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ANNEXE 5.1

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Polyhedra with simple dense geodesics

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POLYHEDRA WITH SIMPLE DENSE GEODESICS

JIN-ICHI ITOH, JOËL ROUYER, AND COSTIN VÎLCU

Abstract. We characterize polyhedral surfaces admitting a simple dense geodesic ray and convex polyhedral surfaces with a simple geodesic ray.

1. Introduction

A tetrahedron is said to be isosceles if its faces are congruent to each other. Equivalently, the singular curvature of each of its vertices is exactly π. These simple objects have many interesting properties, especially from viewpoint of their intrinsic geometry. For instance, it is easy to see that they admit infinitely long simple geodesics, as well as arbitrarily long simple closed geodesics. It was proved by V. Yu. Protasov [9] that this latter property characterizes them among convex polyhedra; his result has recently been strenghtened to convex surfaces by A. Akopyan and A. Petrunin [1]. However, there exist non-convex polyhedra with this property, as for example flat tori.

The aim of this paper is to determine (non-necessarily convex) polyhedra having simple dense geodesics. It turns out that the objects we found had already been studied, for they play a key role in the theory of rational billiards. In [7] they are named "surfaces endowed with a *flat structure (with parallel line field)*". One can find many variants of the name in the literature, generally emphasizing the flatness. Since all the surfaces considered in this paper – namely polyhedra – are flat (with conical singularities), we chose to simply call them *parallel polyhedra*.

Our main result (Theorem 1) states that a polyhedron is parallel if and only if it is orientable and admits a simple dense geodesic ray. In this case, at any point, a ray starting in almost any direction will be simple and dense. We prove, moreover (Theorem 2), that a simple geodesic ray on a convex polyhedron P is dense in P.

This paper is about intrinsic geometry of surfaces: when we write "polyhedron", we always mean a surface. More precisely, a compact surface flat everywhere except at finitely many singularities, of conical type. In other words, it is a 2-dimensional manifold obtained by gluing (with length preserving maps) finitely many Euclidean triangles along their edges. The conical points are called *vertices*. The (*singular*) *curvature* of a vertex v, denoted by $\omega(v)$, is defined as 2π minus the total angle around it. This notion of polyhedron is more general than the notion of boundary of solid polyhedra in $\mathbb{R}^3$, for it also includes non-orientable manifolds. Of course, once a polyhedron that can be embedded in $\mathbb{R}^3$, it can be embedded in many different ways.

When we say that a polyhedron is convex, we mean that it is homeomorphic to the sphere and that all its vertices have positive curvature. It is a famous result of A. D. Alexandrov that such a polyhedron admits a unique (up to rigid motions) realization as the boundary of a convex polyhedron in $\mathbb{R}^3$ – in the usual sense

– whence the denomination. Notice that such a surface is also isometric to the boundary of many more non-convex polyhedra, see for instance [8] or [5].

2. Prerequisite

In this section we recall a few definitions and give some basic results destined to be used later.

Let P be a polyhedron and $I \subset \mathbb{R}$ an interval. A geodesic is a map $G : I \to P$ such that for any $t \in I$ and any $s \in I$ close enough to t, we have $d(G(s), G(t)) = |s - t|$. In particular, all geodesics are parametrized by arc-length. When $I = \mathbb{R}$, we speak of a *geodesic line*, and when $I \neq \mathbb{R}$ is an unbounded interval, we speak of a *geodesic ray*. A geodesic is said to be *simple* if it admits no proper self-intersection. That is, there is no pair of instants t_1, t_2 such that $G(t_1) = G(t_2)$ but $G'(t_1) \neq G'(t_2)$, where $G'(t)$ represents the direction of G at point $G(t)$. According to this definition, a periodic geodesic line may be simple, though not injective; we call it a *simple closed geodesic*.

It is well known and easy to see that a geodesic cannot pass through a positively curved vertex (see for instance [4]), but it may pass through a negatively curved one. Indeed, a broken line through a vertex v will be locally minimizing if and only if it separates the space of directions at v into two parts of measure at least π. It follows that geodesics may have branch points. So, there in no good notion of geodesic flow on a polyhedron. A geodesic which avoids all vertices will be called *strict*.

We denote by $V(P)$ the set of all vertices of P, and by P^* the open (Riemannian) flat manifold $P \setminus V(P)$. So P^* carries a natural notion of parallel transport. The parallel transport along a curve $\gamma : I \to P^*$ will be denoted by $||_\gamma$.

Although these polyhedra are not *Alexandrov spaces with bounded curvature*, small enough balls clerly are either Alexandrov spaces with curvature bounded below, or Alexandrov spaces with curvature bounded above, in both cases by 0 (for definitions and basic properties, see for instance [3]). It follows that the definition of *space of directions* of Alexandrov spaces applies here. We denote by Σ_p the space of directions at point $p \in P$ (it is a circle); for any subset Q of P, ΣQ stands for the disjoint union $\coprod_{p \in Q} \Sigma_p$. Hence ΣP^* is naturally identified to the unit tangent bundle over P^*. For $p \in P^*$, we denote by Δ_p the projective line obtained as the quotient of Σ_p by the group $\{\pm id\}$, and by ΔP^* the corresponding bundle over P^*. A section of ΔP^* will be called a *line distribution* on P^*. It is said to be *parallel* if its integral curves form a geodesic foliation and, restricted to any small domain, those integral lines become parallel once the domain is unfolded onto a plane.

Let $u \in \Sigma_p$. The maximal strict geodesic starting at p in direction u is denoted by γ_u. If γ_u is not defined on $[0, \infty[$, *i.e.,* if γ_u meets some vertex in the positive direction, then u is said to be *singular*.

The distance between two points p, $q \in P$ is denoted by $d(p, q)$. For $r > 0$, $B(p, r)$ stands for the open ball of radius r centered at x.

We start with a few simple lemmas.

Lemma 1. *Let $[a_1 b_1]$ and $[a_2 b_2]$ be two segments of length $2l$ in the standard Euclidean plane. Let m_i be the midpoint of $[a_i b_i]$. Assume that $m_1 \neq m_2$ and that a_1 and a_2 are in the same half plane bounded by the line through m_1 and m_2. Put*

$\alpha_i = \measuredangle\left(\overrightarrow{m_1 m_2}, \overrightarrow{a_i b_i}\right)$ *and* $\delta = \alpha_2 - \alpha_1$. *Then the segments intersect if and only if*

$$d\left(m_1, m_2\right) \max\left(\left|\sin\alpha_1\right|, \left|\sin\alpha_2\right|\right) \leq l \left|\sin\delta\right|.$$

In particular, if the segments do not intersect, then

$$\left|\sin\delta\right| < \frac{d\left(m_1, m_2\right)}{l}.$$

The proof is elementary and is left to the reader.

Lemma 2. *Assume there exists on P a simple dense ray G. Then there exists a locally Lipschitz line distribution D on P^* such that for any t, $D_{G(t)}$ is tangent to G.*

Proof. First define D on $\operatorname{Im}(G)$ in the obvious way. Denote by P^ε the compact set $P \setminus \bigcup_{v \in V(P)} B(v, \varepsilon)$. From Lemma 1, there is a constant K_ε depending only on ε such that D is K_ε-Lipschitz continuous on $P^\varepsilon \cap \operatorname{Im}(G)$. Since $\operatorname{Im}(G)$ is dense, D admits a unique K_ε-Lipschitz continuous extension to P_ε for any ε, and consequently an unique continuous extension to P^*. This extension is obviously locally Lipschitz continuous. □

Lemma 3. *Let p, q be two (possibly coinciding) points on P. Then, the set $C \subset \Sigma_p$ of those u such that γ_u meets q is at most countable.*

Proof. It is sufficient to prove that, for any $a \in \mathbb{N}^*$, the set of directions u such that $\gamma_u|[0, a]$ meets q is finite. Assume on the contrary that there are infinitely many distinct $u_n \in \Sigma_x$ such that γ_{u_n} meets q at time $t_n \leq a$. By the Ascoli theorem, one can extract from $\left\{\gamma^n \stackrel{\text{def}}{=} \gamma_{u_n}|[0, t_n]\right\}_n$ a subsequence (still denoted by γ^n) converging to a curve γ from p to q. Let τ_n be the supremum of those $t > 0$ such that $\gamma^n|]0, t]$ does not intersect $\operatorname{Im}\left(\gamma^{n+1}\right)$, and τ'_n be such that $\gamma^{n+1}\left(\tau'_n\right) = \gamma\left(\tau_n\right) \stackrel{\text{def}}{=} p_n$. It is clear that τ_n and τ'_n are bounded by a, and on the other hand, cannot approach 0. Hence, the geodesic digon $D_n \stackrel{\text{def}}{=} \operatorname{Im}\left(\gamma^n|[0, \tau_n]\right) \cup \operatorname{Im}\left(\gamma^{n+1}|[0, \tau_{n+1}]\right)$ tends to a geodesic arc of γ. It follows that the sum s_n of the angles of D_n must tend to 0. By the Gauss-Bonnet formula, this sum equals the sum of the curvatures of the vertices included inside the digon, and consequently s_n may take only finitely many distinct value. Hence $s_n = 0$ for n large enough. Since u_n and u_{n+1} were supposed to be distinct, we get a contradiction. □

Corollary 1. *For any $p \in P^*$, the set $S_p \subset \Sigma_p$ of singular directions is at most countable.*

Lemma 4. *Let σ be a segment on P, p a point of σ and $u \in \Sigma_p$ transverse to σ. For $y \in \sigma$, denote by $u_y \in \Sigma_y$ the parallel transport of u along σ. Then, the set of those points $y \in \sigma$ such that u_y is singular is at most countable.*

Proof. The proof is similar to the one of Lemma 3. It is sufficient to prove that, for any $a \in \mathbb{N}^*$, the set of points y such that γ_{u_y} is not well defined on $[0, a]$ is finite. Assume on the contrary that there are infinitely many $y_n \in \sigma$ such that $\gamma_{u_{y_n}}$ meets a vertex at time $t_n \leq a$. Possibly passing to a subsequence, we can assume that it is the same vertex for all those points. Put $\gamma^n \stackrel{\text{def}}{=} \gamma_{u_{y_n}}|[0, t_n]$. By extracting a subsequence, one can assume that γ^n is converging to a curve γ. Let q_n be the first intersection point of γ^n and γ^{n+1} along γ^n. Let T_n be the triangle T_n whose vertices are y_n, y_{n+1} and q_n, and the sides are parts of γ^n, γ^{n+1} and σ. By the Gauss-Bonnet theorem, the angle of T_n at q_n equals the sum of the curvatures of the vertices included in T_n, and so, it can take only finitely many distinct values. On the other hand, since γ^n tends to γ, this angle should tend to zero. Hence, it must vanish for large n, in contradiction with the fact that γ^n and γ^{n+1} are distinct. □

3. Parallel polyhedra

In this section we introduce the notion of parallel polyhedra, and give their basic properties.

If the holonomy group of P^* is either trivial, or equal to $\{id, -id\}$, P is said to be *parallel*. In this case, the parallel transport of lines does not depends on the path. In other words, there exist natural bijections $\tau_p^q : \Delta_p \to \Delta_q$ such that, for any path γ from p to q and any tangent line $l \in \Delta_p$, the parallel transport of l along γ is $\tau_p^q(l)$. It follows that there is a well defined notion a line direction that does not depends on the point $p \in P^*$. We denote by $\tilde{\Delta} P^*$ the set of line directions on P^*.

Examples of parallel polyhedra are given after Proposition 1.

In order to give a first characterization of parallel polyhedra, we need the following lemma.

Lemma 5. *If P admits two parallel line distributions that are not orthogonal at some point, then it is orientable.*

Proof. Let D and D' be such distributions and assume that P is not orientable. So there exists a loop $\gamma : [a,b] \to P^*$ such that $||_\gamma$ is a reflection. Let $\tilde{\gamma} : [a,b] \to P^*$ be a path homotopic to γ in P^*, consisting in finitely many segments which are either parallel or normal to D. Since P^* is flat, $||_{\tilde{\gamma}} = ||_\gamma$, whence $||_\gamma D_{\gamma(0)} = D_{\gamma(0)}$. The same holds for D', whence D and D' are either equal or orthogonal. □

Proposition 1. *The following statements are equivalent.*

1. *P is parallel.*
2. *P is orientable and admits a parallel line distribution.*
3. *P admits two parallel line distributions that are not orthogonal at some point.*

Proof. Assume that P is parallel. If P was not orientable, then its holonomy group would contain a reflection, in contradiction with the definition of parallel polyhedra. Choose $p \in P^*$ and $l \in \Delta_p P^*$. The line distribution is given by $D_q = \tau_p^q(l)$.

Conversely, assume that P is orientable and let D be a parallel line distribution. Choose a piecewise smooth loop $\gamma : [0,1] \to P^*$ with basepoint $p \stackrel{\text{def}}{=} \gamma(0) = \gamma(1)$, and $u \in \Sigma_p$. Denote by $\tau_t : \Sigma_p \to \Sigma_{\gamma(t)}$ the parallel transport along $\gamma|[0,t]$. Since D is parallel, the (mod π) angle $\measuredangle\left(D_{\gamma(t)}, \pm\tau_t(u)\right)$ is constant with respect to t, whence $\tau^1(u) \in \left\{\pm u, \pm S_{D_p}(u)\right\}$, where S_l stands for the reflection with respect to l. Now, since P is orientable, $\tau^1 = \pm id$. Lemma 5 ends the proof. □

Example 1. *Any isosceles tetrahedron is parallel. Figure 1 shows an unfolding and a parallel line distribution. By Proposition 2 below, (possibly degenerated) isosceles tetrahedra are the only parallel convex polyhedra.*

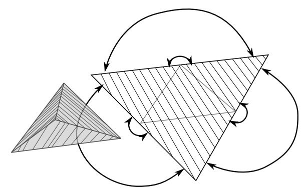

FIGURE 1. Isoceles tetrahedra with a parallel line distribution.

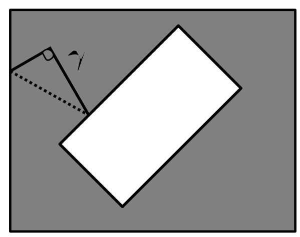

FIGURE 2. Example of a non-parallel surface with vertices of curvature $\pm\pi$.

Example 2. *Any flat torus is clearly a parallel polyhedron.*

Example 3. *Consider a polygonal domain Δ in $\mathbb{R}^2$ such that each segment of its boundary is parallel either to the x-axis or the y-axis. Glue two copies of Δ alomg their boundaries, by identifying the corresponding points. Then the obtained polyhedron (called the double of Δ) is parallel.*

Proposition 2. *If P is parallel, the curvature of any vertex belongs to $\mathbb{Z}\pi$.*

Proof. Let $\gamma : [0,1] \to P$ be a Jordan polygonal curve enclosing a vertex v. Let s be the sum of its n angles, measured toward the domain containing v. On one hand, $\measuredangle\left(\dot\gamma\left(0\right), ||_\gamma\dot\gamma\left(0\right)\right) = n\pi - s$, and on the other hand, by the Gauss-Bonnet theorem, $\omega\left(v\right) = s - (n-2)\pi$. Since P is parallel, $\measuredangle\left(\dot\gamma\left(0\right), ||_\gamma\dot\gamma\left(0\right)\right)$ equals 0 or π, and the conclusion follows. □

There is no converse of Proposition 2 (see Example 4 below), except in the case of polyhedra homeomorphic to the sphere.

Proposition 3. *If P is homeomorphic to the sphere and all vertices have curvature in $\mathbb{Z}\pi$ then P is parallel.*

Proof. Let $\gamma : [0,1] \to P^*$ be a closed loop. The parallel transport along γ is a rotation of angle $\alpha = 2\pi - \omega\left(D\right)$, where D is the domain included on the left side of γ. Since by hypothesis, the curvature of each vertex is divisible by π, so is α. Consequently, the holonomy group of P can contain only id and $-id$, that is, P is parallel. □

Example 4. *Consider the double of the gray polygonal domain shown in Figure 2. Each vertex has curvature either π or $-\pi$. However, the parallel transport along the curve γ is a rotation of angle $\pi/2$.*

Proposition 4. *On a parallel polyhedron, all strict geodesics are simple.*

Proof. If there is a non-simple well defined geodesic, one portion of it is a (well defined) closed geodesic arc $\gamma : [0,a] \to P^*$ making an angle $\alpha \not\equiv 0\ [\pi]$ at its base point. But, since γ is a geodesic $\dot\gamma(a) = ||_\gamma \dot\gamma(0) \in \{\pm\dot\gamma(0)\}$ – for P is parallel – and we get a contradiction. □

We have already mentioned that parallel polyhedra have been studied because of their relation with rational billiards. One result of special interest for us is the following lemma.

Lemma 6. [7] *There is a countable set $C \subset \tilde\Delta P^*$ such that, for any $u \in \Sigma P^*$ whose direction does not belong to C, γ_u is a simple dense ray.*

4. Simple dense geodesics

In this section we prove our main result.

Lemma 7. *If P admits a simple dense strict geodesic ray G, it admits a parallel line distribution.*

Proof. By Lemma 2, P admits a Lipschitz continuous line distribution D. We shall show that it is actually parallel.

Clearly, the integral lines of this distribution are geodesics, for any arc of such a line is limit of arcs of G. We claim that only finitely many integral lines meet some vertex. Indeed two integral lines meeting one vertex v must form an angle at least π, otherwise an arc of G through a point inside the sector they determine should intersect one of them. Hence all but a final number of integral lines are infinite in both direction, we call them *complete integral lines*. The union of all complete integral lines is denoted by C.

Consider $x \in P^*$ and B a ball centered at P that does not contain any vertex. Restricted to this ball, one can define Lipschitz unit vector fields τ, ν such that τ is parallel to D and ν is normal to τ. Define $s_p(y)$ as the slope of $\tau_{p+y\nu_p}$ in the basis (τ_p, ν_p). Set

$$\phi(p) = \limsup_{h\to 0} \left| \frac{s_p(y)}{y} \right|.$$

We claim that $\phi(p) = 0$ almost everywhere, and so, s_p is derivable at 0 and its derivative is 0 for almost all p.

Take $p \in C$ and choose a sequence y_n of real numbers, tending to 0, such that $p_n \overset{\text{def}}{=} p + y_n\nu_p$ belongs to C. Let s (respectively s_n) be the segment of the integral line through p (respectively p_n) of lengh $2l$, whose midpoint is p (respectively p_n). Then s admits a neighborhood N which, endowed with its own intrinsic metric, is isometric to a neighborhood of a $2l$ long segment in $\mathbb{R}^2$. For n large enough, s_n is included in N. Since s_n and s don't intersect, we have $\left|\frac{s_p(y_n)}{y_n}\right| \le \frac{1}{l}$. Hence, any adherence value of $\left|\frac{s_p(y_n)}{y_n}\right|$ is at most $\frac{1}{l}$, for arbitrarily large l, whence $\left|\frac{s_p(y_n)}{y_n}\right| \to 0$.

Now, we drop the assumption $p_n \in C$ and consider $q_n = p + z_n \nu_p \in C$ such that $d(p_n, q_n) < y_n^2$. Since s_p is L-Lipschitz continuous, we get

$$\begin{aligned}\left|\frac{s_p(y_n)}{y_n} - \frac{s_p(z_n)}{z_n}\right| &\leq \left|\frac{s_p(y_n)}{y_n} - \frac{s_p(z_n)}{y_n}\right| + \left|\frac{s_p(z_n)}{y_n} - \frac{s_p(z_n)}{z_n}\right| \\ &\leq L\left|\frac{y_n - z_n}{y_n}\right| + \left|\frac{s_p(z_n)}{z_n}\right|\left|\frac{y_n - z_n}{y_n}\right| \\ &\leq (L + 2\varepsilon)|y_n| \to 0.\end{aligned}$$

It follows that $\left|\frac{s_p(y_n)}{y_n}\right|$ still tends to 0, whence $\phi(p) = s_p'(0) = 0$ for any $p \in C$.

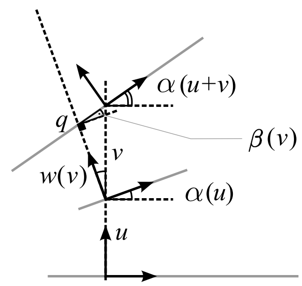

FIGURE 3. Computation of $s_p'(u)$ in the proof of Lemma 7.

Now, we prove that $s_p'(u) = 0$ for any p, u such that $p + u\nu_p \in C$. Let $\alpha(y) = \arctan s_p(y)$, so

$$\alpha(u+v) - \alpha(u) = \arctan s_{p+uN}(w(v)) \stackrel{\text{def}}{=} \beta(v),$$

where $w(v)$ is the distance between $p + u\nu_p$ and the intersection point q between the lines $p + (u+v)\nu_p + \mathbb{R}\tau_{p+(u+v)\nu_p}$ and $p + u\nu_p + \mathbb{R}\nu_{p+u\nu_p}$ (see Figure 3).

On the one hand $\lim \frac{\beta(v)}{w(v)} = s'_{p+u\nu_p}(0) = 0$, on the other hand the law of sines in the triangle $p + u\nu_p$, $p + (u+v)\nu_p$, q gives

$$\frac{w(v)}{v} = \frac{\cos(\beta(v) + \alpha(u))}{\cos\beta(v)} \underset{v\to 0}{\to} \cos\alpha(u),$$

whence

$$\frac{\alpha(u+v) - \alpha(u)}{v} = \frac{\beta(v)}{w(v)}\frac{w(v)}{v} \underset{v\to 0}{\to} 0.$$

Hence, for any p and almost all u we have $s_p'(u) = 0$. Now, since s_p is Lipshitz continuous, we have

$$s_p(y) = s_p(0) + \int_0^y s_p'(u)\,du = 0 + 0$$

for any p, y, and therefore D is parallel. □

Now, we are in a position to state the main result of the paper.

Theorem 1. *The following statements are equivalent.*

(1) *P is parallel.*
(2) *P is orientable and admits a simple dense geodesic ray.*

(3) *P admits two simple dense geodesic rays which are not orthogonal to each other at some point.*
(4) *For any $p \in P^*$ there exists a countable set $C \subset \Sigma_p$ such that for any $u \in \Sigma_p \setminus C$, γ_u is a strict simple ray.*

Proof. By Proposition 1 and Lemma 7, (1), (3) and (2) are equivalent. By Lemma 6, (1) implies (4) which obviously implies (3) and (2). □

5. Convex case

The aim of this section is to supplement Theorem 1 in the convex case by adding a new statement, namely the existence of a (not necessarily dense) simple geodesic ray.

Lemma 8. *Let P be a convex polyhedron. There is a finite set F such that, for any simple geodesic G of P and any two arcs of G lying on the same face of P, the angle between them belongs to F.*

Proof. The two arcs can be seen as the external parts of a longer arc of G. Joining the endpoints of this arc by a segment produces a geodesic digon. By the Gauss Bonnet theorem, the sum of its angles, that is, the angle between the arcs, equals the curvature included in the digon, and so belongs to $F = \left\{\sum_{v \in W} \omega(v) \,|W \subset V\right\}$, where V denotes the set of vertices of P. □

Let G be a simple geodesic ray of P. A point of $x = G(t)$ is said to be of the *first kind*, if there exists arbitrary small $\varepsilon > 0$ such that the intersection $\mathrm{Im}(G) \cap B(x, \varepsilon)$ is arcwise connected. It is said to be of the *second kind* if there exits points $x_n = G(t_n)$ and $x'_n = G(t'_n)$ such that t_n and $t_{n'}$ tend to infinity, x_n and x'_n tend to x, and are locally separated by the arc of G through x. The point x is said to be of the *third kind* if (a) there exist points $x_n = G(t_n)$ tending to x while t_n tends to infinity, and (2) there exits $\varepsilon > 0$ such that the intersection of one of the two open halves of $B(x, \varepsilon)$ delimited by the arc of G through x does not intersect G. Let $K_i(G) \subset \mathbb{R}^+$ be the set of t such that $G(t)$ is of the i^{th} kind. It is easy to see that $\mathbb{R}^+ = \bigcup_{i=1,2,3} K_i(G)$.

Lemma 9. *If G is a simple geodesic ray on a convex polyhedron P, then all its points have the same kind.*

Proof. By Lemma 1, if G is simple and $G(t_n)$ converges to $G(t)$ then we have the convergence of arcs $G([t_n - \varepsilon, t_n + \varepsilon]) \to G([t_n - \varepsilon, t_n + \varepsilon])$. It follows that $K_i(G)$ is open, and the conclusion follows from the connectedness of $\mathbb{R}^+$. □

Hence, one can speak of the kind of a simple ray.

Lemma 10. *Let G be a simple geodesic ray of first kind on a convex polyhedron P, then there exists a simple ray of second or third kind.*

Proof. Let p be an accumulation point of the sequence $\{G(n)\}_{n \in \mathbb{N}}$. By Lemma 8, for large n, arcs of G through $G(n)$ are parallel one to another. Let S be a segment through p, parallel to G. There is at least one side of S where arcs of G accumulate; we call it the side of G. We can prolong S in the direction of G as a quasi-geodesic $\overline{S}$ with angle π on the side of G as long as it does not meet any vertex of curvature at least π. But if $\overline{S}$ meets a vertex of curvature more than π, then G should intersect in the vicinicy of this vertex, which is impossible. If $\overline{S}$

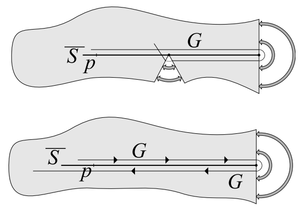

FIGURE 4. Proof of Lemma 10: if $\overline{S}$ meets a π-curved vertex after another one, then G should self-intersect. If it meets the π-curved vertex first, then it can be defined in the other direction.

meets a vertex of curvature π after one or several vertices of curvature less than π, then G should self-intersect in the vicinity of the last of those vertices (see Figure 4). Hence, either G is infinitely prolongable, or it meets a π-curved vertex before any other. In this case, G accumulates on the other side of S too, in the opposite direction (see Figure 4).

So we can define $\overline{S}$ in the other direction. Once again, either $\overline{S}$ is infinitely prolongable, or it meets a π-curved vertex. In this case the prolongation of any arc of G near p should be a closed geodesic, and we get a contradiction.

It follows that $\overline{S}$ can be prolongated as a ray, at least in one direction. Note that $\overline{S}$ cannot have any self-intersection, for it is parallel to G which is simple. If $\overline{S}$ were periodical then, once again, arcs of G would prolong as closed geodesics. Hence $\overline{S}$ visits at most once each vertex, and so, after leaving the last one, becomes a geodesic ray. Since G is parallel to $\overline{S}$ and acumulate along it, then $\overline{S}$ cannot be of the first kind. □

Lemma 11. *On a convex polyhedron, there is no simple ray of third kind.*

Proof. Let $x = G(t)$ a point of third kind. By definition, there exists $\varepsilon > 0$ such that one half of $B_0 = B(x, \varepsilon)$ does not intersect G, and there exist points $x_n = G(t_n)$ tending to x when t_n tends to infinity. Let $A_n = G|[t_n - \varepsilon, t_n + \varepsilon]$ and $A = G|[t - \varepsilon, t + \varepsilon]$. By Lemma 8, those arcs are all parallel for large n. Equip B with an orthonormal coordinate system such that $A(t) = (t, 0)$ and each A_n lies in the positive ordinate half plane. By extracting subsequences, we can assume without loss of generality that (1) the sequence $\{t_n\}_n$ is increasing, (2) the sequence $\{d(x_n, A)\}_n$ is decreasing, and (3) the arcs A_n are all oriented in the same direction, say as the x-axis.

Prolong A in the direction of the x-axis as a quasi-geodesic $\overline{A}$ with angle π on the A_n's side, as long as $\overline{A}$ does not meet any vertex of curvature greater than or equal to π. Indeed, if it meets a vertex with curvature more than π, then the prolongation of A_n should self-intersect in the vicinity of this vertex, for n large enough. Assume now that $\bar{A}$ meets a π-curved vertex. Then either $\overline{A_n}$ will self-intersect (in the case that $\overline{A}$ passes through a vertex) or will intersect the half of B_0 of negative ordinates (see Figure 5). In both cases we get a contradiction. Hence $\overline{A}$ is a quasi-geodesic ray.

Assume that $\overline{A}$ is parametrized in such a way that $\overline{A}(t) = (t, 0)$ for t small enough. Denote by $\overline{A_n}$ the parametrization of G such that $\overline{A_n}(t) = (t, \varepsilon_n)$ for small t. Put $f_n(t) = d\left(\overline{A}(t), \overline{A_n}(t)\right)$. We claim that, for n large enough, $f_n(t) \leq$

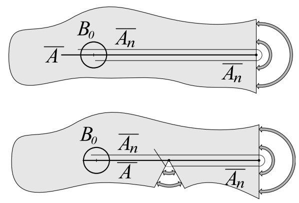

FIGURE 5. Proof of Lemma 11: if $\overline{A}$ meets a π-curved vertex, then $\overline{A_n}$ should either self-intersect or enter the negative ordinate part of B_0.

$f_n(0) = \varepsilon_n$, with equality for any t such that $\overline{A}(t)$ is not too close from a π-curved vertex. Indeed, f_n is constant as long as the "strip" delimited by $\overline{A}$ and $\overline{A_n}$ contains no vertices. But if it happens that some vertex of curvature distinct from π is included in the "strip", then $\overline{A_n}$ and $\overline{A}$ should intersect in its vicinity, provided there are no other vertices around. Moreover, the distance between the vertex and the intersection point depends only on the curvature of the vertex and on ε_n, and tends to 0 when ε_n becomes smaller and smaller. Hence, by choosing n large enough, we can ensure that no other vertex will interfere. If $\overline{A_n}$ intersected $\overline{A}$, then it would also intersect $\overline{A_m}$ for $m > n$, in contradiction to the simpleness of G . Hence only π-curved vertices may interfere. But in this case, it is easy to see that $f_n(t)$ will become again constantly equal to ε_n, when $\overline{A}$ leaves the vicinity of the vertex.

Now A_{n+1} is a subarc of $\overline{A_n}$, meaning that $\overline{A_n}$ will enter again into B_0 at ordinate $\varepsilon_{n+1} < \varepsilon_n$. By the claim, $\overline{A}$ will also enter again into B, at ordinate $\varepsilon_{n+1} \pm \varepsilon_n$. Indeed, $\varepsilon_{n+1} + \varepsilon_n$ is impossible, because it would imply that the strip between $\overline{A}$ and $\overline{A_n}$ is twisted, in contradiction to the orientability of P. It follows that $\overline{A}$ enters into B_0 with an ordinate $y < 0$. Let m be large enough to ensure $\varepsilon_m < -y$; due to the claim, $\overline{A_m}$ should enter too into the negative ordinate part of B_0, and we get a contradiction. Hence $K_3(G)$ is empty. □

Theorem 2. *Let G be a simple geodesic ray on a convex polyhedron P. Then G is dense in P.*

Proof. By Lemma 11, G is not of the third kind. Assume first that G is of the second kind.

Assume that $\mathrm{cl}(\mathrm{Im}(G))$ is not the whole polyhedron and let o be a point in $P \setminus \mathrm{cl}(\mathrm{Im}(G))$. Let $B_0 = B(o, d(o, \mathrm{cl}(S)))$ and $x \in \partial\mathrm{cl}(S) \cap \partial B_0$. By choosing o close enough to a flat point of $\mathrm{cl}(\mathrm{Im}(G))$, one can assume without loss of generality that x is not a vertex. Choose $\varepsilon > 0$ smaller than the half-distance between x and its closest vertex. Let $x_n = G(\tau_n)$ be a point in $\mathrm{Im}(G)$ tending to x such that $\tau_n \to \infty$. Let $A_n = G|[\tau_n - \frac{\varepsilon}{2}, \tau_n + \frac{\varepsilon}{2}]$. By Lemma 8, those arcs are all parallel for large n. Denote by A the limit of A_n. Let B be a ball centered at p, small enough to ensure that all A_ns that intersect B are parallel. Equip B with an orthonormal coordinate system such that $A(t) = (t, 0)$ and A_n lies in the positive ordinate half plane. By Lemma 11, all points of G are of the second kind, meaning that $x_n \notin A$, for otherwise there would be some arcs of G on both sides of A, and those with negative coordinates should intersect B_0. Hence, by extracting subsequences, we can assume without loss of generality that (1) the sequence $\{\tau_n\}_n$ is increasing, (2) the sequence $\{d(x_n, A)\}_n$ is decreasing, and (3) the arcs A_n are all oriented in the same direction, say as the x-axis.

Now, we get a contradiction using exactely the same construction as in the end of the proof of Lemma 11.

If G were of the first kind, then we could apply the above argument to the ray $\overline{S}$ provided by Lemma 10. Hence $\overline{S}$ would be dense, and then G too; in contradiction to the fact that it is of first kind. □

6. Examples and questions

1. As mentioned in the introduction, isosceles tetrahedra are the only convex polyhedra admitting an unbounded length spectrum. In the light of our result, one can ask if a polyhedron with an unbounded length spectrum should be parallel. This question seems especially natural knowing that any parallel polyhedron admits such a length spectrum [6]. The answer, however, is negative, as shown by the following example.

Example 5. *Let $a \in]0,1[$ be an irrational number. Consider the double D of the unit square $[0,1] \times [0,1]$. As a double of rectangle, D is a degenerate isosceles tetrahedron, and so is parallel. Cut D along the segment σ of the upper face (say) corresponding to $\{a\} \times [1/3, 2/3]$. We obtain a manifold with boundary. Glue any polygon of perimeter $2/3$ along its boundary in order to obtain a polyhedron P homeomorphic to the sphere. P is clearly not parallel, for it has vertices whose curvature is not divisible by π. There is a natural injection $i : D \setminus \sigma \to P$. Let γ_n be the maximal geodesic of D starting at $(a, 0)$ on the upper face (say), directed by $(1/n, 1)$. It is easy to see that γ_n will never cross σ, and so, is the image under i of a closed geodesic of D of length $2n\sqrt{1 + \frac{1}{n^2}}$.*

2. The existence of only one simple dense geodesic ray does not guarantee that P is parallel, as illustrated by the following example. We do not know if there exists non-orientable polyhedra with two (orthogonal) simple dense rays.

Example 6. *Let $a \in [0,1] \setminus \mathbb{Q}$. Consider the unit square $[0,1] \times [0,1]$ with boundary identified as shown on Figure 6: A on A, B on B and C on C, respecting the orientation given by the small white triangles. The resulting polyhedron has non-orientable genus 3, and a unique vertex of curvature -2π. It is easy to see that a line starting at any point of abscissa $x \notin \mathbb{Q} + a\mathbb{Q}$ in the direction of the y-axis is dense.*

3. Without the convexity hypothesis, the existence of a simple geodesic ray does not guarantee the parallelism.

Example 7. *Let P be a parallel polyhedron and G a simple dense geodesic on P. Let σ be a segment disjoint of (and consequently parallel to) $\operatorname{Im}(G)$. Cut P along σ and glue any polygon whose perimeter equals the double of the length of σ. Denote by Q this new polyhedron and by $i : P \setminus \sigma \to Q$ the natural injection. Then $i \circ G$ is a simple geodesic ray of Q, which is not dense in G, for it avoids the glued polygon. Of course, Q is not parallel.*

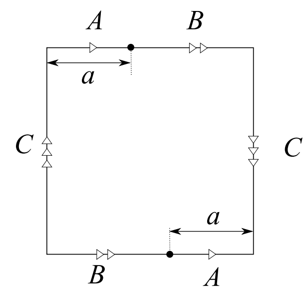

Figure 6. Example of non-parallel polyhedron with simple dense geodesic ray.

4. We saw (Proposition 4) that on a parallel polyhedron all strict geodesics are simple.

Question 1. *Does the fact that all geodesics are simple characterizes parallel polyhedra?*

5. V. Yu. Protasov [9] showed that an unbounded length spectrum characterizes isosceles tetrahedra among convex polyhedra, and his result has recently been strenghtened to convex surfaces by A. Akopyan and A. Petrunin [1]. The following question remains open.

Question 2. *Are isosceles tetrahedra the only convex surfaces with a simple dense geodesic ray? A simple geodesic ray?*

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