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ANNEXE 5.2

Joël Rouyer, Costin Vîlcu:

Farthest points on most Alexandrov surfaces

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Farthest points on most Alexandrov surfaces

Joël Rouyer Costin Vîlcu

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Abstract

We study global maxima of distance functions on most Alexandrov surfaces with curvature bounded below, where *most* is used in the sense of Baire categories.

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1 Introduction

In this article, by *Alexandrov surface*, we always mean a compact 2-dimensional Alexandrov space with curvature bounded below by κ, without boundary. Closed Riemannian surfaces with Gauss curvature at least κ and κ-polyhedra are important examples of such surfaces. Other significant examples, for $\kappa = 0$, are convex surfaces (*i.e.*, boundaries of compact convex sets with non-empty interior) in $\mathbb{R}^3$. Roughly speaking, these surfaces are 2-dimensional topological manifolds endowed with an intrinsic metric which verifies Toponogov's comparison property. For the precise definition and basic properties of Alexandrov spaces, see for example [3].

Let $\mathcal{A}(\kappa)$ be the set of Alexandrov surfaces with curvature bounded below by κ. Before going further, we recall the following simple fact.

Remark 1 *Multiplying all distances in $A \in \mathcal{A}(\kappa)$ by a constant $\delta > 0$ provides an Alexandrov surface $\delta A \in \mathcal{A}(\frac{\kappa}{\delta^2})$. Moreover, for $A' \in \mathcal{A}(\kappa)$, the Gromov-Hausdorff distance between δA and $\delta A'$ is exactly δ times the Gromov-Hausdorff distance between A and A'. Therefore, the spaces $\mathcal{A}(\kappa)$ and $\mathcal{A}(\frac{\kappa}{\delta^2})$ are homothetic, and we may assume without loss of generality that $\kappa \in \{-1, 0, 1\}$.*

Two Alexandrov surfaces belong to the same connected component of $\mathcal{A}(\kappa)$ if and only if they are homeomorphic to each other [14]. Denote by $\mathcal{A}(\kappa, \chi)$ the set of all surfaces in $\mathcal{A}(\kappa)$ whose Euler-Poincaré characteristic is χ. Then $\mathcal{A}(\kappa, \chi)$ (if non-empty) is a connected component of $\mathcal{A}(\kappa)$ if χ is positive or odd, and is the union of two components otherwise. In particular, $\mathcal{A}(0)$ has four components, consisting respectively of flat tori, flat Klein bottles (both of these in $\mathcal{A}(0, 0)$), convex surfaces (in $\mathcal{A}(0, 2)$), and non-negatively curved projective planes (in $\mathcal{A}(0, 1)$). Sometimes it is necessary to exclude the components consisting of flat surfaces; we shall then replace $\mathcal{A}(\kappa)$ by

$$\mathcal{B}(\kappa) = \begin{cases} \mathcal{A}(\kappa) & \text{if } \kappa \neq 0, \\ \mathcal{A}(0,1) \cup \mathcal{A}(0,2) & \text{if } k = 0. \end{cases}$$

It is known that, endowed with the topology induced by the Gromov-Hausdorff distance, the set $\mathcal{A}(\kappa)$ is a Baire space [8]. In any Baire space, one says that *most elements* enjoy, or that a *typical element* enjoys, a given property if the set of those elements which do not satisfy it is of first category.

The investigation of typical properties of Alexandrov surfaces from Baire category viewpoint is very recent, see [1], [8], [15]; it generalizes a similar, well-established research direction for convex surfaces, see *e.g.* the survey [6].

The study of farthest points on convex surfaces originated from some questions of H. Steinhaus, presented by H. T. Croft, K. J. Falconer and R. K. Guy in the chapter A35 of their book [4]. The questions have been answered since then, mainly by T. Zamfirescu in a series of papers [22], [23], [24], [25], and yielded several results about farthest points on most convex surfaces; see the survey [18]. A few results have also been obtained for general Alexandrov surfaces [19], [21]. The framework of a typical Riemannian metric was considered in [12].

In this paper we employ Baire categories to obtain properties of farthest points on Alexandrov surfaces, thus contributing to the study proposed by H. Steinhaus. We generalize results about convex surfaces from [23] by showing that, on most surfaces $A \in \mathcal{B}(\kappa)$, most points $x \in A$ have a unique farthest point (Theorem 1) which is joined to x by precisely 3 segments (Theorem 2). In particular, Theorem 1 gives an answer to the last open problem in [25].

The restriction to $\mathcal{B}(\kappa)$ concerning Theorem 1 is mandatory. Indeed, all points on a typical flat torus have two farthest points, while in the connected component $\mathcal{K} \subset \mathcal{A}(0)$ of flat Klein bottles there is no typical behavior. More precisely, there exists open sets $\mathcal{U}$, $\mathcal{V}$ in $\mathcal{K}$ such that, on any surface in $\mathcal{U}$ most (but not all) points have two antipodes, while on any surface in $\mathcal{V}$ there exits open sets of points U_1 and U_2 such that any point of U_i has precisely i farthest points, $i = 1, 2$. This is proven using only elementary methods [16].

The paper is organized as follows. In Section 2, we recall some useful facts about Alexandrov surfaces. In Section 3, we investigate the typical number of farthest points. There are certain similarities with the original proof in [23] but, as the central argument there does not hold in our framework, we replaced it by a new one derived from the main result in [12]. The last two sections are devoted to the typical number of segments between a point and its unique farthest point. Once again, the original argument does not apply in our framework. The change of (the sign of) the curvature bound forces us to develop a new geometric argument, while the lack of extrinsic geometry leads us to consider, as an auxiliary space, the space of all Alexandrov metrics on a given topological surface.

2 Alexandrov surfaces

In this section we give necessary prerequisites and notation for the rest of the paper. Other basic facts about Alexandrov surfaces, implicitly used in the paper, can be found in [3], [9], [17].

For $A \in \mathcal{A}(\kappa)$ and $x \in A$, $\rho_x = d(x, \cdot)$ denotes the *distance function from* x, and F_x stands for the set of all *farthest points from* x, that is, the global maxima of ρ_x. The set of local maxima of ρ_x will be denoted by M_x. Moreover, $B(x, r)$ and $\bar{B}(x, r)$ stand respectively for the open and closed ball of radius r centered at x.

The above notation assumes that the space A and its metric are clear from the context. In some cases, however, it will be necessary to specify the metric space A or the metric d itself (*e.g.*, when several metrics on the same space are considered), and we shall add a superscript: F_x^d, M_x^A, ρ_x^A, etc.

Let $f : X \to Y$ be a map between metric spaces; the *distortion* of f is given by

$$\operatorname{dis}(f) = \sup_{x,x' \in X} \left| d(x, x') - d\left(f(x), f(x')\right) \right|.$$

Lemma 2 (Perel'man's stability theorem) [10] *Let A_n, $A \in \mathcal{A}(\kappa)$ and suppose that there exist functions $f_n : A \to A_n$ such that* $\operatorname{dis}(f_n) \to 0$. *Then, for n large enough, there exist homeomorphisms $h_n : A \to A_n$ such that* $\sup_{x \in A} d(f_n(x), h_n(x)) \to 0$.

We denote by $d_H^Z(H, K)$ the usual *Pompeiu-Hausdorff distance* between compact subsets H and K of a metric space Z, and omit the superscript Z whenever no confusion is possible. If X and Y are compact metric spaces, we denote by $d_{GH}(X, Y)$ the *Gromov-Hausdorff distance* between X and Y. Therefore $d_{GH}(H, K) \le d_H^Z(H, K)$, and a partial converse is given by the following lemma.

Lemma 3 [13] *Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of compact metric spaces converging to X with respect to the Gromov-Hausdorff metric, and let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers. Then there exist a compact metric space Z, an isometric embedding $\varphi : X \to Z$ and, for each non-negative integer n, an isometric embedding $\varphi_n : X_n \to Z$, such that*

$$d_H^Z(\varphi_n(X_n), \varphi(Y)) < d_{GH}(X_n, X) + \varepsilon_n.$$

Consider two surfaces S and S' with boundaries ∂S and $\partial S'$, and two arcs $I \subset \partial S$ and $I' \subset \partial S'$ having the same length. By *gluing S to S' along I* we mean identifying the points $x \in I$ and $\iota(x) \in I'$, where $\iota : I \to I'$ is a length preserving map between I and I'.

Lemma 4 (Alexandrov's gluing theorem) [11] *Let S be a closed topological surface obtained by gluing finitely many geodesic polygons cut out from surfaces in $\mathcal{A}(\kappa)$, in such a way that the sum of the angles glued together at each point is at most 2π. Then, endowed with the induced intrinsic metric, S belongs to $\mathcal{A}(\kappa)$.*

For any surface $A \in \mathcal{A}(\kappa)$, and any $x \in A$, we denote by Σ_x the set of directions at $x \in A$. It is known that Σ_x is isometric to a circle of length at most 2π [3]. The point x is said to be *conical* if the length of Σ_x is less than 2π. The *singular curvature* at the point x is defined as 2π minus the length of Σ_x.

Let $\mathbb{M}_\kappa$ denote the simply-connected and complete Riemannian surface of constant curvature κ. A *κ-polyhedron* is an Alexandrov surface obtained by gluing finitely many geodesic polygons from $\mathbb{M}_\kappa$.

Denote by $\mathcal{R}(\kappa)$ the set of all closed Riemannian surfaces with Gauss curvature at least κ, and by $\mathcal{P}(\kappa)$ the set of κ-polyhedra. A formal proof for the next result can be found, for instance, in [8].

Lemma 5 *Both $\mathcal{P}(\kappa)$ and $\mathcal{R}(\kappa)$ are dense in $\mathcal{A}(\kappa)$.*

A *segment* on the surface A is a shortest path on A. The set of all segments between two points $x, y \in A$ will be denoted by $\mathfrak{S}_{xy}$.

The *cut locus* $C(x)$ of a point $x \in A$ is the set of all extremities, different from x, of maximal (with respect to inclusion) segments starting at x. It is known that $C(x)$ is locally a tree with at most countably many ramifications points [17].

The following lemma is a direct consequence of Theorem 2 in [26] and the main result in [1].

Lemma 6 *For most $A \in \mathcal{B}(\kappa)$ and any $x \in A$, most points of A belong to $C(x)$ and are joined to x by a unique segment.*

3 Typical number of farthest points

The aim of this section is to prove the following partial generalization of Theorem 2 in [23].

Theorem 1 *On most $A \in \mathcal{B}(\kappa)$, most $x \in A$ have precisely one farthest point.*

T. Zamfirescu strengthened Theorem 2 in [23] by proving that on any convex surface S, the set of those $x \in S$ for which F_x contains more than one point is σ-porous [25]. This statement is not true for all Alexandrov surfaces, as one can easily see for the standard projective plane, but it would be interesting to prove it for most surfaces $A \in \mathcal{B}(\kappa)$. With this respect, our Theorem 1 gives a partial answer to the last open problem in [25].

Note that Theorem 1 also admits the following variant in the framework of Riemannian geometry.

Lemma 7 [12, Theorems 1 and 2] *Let S be a closed differentiable manifold and let $\mathcal{G}$ be the set of all C^2 Riemannian structures on S, endowed with the C^2 topology. Then, for most $g \in \mathcal{G}$ and most $x \in S$, x admits a unique farthest point with respect to g, to which it is joined by at most three segments.*

For the proof of Theorem 1 we need two more lemmas.

Lemma 8 *Let A_n, $A \in \mathcal{A}(\kappa)$ be isometrically embedded in some compact metric space Z. Assume that $A_n \to A$ with respect to the Pompeiu-Hausdorff distance in Z. Let $x_n \in A_n$ converge to $x \in A$. If $y_n \in F_{x_n}^{A_n}$ and $y \in A$ such that $y_n \to y$ then $y \in F_x^A$.*

Proof. If X is a compact subset of Z, and x belongs to X, we denote by ϱ_x^X the radius of X at x, *i.e.*, the maximal value of the distance function from x restricted to X. We claim that ϱ is continuous with respect to x and X. Let X, X' be two compact subsets of Z, let $y \in X$ be a farthest point from $x \in X$, let $x' \in X'$, and let $y' \in X'$ be a closest point to y. Then

$$\begin{aligned}\varrho_x^X &= d^X(x,y) \\ &\leq d^Z(x,x') + d^{X'}(x',y') + d^Z(y',y) \\ &\leq d^Z(x,x') + \varrho_{x'}^{X'} + d_H^Z(X,X').\end{aligned}$$

Exchanging the roles of X and X' we get

$$\left|\varrho_x^X - \varrho_{x'}^{X'}\right| \leq d^Z(x,x') + d_H^Z(X,X'),$$

thus proving the claim.

Now $d^A(x,y) = \lim d^{A_n}(x_n,y_n) = \lim \varrho_{x_n}^{A_n} = \varrho_x^A$, whence $y \in F_x^A$. ■

Lemma 9 *Let $A \in \mathcal{B}(\kappa)$. Then A can be approximated by Riemannian surfaces with Gaussian curvature strictly larger than κ.*

Proof. If $\kappa \neq 0$, the conclusion is reached by contractions or dilations (see Remark 1) and by the density of $\mathcal{R}(\kappa')$ in $\mathcal{A}(\kappa')$. We prove here that each $A \in \mathcal{A}(0,1) \cup \mathcal{A}(0,2)$ can be approximated by Riemannian surfaces of positive curvature.

We shall show that A can be approached by κ-polyhedra of $\mathcal{A}(\kappa)$, with $\kappa > 0$ (but of course tends to 0). Then the approximation of such polyhedra by surfaces of $\mathcal{R}(\kappa)$ will follow from Lemma 5.

Suppose first that $A \in \mathcal{A}(0,2)$. By Alexandrov's realization theorem [2], A is isometric to a convex surface in $E \overset{\text{def}}{=} \mathbb{R}^3$. This surface can be approached by convex polyhedra $P \subset E$. Assume that E is embedded in $\mathbb{R}^4$ as the affine hyperplane $\{R\} \times \mathbb{R}^3$. Let S be the sphere of radius R centered at 0, and consider the radial projection $\phi_R : E \to S$, defined by $\phi_R(x) = \frac{Rx}{\|x\|}$, where $\|x\|$ denotes the Euclidean norm of x as a vector in $\mathbb{R}^4$. It is clear that ϕ_R maps lines of E on great circle of S, planes of E on totally geodesic subspaces of S, and consequently 0-polyhedra of E on $\frac{1}{R^2}$-polyhedra of S. Rather obviously too, $\phi_R(P)$ tends to P when R tends to infinity. (Indeed, a straightforward computations shows that $d^S(\phi_R(a), \phi_R(b)) = \|b-a\| + o\left(\frac{1}{R}\right)$).

If $A \in \mathcal{A}(0,1)$, then one can apply the same construction to its universal covering C, which is a centrally symmetric convex surface. Since ϕ_R obviously preserves this symmetry, $\phi_R(C)$ can be quotiented to obtain an approximation of A. ■

Proof of Theorem 1. For $A \in \mathcal{B}(\kappa)$, define the following sets:

$$T(A) \overset{\text{def}}{=} \{x \in A \,|\, \#F_x \geq 2\},$$

$$T(A,\varepsilon) \overset{\text{def}}{=} \{x \in A \,|\text{diam}(F_x) \geq \varepsilon\}.$$

We have $T(A) = \bigcup_{k\in\mathbb{N}^*} T(A,1/k)$ and $T(A,1/k)$ is obviously closed in A. Further define:

$$\mathcal{M} \overset{\text{def}}{=} \{A \in \mathcal{B}(\kappa) \,|T(A) \text{ is not meager}\},$$

$$\mathcal{M}(\varepsilon) \overset{\text{def}}{=} \{A \in \mathcal{B}(\kappa) \,|\text{int}(T(A,\varepsilon)) \neq \varnothing\},$$

$$\mathcal{M}(\varepsilon,\eta) \overset{\text{def}}{=} \left\{A \in \mathcal{B}(\kappa) \,\middle|\exists x \in A \; \bar{B}(x,\eta) \subset T(A,\varepsilon)\right\},$$

whence

$$\mathcal{M} = \bigcup_{k\in\mathbb{N}^*} \mathcal{M}\left(\frac{1}{k}\right) \subset \bigcup_{\substack{k\in\mathbb{N}^*\\ r\in\mathbb{N}^*}} \mathcal{M}\left(\frac{1}{k},\frac{1}{r}\right).$$

It remains to prove that $\mathcal{M}(\varepsilon,\eta)$ is nowhere dense in $\mathcal{B}(\kappa)$.

We first show that it is closed. Let A_n be a sequence of Alexandrov surfaces of $\mathcal{M}(\varepsilon,\eta)$ converging to $A \in \mathcal{B}(\kappa)$. Assume those surfaces embedded in a same compact metric space (see Lemma 3). Let $x_n \in A_n$ be such that $\bar{B}(x_n,\eta) \subset T(A_n,\varepsilon)$. Select a converging subsequence of $\{x_n\}$ and denote by x its limit.

We claim that $\bar{B}(x,\eta) \subset T(A,\varepsilon)$. Choose $y \in \bar{B}(x,\eta)$, hence y is limit of $\{y_n\}$, with $y_n \in \bar{B}(x_n,\eta) \subset T(A_n,\varepsilon)$. Consequently, there exist $u_n, v_n \in F_{y_n} \subset A_n$ such that $d^{A_n}(u_n,v_n) \geq \varepsilon$. By extracting subsequences, one can assume that $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ converge to $y \in \bar{B}(x,\eta)$, $u \in A$, and $v \in A$. By Lemma 8 and the continuity of the metric function we get $u, v \in F_y \subset A$ and $d^A(u,v) \geq \varepsilon$. That is, $y \in T(A,\varepsilon)$, the claim is proved, and $\mathcal{M}(\varepsilon,\eta)$ is closed.

Suppose now that $\mathcal{M}(\varepsilon,\eta)$ has an interior point A. By Lemma 9, there exists a sequence $R_n \in \mathcal{R}(\kappa_n)$ tending to A in $\mathcal{R}(\kappa)$ for some sequence of numbers $\kappa_n > \kappa$. For n large enough, R_n is also interior to $\mathcal{M}(\varepsilon,\eta)$. Put $\mathcal{R}_1 = \{R \in \mathcal{R} \,|\text{for most } x \in R, \#F_x = 1\}$. By Lemma 7, there exists a sequence of $R_{n,p} \in \mathcal{R}_1$ converging to $R_n \in \mathcal{R}(\kappa_n)$ for the C^2 topology. Hence for p large enough, $R_{n,p} \in \mathcal{M}(\varepsilon,\eta) \cap \mathcal{R}(\kappa)$, which is in contradiction with $R_{n,p} \in \mathcal{R}_1$. Hence $\mathcal{M}(\varepsilon,\eta)$ has empty interior. This completes the proof of Theorem 1. ■

4 Two lemmas

In this section we give two auxiliary results which will be invoked in Section 5; the first one seems to have an interest in its own.

Let S be a closed topological surface. We denote by $\mathfrak{M}_k(S)$ the space of continuous metrics d on S such that $(S,d) \in \mathcal{A}(\kappa)$. $\mathfrak{M}_k(S)$ is naturally endowed with the metric δ defined by

$$\delta\left(d,d'\right) = \max_{x,y\in S} \left\|d(x,y) - d'(x,y)\right\|.$$

Lemma 10 *The space $\mathfrak{M}_\kappa(S)$ is Baire.*

Proof. Consider in $\mathbb{M}_\kappa$ a triangle xyz such that $d(x,y) = a$, $d(x,z) = b$, and $d(y,z) = c$. We denotes by $\Theta_\kappa(a;b,c)$ the angle $\measuredangle xzy$. Moreover, we set $\Theta_\kappa(a;b,c) = 0$ whenever $bc = 0$ or the triple (a,b,c) does not satisfy one of the three triangle inequalities. Hence Θ_κ is defined on $\mathbb{R}^3_+$ and is lower semi-continuous.

Since S is compact, the set $\mathfrak{F}(S)$ of all non-negative valued continuous functions on $S \times S$ is complete. Denote by $\mathfrak{F}'_\kappa(S)$ the set of those functions $f \in \mathfrak{F}(S)$ such that

$$\forall x, y \in S \ f(x, y) = f(y, x) \tag{1}$$
$$\forall x \in S \ f(x, x) = 0 \tag{2}$$
$$\forall x, y, z \in S \ f(x, y) \leq f(x, z) + f(z, x) \tag{3}$$
$$\forall x, y \in S \ \exists z \in A \ f(x, z) = f(z, y) = \frac{1}{2} f(x, y) \tag{4}$$
$$\begin{aligned} &\forall x, y, z, p \in S \ x \neq p \text{ and } y \neq p \text{ and } z \neq p \Longrightarrow \\ &\quad \Theta_k(f(x, y); f(p, x), f(p, y)) \\ &\quad + \Theta_k(f(y, z); f(p, y), f(p, z)) \\ &\quad + \Theta_k(f(z, x); f(p, z), f(p, x)) \leq \pi. \end{aligned} \tag{5}$$

The set $\mathfrak{F}'_\kappa(S)$ is obviously closed in $\mathfrak{F}(S)$ and consequently complete. Now, notice that (1), (2), (3) are standard axioms of metrics, (4) is well-known to imply that the metric is intrinsic, and (5) is the so-called 4 points property in [3], which is one of the alternative definitions of Alexandrov spaces with curvature bounded below by κ. It follows that a function $f \in \mathfrak{F}'_\kappa(S)$ belongs to $\mathfrak{M}_\kappa(S)$ if and only if it satisfies furthermore $f(x, y) = 0 \Rightarrow x = y$. In other words

$$\mathfrak{F}'_\kappa(S) \setminus \mathfrak{M}_\kappa(S) \subset \bigcup_{n \in \mathbb{N}} \mathfrak{H}_n,$$

where

$$\mathfrak{H}_n = \left\{ f \in \mathfrak{F}'_\kappa(S) \,\middle|\, \exists x, y \in S \text{ s.t. } d_0(x, y) \geq \frac{1}{n} \text{ and } f(x, y) = 0 \right\}$$

and d_0 is any fixed distance on S. The sets $\mathfrak{H}_n$ are closed in $\mathfrak{F}'_\kappa(S)$, hence $\mathrm{cl}(\mathfrak{M}_\kappa(S)) \setminus \mathfrak{M}_\kappa(S)$ is included in a countable union of closed sets in $\mathrm{cl}(\mathfrak{M}_\kappa(S))$. On the other hand, the interior of $\mathrm{cl}(\mathfrak{M}_\kappa(S)) \setminus \mathfrak{M}_\kappa(S)$ is obviously empty, whence $\mathfrak{M}_\kappa(S)$ in residual in its closure, which is complete. Therefore it is a Baire space. ■

The next elementary result is a small step in the proof of Theorem 2.

Lemma 11 *Let κ be 0 or 1. Consider in $\mathbb{M}_\kappa$ a non-convex quadrilateral $x_0x_1zx_2$ depending on two parameters l and ε, such that the respective midpoints y_1 and y_2 of x_0x_1 and x_0x_2 satisfy $x_0x_1 = x_0x_2 = 2l$, $x_1z = x_2z = l$, $y_1y_2 = \varepsilon$, see Figure 1. In the case $\kappa = 1$, assume moreover that $l < \frac{\pi}{2}$. Then, for ε small enough, the circumcenter of the triangle $x_0x_1x_2$ lies inside the triangle y_1y_2z.*

Proof. Let α, β, θ, ϕ and ψ be the angles defined by Figure 1; we have to prove that $\beta > \alpha$.

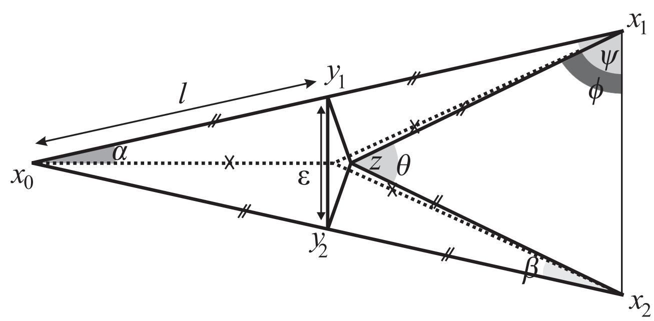

Figure 1: A configuration of isosceles triangles in $\mathbb{M}_\kappa$.

In the case $\kappa = 0$, we may assume without loss of generality that $l = 1$, hence $\alpha = \arcsin \frac{\varepsilon}{2}$ and $\theta = 2 \arcsin \varepsilon$. It follows that

$$\beta - \alpha = \phi - \psi - \alpha = \frac{\pi}{2} - \alpha - \frac{\pi - \theta}{2} - \alpha = \frac{\theta}{2} - 2\alpha > 0.$$

Now we assume $\kappa = 1$. A cosine law in the triangle $x_0x_1x_2$ gives

$$\cos \varepsilon = \cos^2 l + \sin^2 l \cos 2\alpha,$$

whence

$$\alpha = \frac{1}{2 \sin l} \varepsilon \left(1 + \frac{1}{24 \tan^2 l} \varepsilon^2\right) + o\left(\varepsilon^4\right).$$

Let a be the distance between x_1 and x_2. In the triangle $x_0x_1x_2$, a cosine law gives

$$\cos a = \cos^2 2l + \sin^2 2l \cos 2\alpha,$$

whence

$$a = 2 \cos l \varepsilon + \frac{2 \cos l + \cos 3l}{12} \varepsilon^3 + o\left(\varepsilon^4\right).$$

A second cosine law in the same triangle gives

$$\cos 2l = \cos 2l \cos a + \sin 2l \sin a \cos \phi,$$

whence

$$\phi = \frac{\pi}{2} - \frac{\cos l}{\tan 2l} \varepsilon - \frac{\cos^2 l \cos 2l}{48 \sin^3 l} \left(1 + 8 \sin^2 l\right) \varepsilon^3 + o\left(\varepsilon^4\right).$$

In the triangle x_1x_2z we have

$$\cos l = \cos l \cos a + \sin l \sin a \cos \psi,$$

therefore

$$\psi = \frac{\pi}{2} - \frac{\cos l}{\tan l} \varepsilon - \frac{\cos^2 l}{24 \sin^3 l} \left(2 \sin^2 2l + 5 \cos^2 l - 1\right) \varepsilon^3 + o\left(\varepsilon^4\right).$$

Finally,

$$\begin{aligned} \beta - \alpha &= \phi - \psi - \alpha \\ &= \frac{\cos^2 l}{8 \sin^3 l} \varepsilon^3 + o\left(\varepsilon^4\right) > 0. \end{aligned}$$

■

5 Typical number of segments

In this section we generalize the last part of Theorem 2 in [23], from convex surfaces to Alexandrov surfaces. Results in the same direction were obtained by the first author [12], for a manifold endowed with a typical Riemannian metric (see Lemma 7), and by P. Horja [7] and T. Zamfirescu [27], for upper curvature bounds.

Theorem 2 *For most $A \in \mathcal{A}(\kappa)$, most $x \in A$ and are joined to each of their farthest point by exactly* 3 *segments.*

The proof of our last theorem consists of Lemmas 12, 13 and 17.

The next lemma follows from an explicit computation of cut loci of flat surfaces [16].

Lemma 12 *On most $A \in \mathcal{A}(0,0)$, most points are joined to any of their farthest point by exactly three segments.*

Lemma 13 *For most $A \in \mathcal{B}(\kappa)$, for most $x \in A$, x is joined to each of its farthest points by at most three segments.*

Proof. For $x, y \in A$, recall that $\mathfrak{S}_{xy}$ denotes the set of segments from x to y. For $A \in \mathcal{B}(\kappa)$ and $\varepsilon > 0$, define

$$Q(A) \stackrel{\text{def}}{=} \{x \in A \,|\exists y \in F_x \text{ with } \#\mathfrak{S}_{xy} \geq 4\},$$

$$Q(A,\varepsilon) \stackrel{\text{def}}{=} \left\{ x \in A \;\middle|\; \begin{array}{l} \exists y \in F_x \; \exists \sigma^1, \ldots, \sigma^4 \in \Sigma_{xy} \\ \forall 1 \leq i < j \leq 4 \; d_H^A\left(\sigma^i, \sigma^j\right) \geq \varepsilon \end{array} \right\}.$$

Note that $Q(A,\varepsilon)$ is closed, and $Q(A) = \bigcup_{p \in \mathbb{N}^*} Q(A, 1/p)$. Further define

$$\mathcal{N} \stackrel{\text{def}}{=} \{A \in \mathcal{B}(\kappa) \,| Q(A) \text{ is not meager}\},$$

$$\mathcal{N}(\varepsilon) \stackrel{\text{def}}{=} \{A \in \mathcal{B}(\kappa) \,|\text{int}\,(Q(A,\varepsilon)) \neq \varnothing\},$$

$$\mathcal{N}(\varepsilon, \eta) \stackrel{\text{def}}{=} \left\{A \in \mathcal{B}(\kappa) \,\middle|\, \exists x \in A \; \bar{B}(x,\eta) \subset Q(A,\varepsilon)\right\},$$

whence

$$\mathcal{N} = \bigcup_{p \in \mathbb{N}^*} \mathcal{N}(1/p) \subset \bigcup_{p,q \in \mathbb{N}^*} \mathcal{N}(1/p, 1/q).$$

We prove now that $\mathcal{N}(\varepsilon, \eta)$ is closed. Let $\{A_n\}_n$ be a sequence of surfaces in $\mathcal{N}(\varepsilon, \eta)$ converging to $A \in \mathcal{B}(\kappa)$. By Lemma 3, we can assume that A_n and A are embedded in the same compact metric space Z. By hypothesis, there exists $x_n \in A_n$ such that $\bar{B}(x_n, \eta) \subset Q(A_n, \varepsilon)$. We can assume that $\{x_n\}_n$ converges to $x \in A$. A point $z \in \bar{B}(x, \eta)$ is limit of a sequence of points $z_n \in Q(A_n, \varepsilon)$. Hence there exist four segments σ_n^1, ..., σ_n^4 between z_n and $y_n \in F_{z_n}$. By selecting a subsequence, we can assume that $\{y_n\}_n$ converges to $y \in A$. By Lemma 8, $y \in F_z$. We can also assume that $\{\sigma_n^i\}_n$ converges to some segment σ_n^i from z to y, $i = 1, \ldots, 4$. Moreover, $d_H^A\left(\sigma^i, \sigma^j\right) = \lim d_H^A\left(\sigma_n^i, \sigma_n^j\right) \geq \varepsilon$. Hence $\bar{B}(x,\eta) \subset Q(A,\varepsilon)$, and $A \in \mathcal{N}(\varepsilon, \eta)$.

The proof that $\mathcal{N}(\varepsilon, \eta)$ has empty interior is similar to the proof of the emptiness of the interior of $\mathcal{M}(\varepsilon, \eta)$ in the proof of Theorem 1.

It follows that $\mathcal{N}$ is meager, whence the conclusion. ■

Lemma 14 *Let S be a closed topological surface such that $\mathfrak{M}_\kappa(S)$ is nonempty. Choose $d \in \mathfrak{M}_\kappa(S)$ and x, $y \in S$ such that $F_x^d = \{y\}$. Then there exist $d_n \in \mathfrak{M}_\kappa(S)$ converging to d, and $x_n, y_n \in S$ converging to x and y respectively, such that x_n and y_n are conical points on (S, d_n) and $F_{x_n}^{d_n} = \{y_n\}$.*

Proof. Choose d' close to d, such that most points in (S, d') have a unique farthest point to which they are joined by at most three segments, and such that, for any point z, most points in S belong to $C^{d'}(z)$ and are joined to z by unique segments. This is possible by virtue of Lemma 6, Lemma 13, Theorem 1 and Lemma 2. Chose a point x' close to x, whose unique farthest point y' satisfies $\#\mathfrak{S}_{x'y'} \leq 3$. By Lemma 8, y' tends to y when $d' \to d$ and $x' \to x$.

Take a point $v \in C(y')$, distinct from x', joined to y' by a unique segment γ_v. Take $w \in C(y)$ close to v and joined to y by precisely two segments γ_w^1 and γ_w^2; this is possible, because $C(y)$ has at most countably many ramifications points, see [17]. Remove the part Δ of (S, d') bounded by $\gamma_w^1 \cup \gamma_w^2$ and containing v, and glue the rest by identifying the two boundary segments. Denote by (S'', d'') the obtained surface, and by $f : S \setminus \Delta \to S''$ the canonical surjection. When w tends to v, γ_w^1 and γ_w^2 both tends to γ_v (because γ_v is the only segment between y and v), and (S'', d'') obviously tends to (S, d'). Clearly $d''(f(p), f(q)) \leq d'(p, q)$ for any p, $q \in S$.

We claim that, for w close enough to v, $d''(x', y') = d'(x', y')$. Let γ be a segment on (S'', d'') between $x'$and y'. If γ does not intersect $f\left(\gamma_w^1 \cup \gamma_w^1\right)$, then the claim holds. Otherwise, the limit of γ, which is a segment of (S, d') between x' any y', should intersect γ_v, which is impossible.

It follows that $F_{x'}^{(S'', d'')} = \{y'\}$.

By Lemma 2, there exists a homeomorphism $h : S \to S''$ such that $\operatorname{dis}(h) \to 0$, $x^* \overset{\text{def}}{=} h^{-1}(x') \to x'$ and $y^* \overset{\text{def}}{=} h^{-1}(y') \to y'$ when $w \to v$. Define the metric d^* on S by $d^*(p, q) = d''(h(p), h(q))$, we have y^* conical point for d^* and $F_{x^*}^{d^*} = \{y^*\}$.

A similar procedure applied to x produces the desired metric. ■

The following lemma is part of Theorem 3 in [5], see also Theorem 1 in [20]. The theorem was originally stated for all points $x \in A$ and only for F_x, but the proof given in [20] holds here too.

Lemma 15 *If $A \in \mathcal{A}(1)$ and $x \in A$ such that $d(x, F_x) > \pi/2$ then $\#M_x = \#F_x = 1$.*

A *long loop* at $x \in A$ is the union of two segments from some $y \in F_x$ to x, whose directions at y divide Σ_y into two arcs of length at most π.

Lemma 16 *Let S be a closed topological surface such that $\mathfrak{M}(\kappa)$ is nonempty. For most Alexandrov metrics on S, for most $x \in S$ and any $y \in F_x$, there are at least three segments between x and y.*

Proof. For any metric $d \in \mathfrak{M}_\kappa(S)$, the set

$$D(d) \overset{\text{def}}{=} \{x \in S \mid \text{there exists a long loop at } x \text{ with respect to } d\}$$

is clearly closed, from the definition of a long loop.

Consider a countable dense subset Z of S. We have

$$\mathcal{D} \overset{\text{def}}{=} \{d \in \mathfrak{M}_\kappa(S) \,|\text{int}\,(D(d)) \neq \emptyset\} \subset \bigcup_{\substack{z\in Z\\ q\in\mathbb{N}^*}} \mathcal{D}_{z,q}\,,$$

where

$$\mathcal{D}_{z,q} \overset{\text{def}}{=} \left\{ d \in \mathfrak{M}_\kappa(S) \,\middle|\, B\left(z, \frac{1}{q}\right) \subset D(d) \right\}.$$

Here, the balls $B(z, 1/q)$ are understood with respect to any fixed continuous metric on S. The sets $\mathcal{D}_{z,q}$ are clearly closed, it remains to prove that they have empty interior.

Assume on the contrary that there exist $z \in Z$ and $q \in \mathbb{N}^*$ such that $\text{int}\,(\mathcal{D}_{z,q}) \neq \emptyset$; in the rest of the proof we shall derive a contradiction.

By Lemma 2, a dense set in $\mathcal{A}(\kappa)$ corresponds to a dense set in $\mathfrak{M}_\kappa(S)$, so it is possible to choose a metric $d' \in \text{int}\,(\mathcal{D}_{z,q})$ such that (S, d') is typical in $\mathcal{A}(\kappa)$. By Theorem 3.1 in [8], (S, d') has no conical points, and by Lemma 13, we can choose a point $x' \in B(z, \frac{1}{3q})$ which is joined to its unique farthest point y' by at most three segments, two of which are forming a long loop $\Gamma'_{x'}$.

The space of directions at each point in (S, d') is a circle, hence we can define locally around x' and around y' a left and a right side of $\Gamma'_{x'}$. We apply twice the procedure described by Lemma 14 and its proof, one time for each side of $\Gamma'_{x'}$. Doing so allows us to assume that, after cutting, $\Gamma'_{x'}$ still divide $\Sigma_{x'}$ in two almost equally long curves. Here, 'almost equally long' means that the ratio between the length of the two connected components can be chosen arbitrary close to one. For the new metric $d \in \text{int}\,(\mathcal{D}_{z,q})$, there are conical points $x \in B\left(z, \frac{1}{2q}\right)$, $y \in S$ such that $F_x = \{y\}$. A long loop Γ_x at x through y divides Σ_y into two equally long curves. Let γ^1, $\gamma^2 \in \mathfrak{S}_{xy}$ be the two segments from x to y composing Γ_x. We may assume, moreover, that the singular curvatures at x and y verify $\omega_x \leq \omega_y$.

Case 1: $\kappa \in \{-1, 0\}$.

Consider in $\mathbb{M}_0$ a quadrilateral $L = x_0 x_1 z x_2$ defined as in Lemma 11 (see Figure 1) with $l = d(x, y)$ and ε small enough to ensure that $2(\alpha + \beta) \leq \omega_x$.

Cut (S, d) along Γ_x and insert L as follows: identify $x_1 z$ to $x_2 z$, identify $x_0 y_1$ to one image of γ^1 and $x_0 y_2$ to the other image of γ^1, and identify $x_1 y_1$ to one image of γ^2 and $x_2 y_2$ to the other image of γ^2. On the resulting Alexandrov surface $A_0 = (S_0, d_0)$ (see Lemma 4), denote by L_0 the image of L, and by the same letters x, y the images of x and y. By Lemma 11, there exists a point y_0 (the circumcenter of $x_0 x_1 x_2$) interior to L_0 such that $d_0(x, y_0) > d_0(x, z)$, for any $z \in L_0$. It follows that $F_x^{d_0} = \{y_0\}$ and there are three segments from x to y_0 which, moreover, make angles smaller that π at y_0. Since y_0 is a smooth point of (S, d_0), there is no long loop at x.

Using Lemma 2, one can define a metric d_1 on S, such that $(S, d_1) = A_0$ and such that the point x_1 corresponding to x belongs to $B\left(z, \frac{1}{q}\right)$. Hence d_1 does not belongs to $\mathcal{D}_{z,q}$ and a contradiction is obtained.

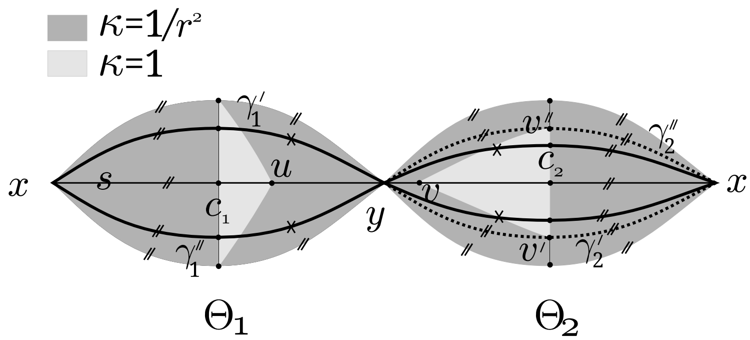

Figure 2: Approximation procedure in the proof of Lemma 15.

Case 2: $\kappa = 1$.

Subcase 2.1: $d(x, y) < \pi/2$.

The construction is similar to that at Case 1, but this time we take L in $\mathbb{M}_1$.

Subcase 2.2: $d(x, y) = \pi/2$.

Apply Subcase 2.1 to $(S, \lambda d)$, where λ is slightly less than one.

Subcase 2.3: $d(x, y) > \pi/2$. This cannot be done similarly to Subcase 2.1, because now the isosceles triangles $x_0y_1y_2$, x_1y_1z and x_2y_2z have base angles larger that π.

It is known that $\text{diam}^d(S) \leq \pi$ for any d such that $(S, d) \in \mathcal{A}(1)$ [3, Theorem 3.6]. By replacing d by λd (with $\lambda < 1$ close to 1), we may assume without loss of generality that $\text{diam}^d(S) < \pi$.

Consider the sphere Λ of radius $r \overset{\text{def}}{=} d(x, F_x)/\pi < 1$; put $\kappa' = \frac{1}{r^2}$. Then $\Lambda \in \mathcal{A}(\kappa', 2) \subset \mathcal{A}(1, 2)$. Consider a slice Θ of Λ determined by great half-circles making an angle of ϕ at their intersection point, such that $2\phi = \omega_y$.

Cut (S, d) along Γ_x and insert two copies of Θ. Denote by A_1 the resulting Alexandrov surfaces (see Lemma 4) and by Θ_1, $\Theta_2 \subset A_1$ the two copies of Θ on A_1. Figure 2 illustrates the approximation procedure for this subcase.

Denote by s_i the median great half-circle of Θ_i ($i = 1, 2$). By considering only subslices of Θ_1, Θ_2, we may assume without loss of generality that, on the one hand, the angle at y between the third segment to x (if it exists) and any direction toward $\Theta_1 \cup \Theta_2$ is distinct from π, and on the other hand, the angle at y between s_1 and s_2 is exactly π. Put $s \overset{\text{def}}{=} s_1 \cup s_2$ and denote by c_1, c_2 the arcs of great circles in Θ_1, Θ_2 orthogonal to s, through the midpoints of s_1 and s_2 respectively. Consider a point $u \in s$ close to c_1 and replace the triangle whose vertices are u and the endpoints of c_1 by a triangle of constant curvature 1 with the same side lengths. After this, the distances on Θ_1 slightly decrease and, by Toponogov's comparison property, u becomes a conical point. Due to the symmetry of Θ_1 with respect to s, on the resulting surface A_2 there are two segments from y to x, say of length l_1. Denote them by γ_1', γ_1'' and denote by γ_2', γ_2'' the segments on Θ_2 of tangent directions at y opposite to those of γ_1', γ_1''. Let v', v'' be the intersection points of c_2 with γ_2', γ_2'', and consider a variable point $v \in s_2$. Replace the triangle $vv'v''$ by a triangle of constant curvature 1. Then, on the new surface $A_2 = A_2(v)$, there are two shortest path on Θ_2 from y to x, of length $l_2(v)$. Note that $l_2(c_2 \cap s_2) = \pi r$, and $l_2(y) = (1+r)\pi/2$. Hence there exists $v_0 \in s$ such that $l_2(v_0) = l_1$.

On $A_2(v_0)$ we have four segments from x to y such that no two of them are composing a loop, and the angle at y between any two consecutive segments is less than π. By virtue of the first variation formula [9, Theorem 3.5], y is a strict local maximum for $\rho_x^{A_2(v_0)}$, and Lemma 15 shows that $F_x = M_x = \{y\}$.

Now, using Lemma 2, one can define a metric d_2 on S such that $(S, d_2) = A_2(v_0)$, and such that $x \in B\left(z, \frac{1}{q}\right)$.

Therefore, the Alexandrov metric d_2 does not belongs to $\mathcal{D}_{z,q}$, and a contradiction is obtained. ■

Lemma 17 *For most $A \in \mathcal{B}(\kappa)$, for most $x \in A$ and any $y \in F_x$, there are at least three segments between x and y.*

Proof. For any surface $A \in \mathcal{B}(\kappa)$, the set

$$L(A) \stackrel{\text{def}}{=} \{x \in A \,|\text{there exists a long loop at } x\},$$

is closed, from the definition of a long loop and the semi-continuity of angles.

We have

$$\mathcal{L} = \{A \in \mathcal{B}(\kappa) \,|\text{int}\,(L(A)) \neq \emptyset\} = \bigcup_{q \in \mathbb{N}^*} \mathcal{L}_q,$$

where

$$\mathcal{L}_q \stackrel{\text{def}}{=} \left\{A \in \mathcal{B}(\kappa) \,\middle|\, \exists x \in A \; \bar{B}\left(x, \frac{1}{q}\right) \subset L(A)\right\}.$$

The sets $\mathcal{L}_q$ are clearly closed, and they have empty interior by Lemma 16, hence $\mathcal{L}$ is of first category. ■

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Joël Rouyer
Institute of Mathematics “Simion Stoilow” of the Romanian Academy,
P.O. Box 1-764, Bucharest 70700, ROMANIA
Joel.Rouyer@ymail.com, Joel.Rouyer@imar.ro

Costin Vîlcu
Institute of Mathematics “Simion Stoilow” of the Romanian Academy,
P.O. Box 1-764, Bucharest 70700, ROMANIA
Costin.Vilcu@imar.ro