cases} \lambda\Phi_{xx} + \Phi_{xxxx} = \sigma\Phi, & x \in (0, L), \\ \Phi(0) = 0,\ \Phi_{xx}(0) = 0, \\ \Phi_x(L) = \Phi_{xxx}(L) = 0, \end{cases}$$

respectively. Recall that Lemma 3.1 applies for (69) whereas Lemma 3.2 applies for (70). Coming back to our spectral problem (66), we introduce the functions

$$(71)\qquad S := \sum_{k=1}^{N} \phi^k,$$

$$(72)\qquad D^k := \phi^k - \frac{S}{N}, \quad k \in \{1, \dots, N\}.$$

The motivation for analyzing systems (69) and (70) is due to the fact that S verifies (69) whereas D^k satisfies (70) for all $k \in \{1, \dots, N\}$.

Next we state and prove some preliminary lemmas.

Lemma 3.5. *For any $\lambda > 0$ the eigenvalue problems (69) and (70) have no any common positive eigenvalues. The value $\sigma = 0$ is a common eigenvalue if and only if λ belongs to $\mathcal{N}_3$. Moreover, problems (69) and (70) have no common negative eigenvalues if and only if $\lambda \notin \mathcal{N}_4$.*

Proof. Assume that $\sigma > 0$ is a common eigenvalue for (69) and (70). Then, according to the precise analysis in subsections 3.1 and 3.2 we must necessary have

$$\sin(\beta L) = \cos(\beta L),$$

which never may happen. Again, in view of the subsections above we obtain that $\sigma = 0$ is an eigenvalue for (69) but it cannot be an eigenvalue for (70) unless $\lambda \in \mathcal{N}_3$. Moreover, if some $\sigma < 0$ was a common eigenvalue we should have

$$\sin(\beta L)\sin(\gamma L) = \cos(\beta L)\cos(\gamma L) = 0,$$

which is equivalent to the alternatives

$$\sin(\beta L) = \cos(\gamma L) = 0 \text{ or } \sin(\gamma L) = \cos(\beta L) = 0.$$

This is impossible unless $\lambda \in \mathcal{N}_4$. $\square$

In consequence we have the following partition for the set of eigenvalues of system (66).

Lemma 3.6. *Assume $\lambda \notin \mathcal{N}_{mixt}$ then*

$$(73)\qquad \sigma_p(A) = \sigma_p(E_1) \cup \sigma_p(E_2); \quad \sigma_p(E_1) \cap \sigma_p(E_2) = \emptyset,$$

where $\sigma_p(A)$, $\sigma_p(E_1)$ and $\sigma_p(E_2)$ denote the set of eigenvalues for the spectral problems (66), (69) and (70), respectively. In addition, we can precisely describe the eigenpairs of (66) as follows. If $(\sigma, \phi = (\phi^k)_{k=1,N})$ is an eigenpair of (66) we have the following possibilities:

(1) *If $\sigma > 0$ then we have the alternative*
   (a) *either σ is an eigenvalue of* (69) *and there exists Ψ and eigenfunction of σ in* (69) *such that*
$$\phi = (\Psi, \dots, \Psi).$$
   (In this case σ is a simple eigenvalue of (66) *(i.e. $m(\sigma) = 1$) and a basis for its eigenpspace is given by*
$$\mathcal{B}_\sigma = \{(\Psi, \dots \Psi)\}.$$
   (b) *or σ is an eigenvalue of* (70) *and there exists an eigenpair (σ, Φ) of* (70) *and there exist a nontrivial constant vector $\overrightarrow{C} = (c_i)_{i=1,N}$ with $\sum_{k=1}^N c_k = 0$ such that*
$$\phi = \overrightarrow{C}\Phi.$$
   In this case σ has multiplicity $m(\sigma) = N-1$. A basis for the eigenspace of σ is given by
$$\mathcal{B}_\sigma = \{\Phi \mathbf{e}_l - \Phi \mathbf{e}_{l+1}\}_{l=1,N-1}.$$

(2) *If $\sigma = 0$ then σ is an eigenvalue of* (69)*. We distinguish two cases*
   - *If $\lambda \notin \mathcal{N}_2$ then σ is has multiplicity $m(\sigma) = 1$ and a basis for its eigenspace in* (66) *is*
$$\mathcal{B}_\sigma = \{(1, \dots, 1)\}.$$
   - *If $\lambda \in \mathcal{N}_2$ then σ has multiplicity $m(\sigma) = 2$ and its eigenspace in* (66) *is induced by the basis*
$$\mathcal{B}_\sigma = \{(1, \dots, 1), \quad (\Psi, \dots, \Psi)\},$$
   where 1 and Ψ are two linear independent eigenfunctions of σ in (69).

(3) *If $\sigma < 0$ then we have the alternative*
   (a) *either σ is eigevalue of* (69)*. In this case σ has multiplicity $m(\sigma) = 2$ and a basis for the eigenspace of σ is given by*
$$\mathcal{B}_\sigma = \{(\Psi, \dots, \Psi), \quad (\tilde{\Psi}, \dots, \tilde{\Psi})\},$$
   where Ψ and $\tilde{\Psi}$ are two linear independent eigenfunctions of σ in (69).
   (b) *or σ is an eigenvalue of* (70)*. In this case we distinguish two cases*
      - *If $\lambda \notin \mathcal{N}_{odd}$ there exists an eigenpair (σ, Φ) of* (70) *and there exists the nontrivial constant vector $\overrightarrow{C} = (c_i)_{i=1,N}$ with $\sum_{k=1}^N c_k = 0$ such that*
$$\phi = \overrightarrow{C}\Phi.$$
      In this case σ has multiplicity $m(\sigma) = N - 1$ and a basis for its eigenspace is given by
$$\mathcal{B}_\sigma = \{\Phi \mathbf{e}_l - \Phi \mathbf{e}_{l+1}\}_{l=1,N-1}.$$
      - *If $\lambda \in \mathcal{N}_{odd}$ then there exists two linear independent eigenfunctions $\Phi, \tilde{\Phi}$ of σ in* (70) *and there exists two scalar vectors $\overrightarrow{C} = (c_i)_{i=1,N}$, $\overrightarrow{D} = (d_i)_{i=1,N}$ such that $\sum_{i=1}^N c_i = \sum_{i=1}^N d_i = 0$ and*
$$\phi = \overrightarrow{C}\Phi + \overrightarrow{D}\tilde{\Phi}.$$
      In this case σ has multiplicity $m(\sigma) = 2(N-1)$ and a basis for its eigenspace is given by
$$\mathcal{B}_\sigma = \{\Phi \mathbf{e}_l - \Phi \mathbf{e}_{l+1}, \quad \tilde{\Phi}\mathbf{e}_l - \tilde{\Phi}\mathbf{e}_{l+1}\}_{l=1,N-1}.$$

Remark 3.7. *In fact the analysis in the preliminary sections 3.1 and 3.2 allow us to say more about the eigenfunctions* $\Psi, \tilde{\Psi}, \Phi, \tilde{\Phi}$ *in Lemma 3.6 since they are actually sinus or cosinus type functions.*

Proof of Lemma 3.6. First we proof the partition of the eigenvalues (73). Assume $(\sigma, \phi = (\phi^k)_{k=1,N})$ is an eigenpair of (66) . Then σ verifies (69) for $\Psi = S$ in (71). In addition, σ verifies (70) for any $\Phi = D^k$ in (72). If $S \neq 0$ then (σ, S) is an eigenpair for (69). Otherwise, if $S = 0$, according to Lemma 3.5 we have $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. Consequently, there exists $k_0 \in \{1, \dots, N\}$ such that $\phi^{k_0} \neq 0$ and therefore (σ, ϕ^{k_0}) is an eigenpair for (70).

Conversely, Let (σ, Ψ) be an eigenpair for (69). Then $(\sigma, \phi = (\Psi, \dots, \Psi, \Psi))$ is an eigenpair for (66). Let (σ, Φ) be an eigenpair for (70). Then $(\sigma, \phi = (0, \dots, 0, -\Phi, \Phi))$ is an eigenpair for (66), which completes the first part of Lemma 3.6.

The rest of the proof follows in each one of the cases $\sigma > 0$, $\sigma = 0$ and $\sigma < 0$ from the preliminary analysis in sections 3.1 and 3.2. □

The previous results allow us to conclude the following lemma.

Lemma 3.8. *For any* $\lambda \notin \mathcal{N}_{mixt}$ *any eigenfunction* $\phi = (\phi^k)_{k=1,N}$ *of* A *in* (67) *satisfies* $\phi^k(L) \neq 0$ *for at least two indexes or* $\lambda\phi^k(L) + \phi^k_{xx}(L) \neq 0$ *for at least two indexes* $k \in \{1, \dots, N\}$.

Proof. With the same notations as above we have that S and D^k, $k \in \{1, \dots, N\}$, satisfy (69) and (70). We distinguish two cases as follows.

The case $S \not\equiv 0$. Using Lemma the first part of 3.6 we must have $D^k \equiv 0$, for all $k \in \{1, \dots, N\}$. This means that $\phi = (S/N, \dots, S/N)$ where S is an eigenfunction of problem (69). So, from Lemma 3.1 it holds that $S(L) \neq 0$ or $\lambda S(L) + S_{xx}(L) \neq 0$. This gives the desired result.

The case $S \equiv 0$. In this case we have $D^k = \phi^k$ for all $k \in \{1, \dots, N\}$. Assume that for at least $N-1$ indexes $k \in \{1, \dots, N\}$ we have $\phi^k(L) = 0$ and also $\lambda\phi^k(L) + \phi^k_{xx}(L) = 0$ for at least $N-1$ indexes. Then we must have $\phi^k(L) = 0$ for all $k \in \{1, \dots, N\}$ and $\phi^k_{xx}(L) = 0$ for at least $N-1$ indexes. Since $S \equiv 0$ we also get $\phi^k_{xx}(L) = 0$ for any $k \in \{1, \dots, N\}$. On the other hand, from the hypothesis we know that there exists $k_0 \in \{1, \dots, N\}$ such that $\phi^{k_0} \neq 0$. This implies that ϕ^{k_0} is an eigenfunction for (70). For any $\lambda > 0$, applying Lemma 3.1 we must have $\phi^{k_0}_{xx}(L) \neq 0$ or $\lambda\phi^{k_0}(L) + \phi^{k_0}_{xx}(L) \neq 0$, which leads to a contradiction.

Therefore, the proof is finished. □

We also have

Lemma 3.9. *Let* $\{\sigma_n\}_{n\geq 0}$ *be the family of eigenvalues for the operator* A *in* (67) *and let* ϕ_n *be a corresponding eigenfunction of the spectral problem* (66) *with* $\|\phi_n\|_{L^2(\Gamma)} = 1$. *Also, let* $\{\sigma_n\}_{n\geq n_0}$ *be the set of positive eigenvalues. We claim*

(1) *The positive eigenvalues of problem* (66) *satisfy the asymptotic property*

$$\sigma_n = \left(\frac{\pi}{2L}\right)^4 \left(n - n_0 + 2\left[\frac{L\sqrt{\lambda}}{2\pi}\right] + 1\right)^4 + o(n^3), \quad n \to \infty.$$

(2) *For any fixed component* $k \in \{1, \dots, N\}$ *there exists a positive constant* $C_{N,L}$ *depending only on* N *and* L *such that*

$$|\phi^k_n(L)| = C_{N,L}.$$

(3) *For any* $\lambda > 0$ *and any fixed component* $k \in \{1, \dots, N\}$ *it holds*

$$\lim_{n\to\infty} \frac{|\lambda\phi^k_n(L) + \phi^k_{n,xx}(L)|}{n^2} = C_{N,L,\lambda},$$

where $C_{N,L,\lambda}$ *is a positive constant depending only on* N, L *and* λ.

Proof of Lemma 3.9. First let us make the notation $n_0(\lambda) = \left[\frac{L\sqrt{\lambda}}{2\pi}\right] + 1$. According to sections 3.1-3.2 and Lemma 3.6 we have the partition of the positive eigenvalues

$$\{\sigma_n \mid n \geq n_0\} = \{\sigma_{1,n} \mid n \geq n_0(\lambda)\} \cup \{\sigma_{2,n} \mid n \geq n_0(\lambda)\},$$

where $\sigma_{1,n}$ are the positive eigenvalues of (64) whereas $\sigma_{2,n}$ are the positive eigenvalues of (65). Then we observe that

$$0 < \sigma_{1,n} < \sigma_{2,n} < \sigma_{1,n+1}, \quad \forall n \geq n_0(\lambda).$$

Let us now define the sequence

$$\tilde{\sigma}_n := \left(\frac{n\pi}{2L}\right)^2 \left(\frac{n^2\pi^2}{4L^2} - \lambda\right), \quad n \geq 2n_0(\lambda).$$

Therefore, $\tilde{\sigma}_n$ is the sequence obtained by concatenating $\{\sigma_{1,n}\}$ and $\sigma_{2,n}$ since

$$\tilde{\sigma}_{2n} = \sigma_{1,n}, \quad \tilde{\sigma}_{2n+1} = \sigma_{2,n}, \quad \forall n \geq n_0(\lambda).$$

Finally we remark that

$$\sigma_n = \tilde{\sigma}_{n-n_0+2n_0(\lambda)}, \quad n \geq n_0,$$

and the asympotic formula is proved. The rest of the proof of Lemma 3.9 is a direct consequence of the analysis done in subsections 3.1 and 3.2. We omit further details here since both the eigenvalues and eigenfunctions are explicitly determined in subsections 3.1-3.2 and easy computations are just to be checked. □

3.5. **Controllability problem.** The control problem (2)-(II) is reduced to solve the following moment problem. Similarly as in Lemma 2.12 we can show

Lemma 3.10. *Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be the set of distinct eigenvalues for system (66) and denote by $m(\sigma_n)$ the multiplicity of σ_n whose eigenspace is generated by linear independent eigenfunctions normalized in $L^2(\Gamma)$, say, $\{\phi_{n,l}\}_{l=1,m(\sigma_n)}$. Since $\{\phi_{n,l}\}_{l=1,m(\sigma_n),n\in\mathbb{N}}$ form an orthonormal basis in $L^2(\Gamma)$ then system (2)-(II) is null-controllable if for any initial data $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$,*

$$y_0 = \sum_{n\in\mathbb{N}} \sum_{l=1}^{m(\sigma_n)} y_{0,n,l}\phi_{n,l}$$

and any time $T > 0$, there exist controls $u = (a^k, b^k)_{k=1,N} \in H^1(\Gamma) \times H^1(\Gamma)$ such that

$$y_{0,n,l}e^{-T\sigma_n} = \sum_{k=1}^{N}(\lambda\phi_{n,l}^k(L) + \phi_{n,l,xx}^k(L)) \int_0^T a^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t$$

$$+ \sum_{k=1}^{N} \phi_n^k(L) \int_0^T b^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t, \quad \forall n \in \mathbb{N}, \quad l = 1, m(\sigma_n). \tag{74}$$

3.6. **Proof of Theorem 1.2.** In view of Lemma 3.10 it is sufficient to ensure the existence of controls $u = (a^k, b^k)_{k=1,N}$ satisfying (74). The construction of such controls is again based on the strategy of Fattorini-Russell [13] implemented also in the proof of Theorem 1.1.

First, without losing the generality, we assume for simplicity that the control does not act on the N-th component, i.e. $u^N = (a^N, b^N) = (0, 0)$. Then the moments problem (74) becomes

$$
\begin{aligned}
y_{0,n,l}e^{-T\sigma_n} = &\sum_{k=1}^{N-1}(\lambda\phi^k_{n,l}(L)+\phi^k_{n,l,xx}(L))\int_0^T a^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t\\
&+\sum_{k=1}^{N-1}\phi^k_{n,l}(L)\int_0^T b^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t, \quad \forall n\in\mathbb{N}., \forall\, l=1,\dots,m(\sigma_n).
\end{aligned}
\tag{75}
$$

We explain now how we construct the sequences a^k, b^k, depending if σ is eigenvalue of problem (69) or (70). Let us consider the first type.

Case I. σ_n is eigenvalue of (69).

(1) $\sigma_n>0$, $m(\sigma_n)=1$, $\phi^k_{n,1}=\Psi$, $k=1,\dots,N$. In view of Lemma 3.1 both terms $\Psi(L)$ and $\lambda\Psi(L)+\Psi_{xx}(L)$ do not vanish so we can choose

$$
\int_0^T a^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t=\frac{1}{2(N-1)}\frac{y_{0,n,1}e^{-T\sigma_n}}{\lambda\Psi(L)+\Psi_{xx}(L)}, \quad k=1,\dots,N-1,
$$

$$
\int_0^T b^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t=\frac{1}{2(N-1)}\frac{y_{0,n,1}e^{-T\sigma_n}}{\Psi(L)}, \quad k=1,\dots,N-1.
$$

(2) $\sigma_n=0$, $\lambda\in\mathcal{N}_2$, $m(\sigma_n)=2$, $\phi^k_{n,1}=\Psi\equiv 1$, $\phi^k_{n,2}=\tilde\Psi\equiv\cos(\sqrt{\lambda}x)$. Choosing $\int_0^T a^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t=A_n$, $\int_0^T b^k(T-t)e^{-t\sigma_n}\,\mathrm{d}t=B_n$, for all $k=1,\dots,N-1$, it remains to solve the system

$$
\begin{pmatrix}\lambda\Psi(L)+\Psi_{xx}(L) & \Psi(L)\\ \lambda\tilde\Psi(L)+\tilde\Psi_{xx}(L) & \tilde\Psi(L)\end{pmatrix}\begin{pmatrix}A_n\\ B_n\end{pmatrix}=\frac{e^{-T\sigma_n}}{N-1}\begin{pmatrix}y_{0,n,1}\\ y_{0,n,2}\end{pmatrix}
$$

Computing explicitly the determinant in the left hand side we find that equals ±1 and the system is compatible.

(3) $\sigma_n<0$, $\lambda\in\mathcal{N}_1$, $m(\sigma_n)=2$, $\phi^k_{n,1}=\Psi\equiv\cos(\beta_n x)$, $\phi^k_{n,2}=\tilde\Psi\equiv\cos(\beta_m x)$. With the same choice as in the previous case we obtain that the determinant in the left hand side satisfies

$$
\begin{vmatrix}\lambda\Psi(L)+\Psi_{xx}(L) & \Psi(L)\\ \lambda\tilde\Psi(L)+\tilde\Psi_{xx}(L) & \tilde\Psi(L)\end{vmatrix}=(\beta_n^2-\beta_m^2)\cos(\beta_n L)\cos(\beta_m L)=(\beta_n^2-\beta_m^2)(-1)^{m+n}
$$

so the system is compatible.

Case II. σ_n is eigenvalue of (70).

(1) $\sigma_n>0$, $m(\sigma_n)=N-1$, and a basis for the associated eigenspace is given by $\{\Phi e_l-\Phi e_{l+1}\}_{l=1,N-1}$ where Φ solves (70).

In this case our system becomes

$$
(\lambda\Phi(L)+\Phi_{xx}(L))A\vec{\mathbf{a}}+\Phi(L)A\vec{\mathbf{b}}=e^{-T\sigma_n}\vec{\mathbf{y}}
$$

where

$$
\vec{\mathbf{a}}=\begin{pmatrix}\int_0^T a^1(T-t)e^{-t\sigma_n}\,\mathrm{d}t\\ \dots\\ \int_0^T a^{N-1}(T-t)e^{-t\sigma_n}\,\mathrm{d}t\end{pmatrix}, \vec{\mathbf{b}}=\begin{pmatrix}\int_0^T b^1(T-t)e^{-t\sigma_n}\,\mathrm{d}t\\ \dots\\ \int_0^T b^{N-1}(T-t)e^{-t\sigma_n}\,\mathrm{d}t\end{pmatrix}, \vec{\mathbf{y}}=\begin{pmatrix}y_{0,n,1}\\ \dots\\ y_{0,n,m(\sigma_n)}\end{pmatrix}
$$

In view of Lemma 3.2 both terms $\Phi(L)$ and $\lambda\Phi(L)+\Phi_{xx}(L)$ do not vanish so we choose

$$
\vec{\mathbf{a}}=\frac{1}{\lambda\Phi(L)+\Phi_{xx}(L)}A^{-1}e^{-T\sigma_n}\vec{\mathbf{y}}, \quad \vec{\mathbf{b}}=\vec{0}.
$$

We remark that the following choice is also possible and

$$\vec{a} = \vec{0}, \quad \vec{\mathbf{b}} = \frac{1}{\Phi(L)} A^{-1} e^{-T\sigma_n} \vec{\mathbf{y}}.$$

(2) $\sigma < 0$, $\lambda \notin \mathcal{N}_{odd}$, $m(\sigma_n) = N-1$. The construction is the same as in the case II (1) above.

(3) $\sigma < 0$, $\lambda \in \mathcal{N}_{odd}$, $m(\sigma_n) = 2(N-1)$ and a basis for the associated eigenspace is given by $\{\Phi e_l - \Phi e_{l+1}\}_{l=1,N-1}$, $\{\tilde{\Phi} e_l - \tilde{\Phi} e_{l+1}\}_{l=1,N-1}$ where $\Phi \equiv \sin(\beta_n x)$, $\tilde{\Phi} \equiv \sin(\beta_m x)$ for some $m \neq n$. In this case $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ solve the system

$$\begin{cases} (\lambda \Phi(L) + \Phi_{xx}(L)) A \vec{\mathbf{a}} + \Phi(L) A \vec{\mathbf{b}} = e^{-T\sigma_n} \vec{\mathbf{y}}_1 \\ (\lambda \tilde{\Phi}(L) + \tilde{\Phi}_{xx}(L)) A \vec{\mathbf{a}} + \tilde{\Phi}(L) A \vec{\mathbf{b}} = e^{-T\sigma_n} \vec{\mathbf{y}}_2 \end{cases}$$

where $\vec{\mathbf{y}} = (\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2)^T$. Explicit computations show that

$$\begin{vmatrix} \lambda \Psi(L) + \Psi_{xx}(L) & \Psi(L) \\ \lambda \tilde{\Psi}(L) + \tilde{\Psi}_{xx}(L) & \tilde{\Psi}(L) \end{vmatrix} = (\beta_n^2 - \beta_m^2) \sin(\beta_n L) \sin(\beta_m L) = (\beta_n^2 - \beta_m^2)(-1)^{m+n} \neq 0$$

and

$$\begin{pmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{b}} \end{pmatrix} = \frac{1}{(\beta_n^2 - \beta_m^2)(-1)^{m+n}} \begin{pmatrix} \tilde{\Psi}(L) A^{-1} & -\Psi(L) A^{-1} \\ -(\lambda \tilde{\Psi}(L) + \tilde{\Psi}_{xx}(L)) A^{-1} & (\lambda \Psi(L) + \Psi_{xx}(L)) A^{-1} \end{pmatrix} \vec{\mathbf{y}}.$$

Summarizing, in order to finish the proof of the main part of Theorem 1.2 it is enough to solve a moment problem for each a^k, b^k, for any $k = 1, N-1$, that is

$$\int_0^T a^k (T-t) e^{-t\sigma_n} \, \mathrm{d}t = c_{n,k}, \quad \forall n \in \mathbb{N},$$

respectively

$$\int_0^T b^k (T-t) e^{-t\sigma_n} \, \mathrm{d}t = d_{n,k} \quad \forall n \in \mathbb{N},$$

for precised sequences $\{c_{n,k}\}_n$, $\{d_{n,k}\}_n$ determined in the analysis above. In consequence, with the help of the asymptotic properties in Lemma 3.9 for both the eigenvalues and their eigenfunctions, as in the proof of Theorem 1.1 we are able to show absolute convergence of the series

$$\sum_{n\in\mathbb{N}} c_{n,k} \prod_{j=1, j\neq n}^{\infty} \frac{\sigma_j + \sigma_n}{\sigma_j - \sigma_n}, \quad \sum_{n\in\mathbb{N}} d_{n,k} \prod_{j=1, j\neq n}^{\infty} \frac{\sigma_j + \sigma_n}{\sigma_j - \sigma_n}, \quad \forall k = 1, N-1.$$

Then, in view of [13] we obtain the null-controllability for system (2)-(II).

Step II. Optimality of (2N-2) controls. Let us suppose that we can control with $2N-3$ controls distributed as in the following two cases. In both cases we prove that for $\lambda \in \mathcal{N}_{odd}$ and $\sigma < 0$ eigenvalue of (66) initial data of the type

$$y_0 = y_{01}(\Phi e_{N-1} - \Phi e_N) + y_{02}(\tilde{\Phi} e_{N-1} - \tilde{\Phi} e_N)$$

cannot be driven to the null state where $\Phi(x) = \sin(\beta_n x)$, $\tilde{\Phi}(x) = \sin(\beta_m x)$, $\lambda = \beta_n^2 + \beta_m^2$. We emphasize that, up to normalization, $\Phi e_{N-1} - \Phi e_N$ and $\tilde{\Phi} e_{N-1} - \tilde{\Phi} e_N$ are elements of the orthonormal basis $\{\phi_{n,l}\}_{l=1,m(\sigma_n),n\in\mathbb{N}}$ in the hypothesis of Lemma 3.10.

Case I. $a^N = a^{N-1} = b^N \equiv 0$. In this case system (75) becomes

$$y_{01} e^{-T\sigma} = \Phi(L) \int_0^T b^{N-1}(T-t) e^{-t\sigma} \, \mathrm{d}t,$$

$$y_{02} e^{-T\sigma} = \tilde{\Phi}(L) \int_0^T b^{N-1}(T-t) e^{-t\sigma} \, \mathrm{d}t.$$

Explicit computations shows that $\Phi(L) = (-1)^n$, $\tilde{\Phi}(L) = (-1)^m$. Choosing $y_{01} = 0$ and $y_{02} = 1$ leads to a contradiction.

Case II. $a^N = b^N = b^{N-1} \equiv 0$. In this case system (75) becomes

$$y_{01}e^{-T\sigma} = (\lambda\Phi(L) + \Phi_{xx}(L)) \int_0^T a^{N-1}(T-t)e^{-t\sigma}\,\mathrm{d}t,$$
$$y_{02}e^{-T\sigma} = (\lambda\tilde{\Phi}(L) + \tilde{\Phi}_{xx}(L)) \int_0^T a^{N-1}(T-t)e^{-t\sigma}\,\mathrm{d}t.$$

Explicit computations shows that $\lambda\Phi(L)+\Phi_{xx}(L) = (-1)^n\beta_m^2$ and $\lambda\tilde{\Phi}(L)+\tilde{\Phi}_{xx}(L)(-1)^m\beta_n^2$. Choosing $y_{01} = 0$ and $y_{02} = 1$ leads again to a contradiction.

Step III. Null-controllability with $(2n-3)$ **controls.** When $\lambda \notin \mathcal{N}_{odd}$ we can easily adapt the proof given in the Step I in order to construct the controls. The details are left to the reader.

4. Further control and stabilization results

4.1. **Null-controllability of systems (2)-(I) and (II).** In the fist part of the paper we have been concerned with studying controllability problems acting with a minimal number of control inputs. However, we have to mention that we can also address the question for which values $\lambda > 0$ systems (2)-(I) and (2)-(II) are null-controllable for a maximal number of control inputs. In fact, we are able to prove that system (2)-(II) is null-controllable for any λ if we act with $2N$ nontrivial controls. In contrast with that, system (2)-(I) is not null-controllable for any λ even if we impose to act with N nontrivial controls. More precisely we obtain

Theorem 4.1 (Null-controllability for model (2)-(I))**.** *Let be* $T > 0$.

(1) Assume $\lambda \notin \mathcal{N}_1 \cup \mathcal{N}_{odd}$ *Then for any initial state* $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$, *there exist controls* $u = (u^k)_{k=1,N} \in (H^1(0,T))^N$ *such that the solution of system (2)-(I) satisfies*

$$y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}. \tag{76}$$

(2) Assume $\lambda \in \mathcal{N}_1 \cup \mathcal{N}_{odd}$. *There exist initial state* $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$ *such that for any control* $u = (u^k)_{k=1,N} \in (H^1(0,T))^N$ *with the solution of system (2)-(I) satisfies*

$$y^{k_0}(T,\cdot) \not\equiv 0,$$

for some $k_0 \in \{1,\dots,N\}$.

Theorem 4.2 (Null-controllability for model (2)-(II))**.** *Let* $T > 0$ *be fixed and let* $\lambda \notin \mathcal{N}_3$. *Then*
For any initial state $y_0 = (y_0^k)_{k=1,N} \in L^2(\Gamma)$ *and any edge* e_i, $i \in \{1,\dots,N\}$, *there exist controls* $a = (a^k)_{k=1,N}, b = (b^k)_{k=1,N} \in (H^1(0,T))^N$ *such that the solution of system (2)-(II) satisfies*

$$y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}. \tag{77}$$

Next we sketch the proofs of both Theorems 4.1 and 4.2 since they follow similar ideas as in Theorems 1.1 and 1.2.

The main ingredients in the proof are the following propositions.

Proposition 4.3. *Let* $\lambda > 0$ *and* $(\sigma, \phi = (\phi^k)_{1,N})$ *be an eigenpair of problem (9). Then*

- *If* $\sigma \geq 0$ *there exists at least one index* $k \in \{1,\dots,N\}$ *such that* $\phi^k_{xx}(L) \neq 0$.
- *If* $\sigma < 0$ *there exists at least one index* $k \in \{1,\dots,N\}$ *such that* $\phi^k_{xx}(L) \neq 0$ *if and only if* $\lambda \notin \mathcal{N}_1 \cup \mathcal{N}_{odd}$.

Proposition 4.4. *Let* $\lambda \notin \mathcal{N}_3$ *and* $(\sigma, \phi = (\phi^k)_{1,N})$ *be an eigenpair of problem* (66)*. Then there exists at least one index* $k \in \{1, \dots, N\}$ *such that* $\phi^k(L) \neq 0$ *or* $\lambda\phi^k(L) + \phi^k_{xx}(L) \neq 0$.

It is then obvious that Propositions 4.3 and 4.4 lead to the proof of theorems above since we can apply the method of moments implemented in the proof of Theorems 1.1 and 1.2 taking into account that the study of controllability is based on the relation (53) and (74). Thus, to conclude it suffices to prove Propositions 4.3 and 4.4.

Proof of Proposition 4.3. Assume by contradiction that there exist eigenpairs $(\sigma, \phi = (\phi^k)_{k=1,N}$ of (9) such that $\phi^k_{xx}(L) = 0$ for any $k \in \{1, \dots, N\}$. Then S in (21) satisfies

$$\begin{cases} \lambda S_{xx} + S_{xxxx} = \sigma S, & x \in (0, L), \\ S_x(0) = S_{xxx}(0) = 0, \\ S(L) = S_x(L) = S_{xx}(L) = 0 \end{cases} \tag{78}$$

whereas D^k in (22) verifies

$$\begin{cases} \lambda D_{xx} + D_{xxxx} = \sigma D, & x \in (0, L), \\ D(0) = D_{xx}(0) = 0, \\ D(L) = D_x(L) = D_{xx}(L) = 0, \end{cases} \tag{79}$$

for any $k \in \{1, \dots, N\}$. Next we distinguish several cases in terms of the sign of σ.
Case $\sigma > 0$**.** As in section 3.1 the boundary conditions at $x = 0$ lead to

$$S(x) = C_1 \cosh(\alpha x) + C_3 \cos(\beta x).$$

Imposing the conditions at $x = L$ in (78) we get

$$\mathfrak{M} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{80}$$

where

$$\mathfrak{M} = \begin{pmatrix} \cosh(\alpha L) & \cos(\beta L) \\ \alpha \sinh(\alpha L) & -\beta \sin(\beta L) \\ \alpha^2 \cosh(\alpha L) & -\beta^2 \sin(\beta L) \end{pmatrix}.$$

Observe that rank $\mathfrak{M} = 2$ since $\cos(\beta L)$ and $\sin(\beta L)$ cannot vanish simultaneously. Therefore $S \equiv 0$. This implies that ϕ^k satisfies (79). From section 3.2 imposing the boundary conditions at the origin we have

$$D(x) = C_2 \sinh(\alpha x) + C_4 \sin(\beta x).$$

The conditions at $x = L$ in (79) give

$$\mathfrak{N} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{81}$$

where

$$\mathfrak{N} = \begin{pmatrix} \sinh(\alpha L) & \sin(\beta L) \\ \alpha \cosh(\alpha L) & \beta \cos(\beta L) \\ \alpha^2 \sinh(\alpha L) & -\beta^2 \sin(\beta L) \end{pmatrix}.$$

Since rank $\mathfrak{N} = 2$ we obtain $D \equiv 0$. Therefore $\phi^k \equiv 0$ for all $k \in \{1, \dots, N\}$ which is in contradiction with the fact that ϕ is an eigenfunction.

Case $\sigma = 0$. From section 3.1 and the conditions at $x = 0$ we necessary have

$$S(x) = C_1 + C_3 \cos(\sqrt{\lambda} x).$$

From the conditions at $x = L$ we get

$$\mathfrak{P} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{82}$$

where

$$\mathfrak{P} = \begin{pmatrix} 1 & \cos(\sqrt{\lambda} L) \\ 0 & -\sqrt{\lambda}\sin(\sqrt{\lambda} L) \\ 0 & -\lambda \cos(\sqrt{\lambda} L) \end{pmatrix}.$$

Again rank $\mathfrak{P} = 2$ which implies $C_1 = C_3 = 0$ and therefore $S \equiv 0$. This implies that ϕ^k satisfy (79). Going back to section 3.2 from the first conditions in (79) we get that

$$D(x) = C_2 x + C_4 \sin(\sqrt{\lambda} x).$$

Applying the conditions at $x = L$ in (79) we obtain

$$\mathfrak{Q} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{83}$$

where

$$\mathfrak{Q} = \begin{pmatrix} L & \sin(\sqrt{\lambda} L) \\ 1 & \sqrt{\lambda}\cos(\sqrt{\lambda} L) \\ 0 & -\lambda \sin(\sqrt{\lambda} L) \end{pmatrix}.$$

Since rank $\mathfrak{Q} = 2$ we obtain $D \equiv 0$ and $\phi^k \equiv 0$ for any k. This is in contradiction with the fact that ϕ is an eigenfunction.

The case $\sigma < 0$**.** From section 3.1 and (78) we necessary have

$$S(x) = C_1 \cos(\gamma x) + C_3 \cos(\gamma x).$$

Then from the conditions at $x = L$ in (79) we obtain

$$\mathfrak{R} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{84}$$

where

$$\mathfrak{R} = \begin{pmatrix} \cos(\gamma L) & \cos(\beta L) \\ -\gamma \sin(\gamma L) & -\beta \sin(\beta L) \\ -\gamma^2 \cos(\gamma L) & -\beta^2 \cos(\beta L) \end{pmatrix}.$$

It is easy to see that rank $\mathfrak{R} = 1$ if and only if $\cos(\gamma L) = \cos(\beta L) = 0$. In other words, rank $\mathfrak{R} = 2$ if and only if $\lambda \notin \mathcal{N}_{odd}$ in which case we get $S \equiv 0$. Then ϕ^k solves (79) for all k. Again, in view of section 3.2 we must have

$$D(x) = C_2 \sin(\gamma x) + C_4 \sin(\beta x).$$

Imposing the conditions at $x = L$ in (79) we obtain

$$\mathfrak{S} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{85}$$

where

$$\mathfrak{S} = \begin{pmatrix} \sin(\gamma L) & \sin(\beta L) \\ \gamma\cos(\gamma L) & \beta\cos(\beta L) \\ -\gamma^2\sin(\gamma L) & -\beta^2\sin(\beta L) \end{pmatrix}.$$

We easily deduce that rank $\mathfrak{S} = 1$ if and only if $\sin(\gamma L) = \sin(\beta L) = 0$ which is equivalent to $\lambda \in \mathcal{N}_1$. Therefore, rank $\mathfrak{S} = 2$ if and only if $\lambda \notin \mathcal{N}_1$ in which case we get $D \equiv 0$. This implies $\phi^k \equiv 0$ for all k which is in contradiction with the election of ϕ as an eigenfunction.

Finally, the proof of Proposition 4.3 is finished. □

We avoid the details of the proof of Proposition 4.4 since it follows the same steps and ideas as the proof of Proposition 4.3. We let the rest of details to the reader.

4.2. **New control results for the linear KS on an interval.** In this section we present some new control results for a single linear KS equation which are direct consequences of the spectral analysis developed in Section 3. For the sake of clarity we will not insist too much on the rigorousness of the technical details since we already did it in the previous sections.

We consider the system

$$(86)\qquad \begin{cases} y_t + \lambda y_{xx} + y_{xxxx} = 0, & (t,x) \in (0,T)\times(0,L) \\ y_x(t,0) = u^1(t), & t \in (0,T) \\ y_{xxx}(t,0) = u^2(t), & t \in (0,T) \\ y_x(t,L) = y_{xxx}(t,L) = 0, & t \in (0,T) \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases}$$

where u^1, u^2 are control inputs. We then obtain

Theorem 4.5. *The system (86) is null-controllable for any $\lambda \notin \mathcal{N}_3$: for any time $T > 0$ and any initial data $y_0 \in L^2(0,L)$ there exist two controls $u^1, u^2 \in L^2(0,L)$ which steer the solution of (86) to the zero state, i.e. $y(T,x) = 0$, for all $x \in (0,L)$.*

Roughly speaking, the null-controllability of (86) is given through the adjoint problem (backward in time)

$$(87)\qquad \begin{cases} -q_t + \lambda q_{xx} + q_{xxxx} = 0, & (t,x) \in (0,T)\times(0,L) \\ q_x(t,0) = q_{xxx}(t,0) = 0, & t \in (0,T) \\ q_x(t,L) = q_{xxx}(t,L) = 0, & t \in (0,T) \\ q(T,x) = q_T(x), & x \in (0,L). \end{cases}$$

As in subsection 2.4 the system (86) is null-controllable if one ensures the existence of the control inputs u^1, u^2 such that

$$(88)\qquad \int_0^L y_0(x)q(0,x)dx + \int_0^T u^1(t)(\lambda q(t,0) + q_{xx}(t,0)) + \int_0^T u^2(t)q(t,0)dt.$$

Note that the spectral problem corresponding to the adjoint system (87) is precisely the eigenvalue problem (64) in subsection 3.1.

Therefore, in view of the method of moments such controls u^1, u^1 satisfying (88) can be build if for any eigenfunction ϕ of system (64) it is verifies that both terms $\phi(0)$ and $\lambda\phi(0)+\phi_{xx}(0)$ cannot vanish simultaneously. This is true as a consequence of Lemma 3.1 applied at the point $x = 0$ instead of $x = L$ (we let the details to the reader to check that Lemma 3.1 is also valid when replacing L with 0).

Another direct application of our spectral results in Section 3 regards the following system

$$\begin{cases} y_t + \lambda y_{xx} + y_{xxxx} = 0, & (t,x) \in (0,T) \times (0,L) \\ y(t,0) = u^1(t), & t \in (0,T) \\ y_{xx}(t,0) = u^2(t), & t \in (0,T) \\ y_x(t,L) = y_{xxx}(t,L) = 0, & t \in (0,T) \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{89}$$

whose adjoint is given by

$$\begin{cases} -q_t + \lambda q_{xx} + q_{xxxx} = 0, & (t,x) \in (0,T) \times (0,L) \\ q(t,0) = q_{xx}(t,0) = 0, & t \in (0,T) \\ q_x(t,L) = q_{xxx}(t,L) = 0, & t \in (0,T) \\ q(T,x) = q_T(x), & x \in (0,L). \end{cases} \tag{90}$$

We easily observe that the spectral problem corresponding to system (90) is nothing else than the eigenvalue problem 65 in subsection 3.2.

By the above considerations system (89) is null-controllable if one can find u^1, u^2 such that

$$\int_0^L y_0(x)q(0,x)dx + \int_0^T u^1(t)(\lambda q_x(t,0) + q_{xxx}(t,0)) + \int_0^T u^2(t)q_x(t,0)dt = 0. \tag{91}$$

This is equivalent to verify whether each eigenfunction ϕ of (65) satisfies that $\phi_x(0)$ and $\lambda\phi_x(0) + \phi_{xxx}(0)$ do not vanish simultaneously. Indeed, as a consequence of the complete determination of the eigenfunctions of (65) in subsection 3.2, this is true unless $\lambda \notin \mathcal{N}_3$. Therefore, we obtain

Theorem 4.6. *The system (89) is null-controllable for any $\lambda \notin \mathcal{N}_3$: for any time $T > 0$ and any initial data $y_0 \in L^2(0,L)$ there exist two controls $u^1, u^2 \in L^2(0,L)$ which steer the solution of (89) to the zero state, i.e. $y(T,x) = 0$, for all $x \in (0,L)$.*

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References

1. H. Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983, Théorie et applications. [Theory and applications].
2. N. Carreño and E. Cerpa, *Local controllability of the stabilized Kuramoto-Sivashinsky system by a single control acting on the heat equation*, J. Math. Pures Appl. **106** (2016), no. 4, 670–694.
3. T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and its Applications, vol. 13, The Clarendon Press, Oxford University Press, New York, 1998, Translated from the 1990 French original by Yvan Martel and revised by the authors.
4. E. Cerpa, *Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation*, Commun. Pure Appl. Anal. **9** (2010), no. 1, 91–102.
5. E. Cerpa, P. Guzman, and A. Mercado, *On the control of the linear Kuramoto-Sivashinsky equation*, ESAIM Control Optim. Calc. Var. **23** (2017), 165–194.
6. E. Cerpa and A. Mercado, *Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation*, J. Differential Equations **250** (2011), no. 4, 2024–2044.
7. E. Cerpa, A. Mercado, and A. F. Pazoto, *On the boundary control of a parabolic system coupling KS-KdV and heat equations*, Sci. Ser. A Math. Sci. (N.S.) **22** (2012), 55–74.
8. ———, *Null controllability of the stabilized Kuramoto-Sivashinsky system with one distributed control*, SIAM J. Control Optim. **53** (2015), no. 3, 1543–1568.
9. L. H. Chen and H. C. Chang, *Nonlinear waves on liquid film surfaces ii. bifurcation analysis of the long-wave equation*, Chem. Eng. Sci. **41** (1986), 2477–2486.
10. Yacine Chitour, Guilherme Mazanti, and Mario Sigalotti, *Stability of non-autonomous difference equations with applications to transport and wave propagation on networks*, Netw. Heterog. Media **11** (2016), no. 4, 563–601.
11. E. Crépeau, *Exact controllability of the Boussinesq equation on a bounded domain*, Differential Integral Equations **16** (2003), no. 3, 303–326.
12. B. Dekoninck and S. Nicaise, *The eigenvalue problem for networks of beams*, Linear Algebra Appl. **314** (2000), no. 1-3, 165–189.
13. H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. **43** (1971), 272–292.
14. M. Gugat, A. Keimer, G. Leugering, and Z. Wang, *Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks*, Netw. Heterog. Media **10** (2015), no. 4, 749–785.
15. M. Gugat, D. Wintergerst, and R. Schultz, *Networks of pipelines for gas with nonconstant compressibility factor: stationary states*, Comp. Appl. Math., to appear.
16. A. P. Hooper and R. Grimshaw, *Two-dimensional disturbance growth of linearly stable viscous shear flows*, Phys. Fluids **8** (1996), no. 6, 1424–1432.
17. Y. Kuramoto and T. Tsuzuki, *On the formation of dissipative structures in reaction-diffusion systems: Reductive perturbation approach*, Prog. Theor. Phys. **54** (1975), no. 3, 687–699.
18. ———, *Persistent propagation of concentration waves in dissipative media far from thermal equilibrium*, Prog. Theor. Phys. **55** (1976), no. 2, 356–369.
19. D. M. Michelson and G. I. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flames. II. Numerical experiments*, Acta Astronaut. **4** (1977), no. 11-12, 1207–1221.
20. G. I. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations*, Acta Astronaut. **4** (1977), no. 11-12, 1177–1206.
21. T. Takahashi, *Boundary local null-controllability of the Kuramoto-Sivashinsky equation*, Math. Control Signals Systems **29** (2017), no. 1, Art. 2, 21.

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