ANNEXE 5.4

Universitatea din Bucuresti

La 13e Conférence Internationale sur les Mathématiques Discrètes :

Géométrie Discrète et Corps Convexes

Bucharest (September 4 – 7, 2017).

Volume des exposés

Bounded distance equivalence of cut-and-project sets

Dirk Frettlöh

Technische Fakultät
Universität Bielefeld

The 13th International Conference on Discrete Mathematics:
Discrete Geometry and Convex Bodies

Bucharest 4. September 2017

joint work with Alexey Garber (UTRGV Brownsville, Texas)

- Basics
- Dimension 1
- Higher dimensions
- New result

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each ball of radius R contains at least one point of Λ (*relatively dense*)

(Aka "separated nets". Can also live in $\mathbb{H}^d$, $(\mathbb{Q}_p)^d$...)

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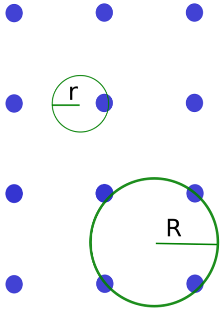
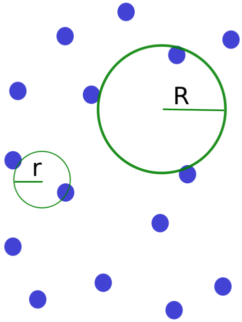


crystallographic



disordered

Relation between Delone sets:

$\Lambda \stackrel{\mathrm{bd}}{\sim} \Lambda'$ *(bounded distance equivalent)*:

There is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$

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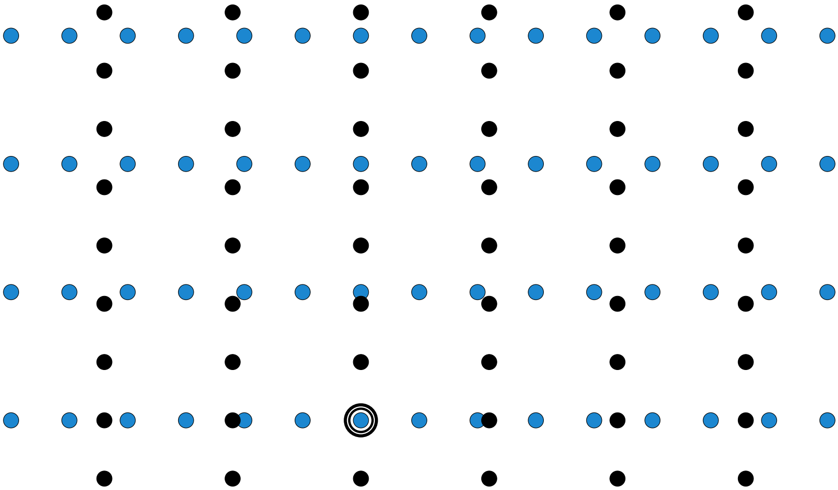
Lemma

Bounded distance equivalence is an equivalence relation.

Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?



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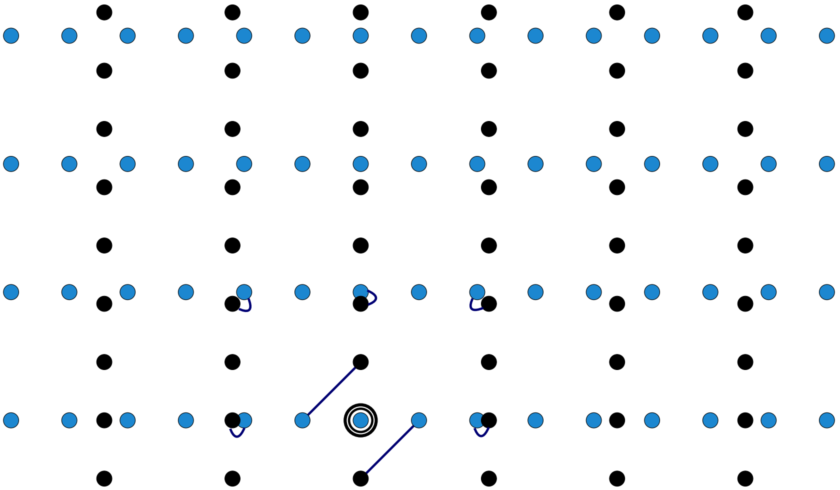

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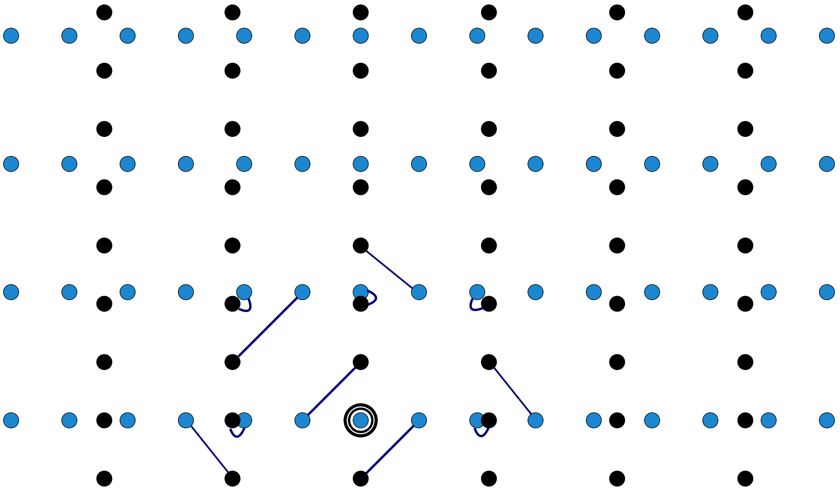

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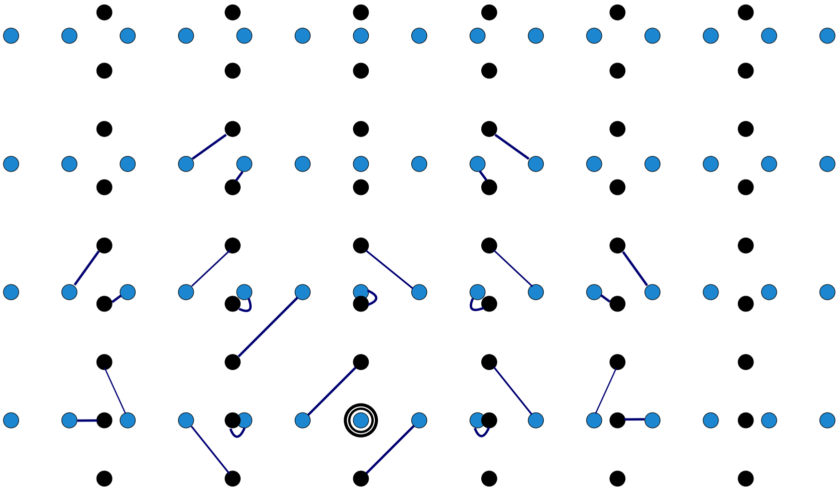

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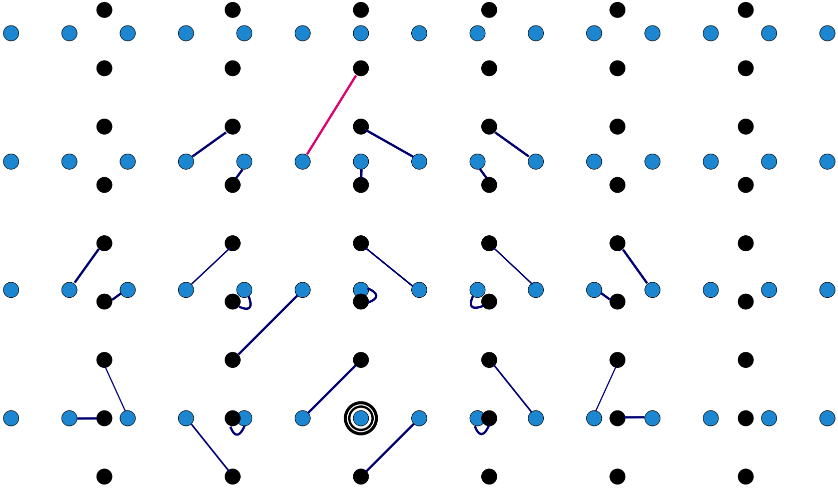

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Question: Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$? Always?

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Examples:

- $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \overset{\mathrm{bd}}{\not\sim} \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$
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$$\text{dens}(\Lambda) := \lim_{r \to \infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]),$$

if it exists.

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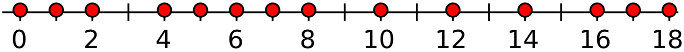

Oscillates between $\frac{2}{3}$ and $\frac{5}{6}$.

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Theorem (Duneau-Oguey 1990)

Let Λ, Λ' be periodic. Then dens(Λ)=dens(Λ') implies $\Lambda \overset{\text{bd}}{\sim} \Lambda'$.
(True even in $\mathbb{R}^d$ for $d \geq 2$)

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Interesting examples are non-periodic.

Theorem (Kesten 1966)

Let $\xi \in [0,1]$, $0 \leq a < b \leq 1$ and define

$$\Lambda := \{k \in \mathbb{Z} \mid a \leq \big(k\xi \bmod 1\big) < b\}.$$

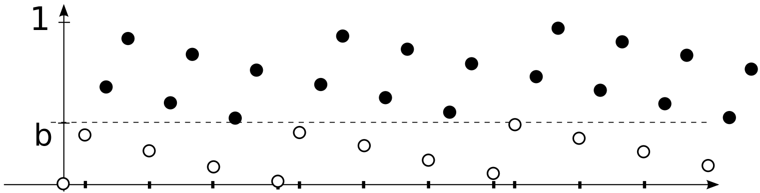
Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - n(b-a)$ is bounded, if and only if $b - a = k\xi \mod 1$ for some $k \in \mathbb{Z}$.

(if-part: Hecke 1921, Ostrowski 1927)

Choose $\xi \in [0,1]$ irrational, let $0 < b \le 1$ and define

$$\Lambda_b := \{k \in \mathbb{Z} \mid 0 \le \big(k\xi \mod 1\big) < b\}.$$

Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - nb$ is bounded, if and only if $b = k\xi \mod 1$ for some $k \in \mathbb{Z}$.



The image shows $\{(k, k\xi \text{ mod } 1) \mid k = 0,1,2,\dots\}$.

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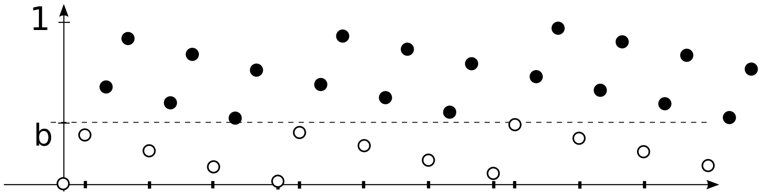

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In particular:

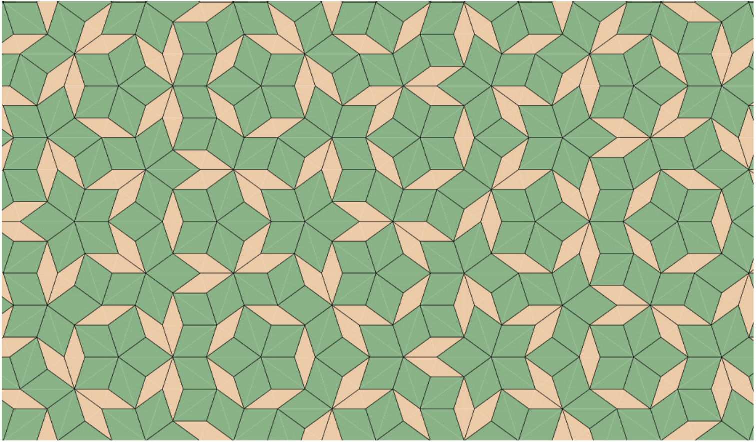
- Deficiency bounded $\Leftrightarrow \Lambda_b \overset{\text{bd}}{\sim} \frac{1}{b}\mathbb{Z}$,
- Any $b \neq k\xi$ mod 1 yields a (nonperiodic!) Delone set Λ_b such that $\Lambda_b \overset{\text{bd}}{\not\sim} c\mathbb{Z}$. Even when $\text{dens}(\Lambda_b)$ exists!

Higher Dimensions

Cool! Alexey Garber and I started to study some problems in this field. E.g.

1. Are the vertices of the Penrose tiling bounded distance equivalent to some lattice?
2. Which cut-and-project sets are bounded distance equivalent to some lattice?
3. Which substitution tilings (resp. their vertex sets) are bounded distance equivalent to some lattice?

Recall: Interesting examples are non-periodic.
Like the Penrose tiling:



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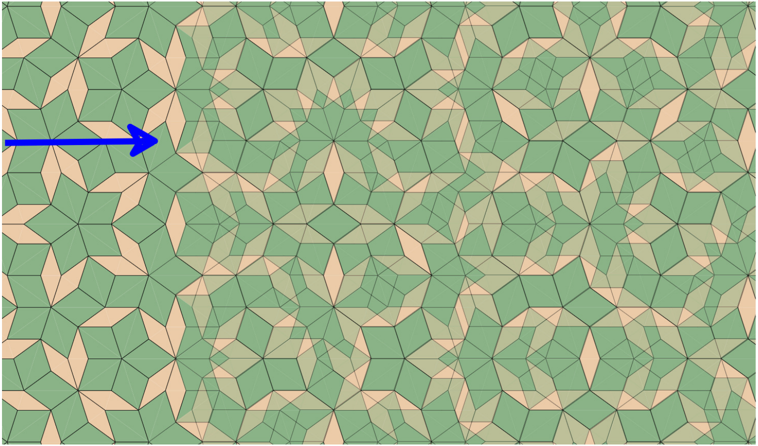

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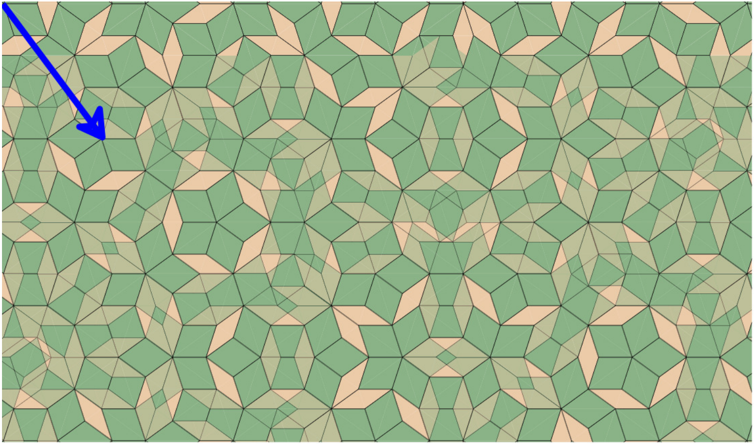

Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \stackrel{\text{bil}}{\sim} \mathbb{Z}^2$.
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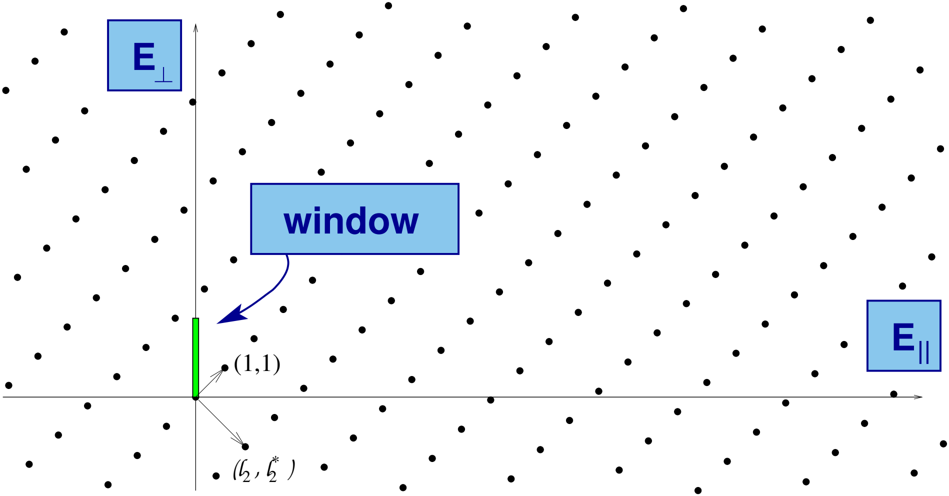
Well. Then let us generalise Kesten's Theorem to higher dimensions.

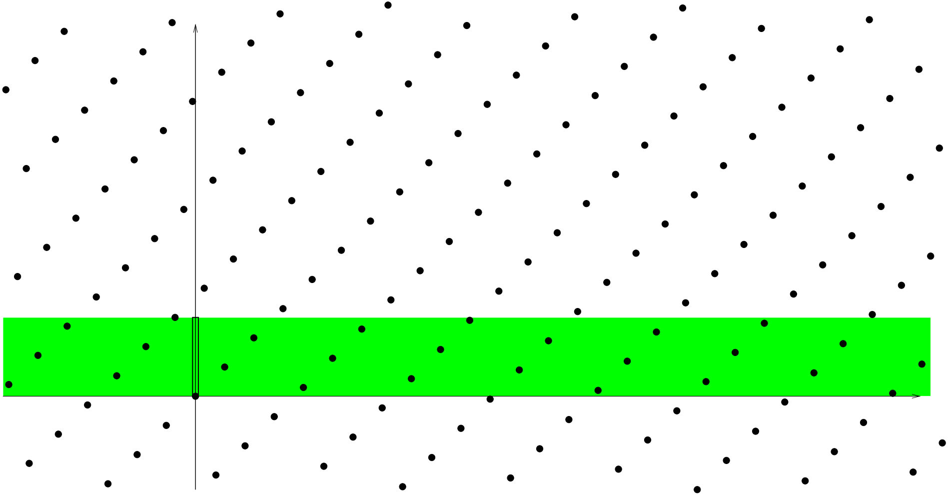
Cut-and-Project Sets

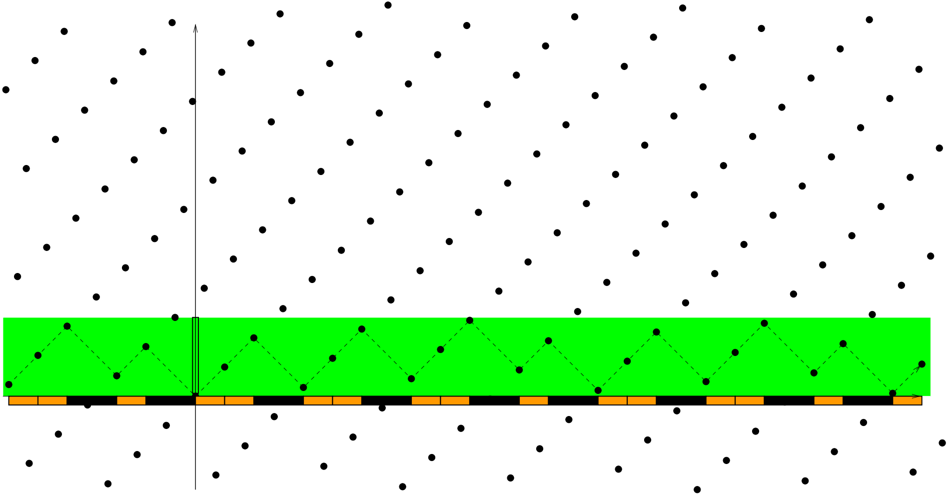
$$\begin{array}{ccccc} E_{||} = \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^e & \xrightarrow{\pi_2} & \mathbb{R}^e = E_\perp \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

- Γ a *lattice* in $\mathbb{R}^d \times \mathbb{R}^e$
- π_1, π_2 *projections*
  - $\pi_1|_\Gamma$ injective
  - $\pi_2(\Gamma)$ dense
- W *compact* ("window", somehow nice, e.g. ∂W has zero measure)

Then $\Lambda = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set* (CPS).


$E_\perp$
window
$E_{||}$
(1,1)
(ℓ_2, ℓ_2^*)





The last one uses $d = e = 1$ ($E_{||} = \mathbb{R}^1, E_\perp = \mathbb{R}^1$).

An example with $d = 1$, $e = 2$:

$$\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML$$

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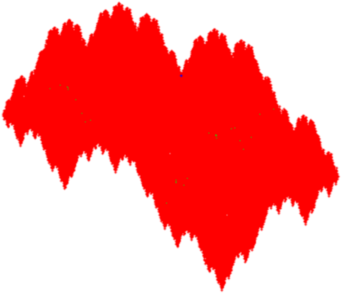

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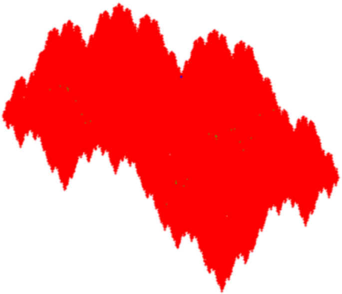
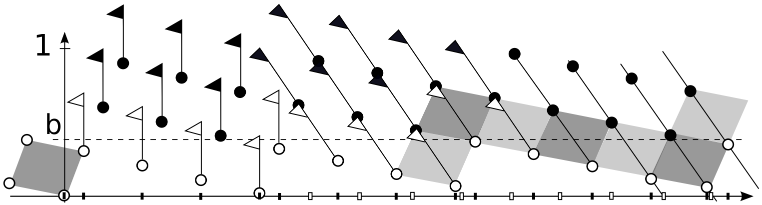
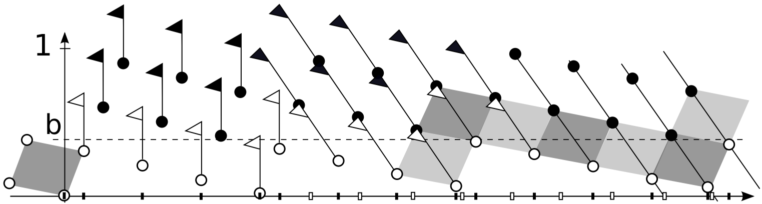


Now let us generalize Kesten to $\mathbb{R}^d$ (at least "if"-part)



(looks almost like a cut-and-project set!)



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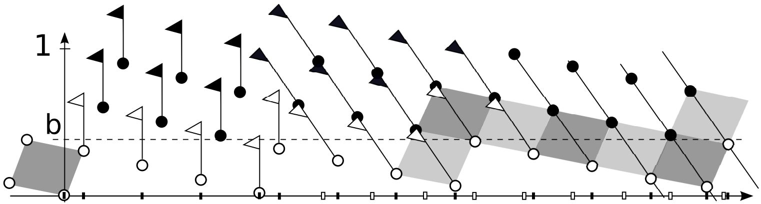
One can state the argument in purely algebraic terms:

- $X = \mathbb{R}^{d+1}$
- $X = V_p + V_i$ (here: horizontal + vertical), $W \subset V_i$ compact set (here $W = [0, b]$),
- π_p projection to V_p (here: $\downarrow$),
- π_i projection to V_i (here: $\leftarrow$),
- Γ discrete cocompact subgroup (here: black and white points)



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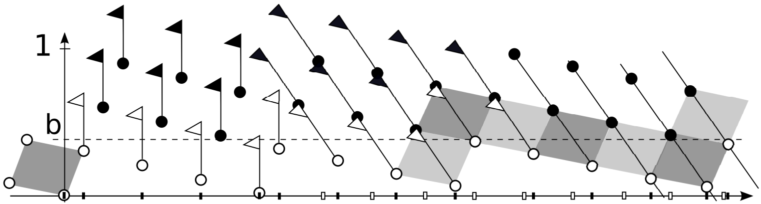

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- $Y = \pi_i^{-1}(W) \cap \Gamma$ (here: white points),
- $\Lambda = \pi_p(Y)$
- Z subgroup of X with $V_p + Z = X$, $Z/(Z \cap \Gamma)$ compact (here "lattice direction" for projection)
- π_Z corresponding projection etc...

...then $\pi_p(Y) \stackrel{\text{bd}}{\sim} \pi_Z(Y)$.

Other colleagues had the same idea: Haynes-Koivusalo 2014, Haynes-Kelly-Koivusalo 2017.

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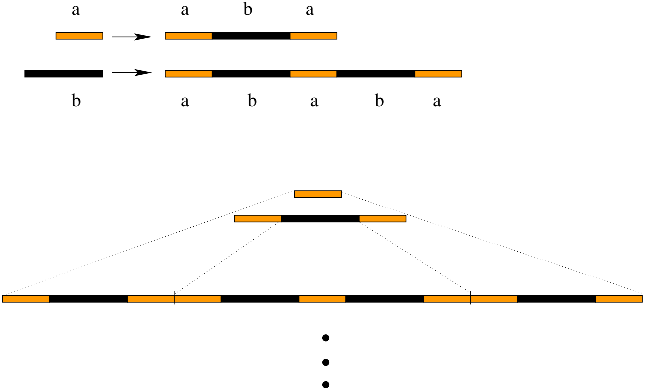
Last October I've learned from Alan Haynes that this was done already in

C. Godrèche and C. Oguey:
Construction of average lattices for quasiperiodic structures by the section method, *J. Phys. France* 51 (1990) 21-37

So much on Question 2.

Only briefly regarding Question 3:

A one-dimensional *tile substitution*, producing tilings of the line by intervals. The endpoints form some Delone set.



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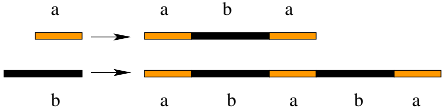

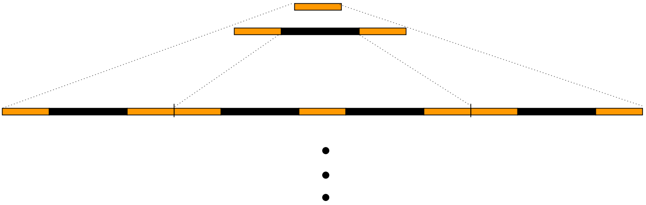

- $M_\sigma = \left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right)$
- Inflation factor $2 + \sqrt{3}$
- $\text{length}(a) = 1$, $\text{length}(b) = \sqrt{3}$

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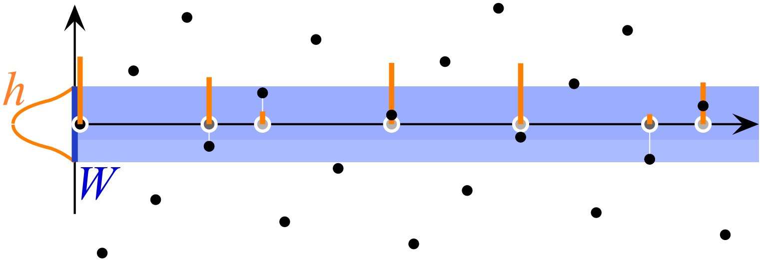
We did not give up....

New result

Take some CPS Λ and give each point a weight. One convenient way to write it: *Dirac comb*

$$\delta_{w,\Lambda} = \sum_{x \in \Lambda} w(x)\delta_x \qquad (w(x) \in \mathbb{R}, \delta_x \text{ the Dirac measure in } x)$$

If $w(x) = h(x^*)$ for $h : W \to \mathbb{R}$ continuous, then $\delta_{w,\Lambda}$ is called a *weighted CPS*.



Theorem (F-Garber 2017 preprint)

Let $\delta_{w,\Lambda}$ *be a weighted CPS with* $e = d = 1$*. Let* $W = [a, b]$*,* $w(x) = h(x^*)$ *and* $h(a) = h(b) = 0$*. If* h *is*

1. *piecewise linear, or*
2. *twice differentiable,*

then $\delta_{w,\Lambda}$ *is bounded distance equivalent to* $c\mu$ *for some* $c > 0$*, where* μ *denotes the one-dimensional Lebesgue measure.*

Finally, our first new result!

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1. *piecewise linear, or*
2. *twice differentiable,*

then $\delta_{w,\Lambda}$ is bounded distance equivalent to $c\mu$ for some $c > 0$, where μ denotes the one-dimensional Lebesgue measure.

Finally, our first new result! (At least we hope so...)

More here:

D.F., Alexey Garber:
Bounded distance and bilipschitz equivalence of Delone sets, preprint,
`www.math.uni-bielefeld.de/~frettloe/papers/bilip-draft.pdf`

and references therein.

*   *
  *

Thank you!

Total mixed curvature of open curves in E^3

Kazuyuki Enomoto
(Tokyo Univ. of Science, Japan)

Jin-ichi Itoh
(Kumamoto Univ. , Japan)

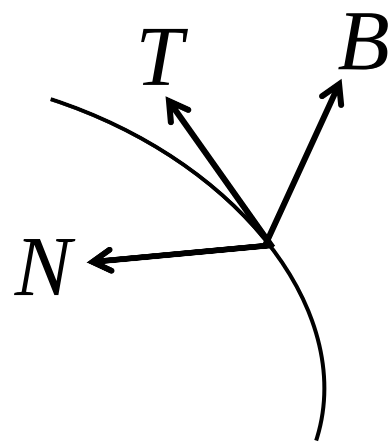
$\Sigma : x(s)$ smooth curve in E^3

s : arclength $(0 \leq s \leq L)$

T : unit tangent vector

N : principal normal vector

B : binormal vector



$$\frac{dT}{ds} = \kappa N$$

$$\frac{dN}{ds} = -\kappa T + \tau B$$

$$\frac{dB}{ds} = -\tau N$$

κ : curvature

τ : torsion

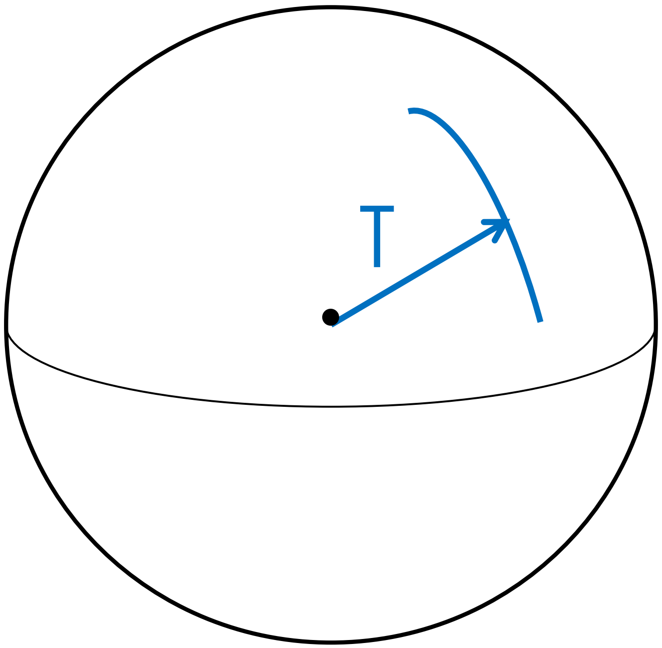
total absolute curvature $$TAC(\Sigma) = \int_0^L \kappa(s)\, ds$$

total absolute torsion $$TAT(\Sigma) = \int_0^L |\tau(s)|\, ds$$

total mixed curvature $$TMC(\Sigma) = \int_0^L \sqrt{\kappa^2 + \tau^2}\, ds$$

$$TAC(\Sigma) = \int_0^L \kappa(s)\,ds = \int_0^L \left|\frac{dT}{ds}\right| ds$$

$$= \text{length of } T(s) \text{ as a curve in } S^2$$



$$TAT(\Sigma) = \int_0^L |\tau(s)|\, ds = \int_0^L \left|\frac{dB}{ds}\right| ds$$

$$= \text{length of } B(s) \text{ as a curve in } S^2$$

$$TMC(\Sigma) = \int_0^L \sqrt{\kappa^2 + \tau^2}\, ds = \int_0^L \left|\frac{dN}{ds}\right| ds$$

$$= \text{length of } N(s) \text{ as a curve in } S^2$$

Fenchel (1929) $\int_{\Sigma} \kappa \geq 2\pi$ if Σ is closed.

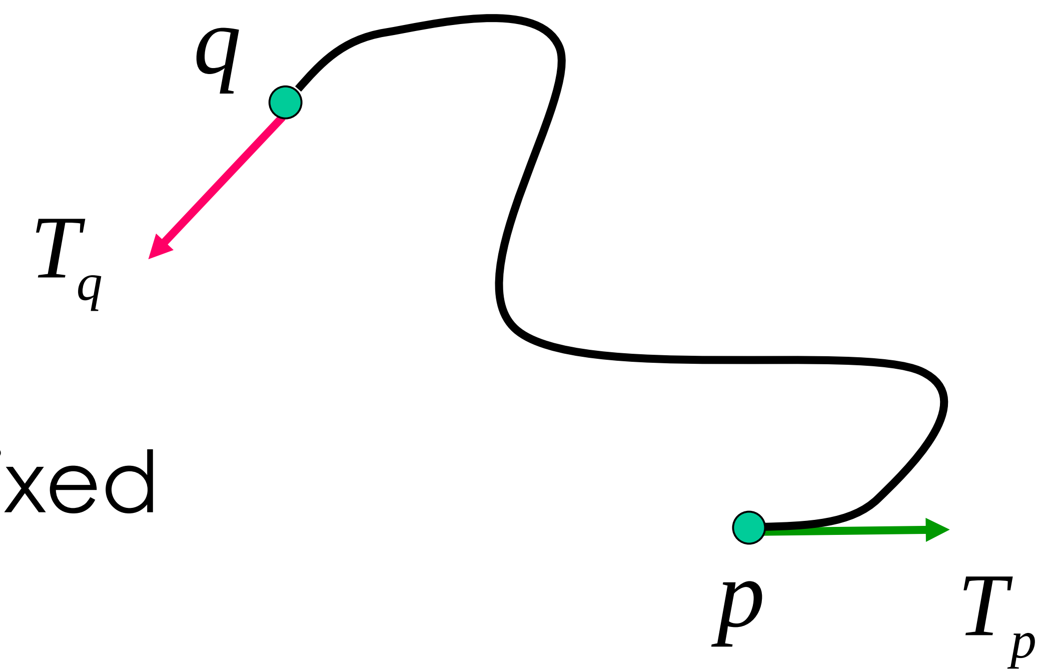
= ⟺ closed convex plane curve

Enomoto-Itoh-Sinclair (2008) determines

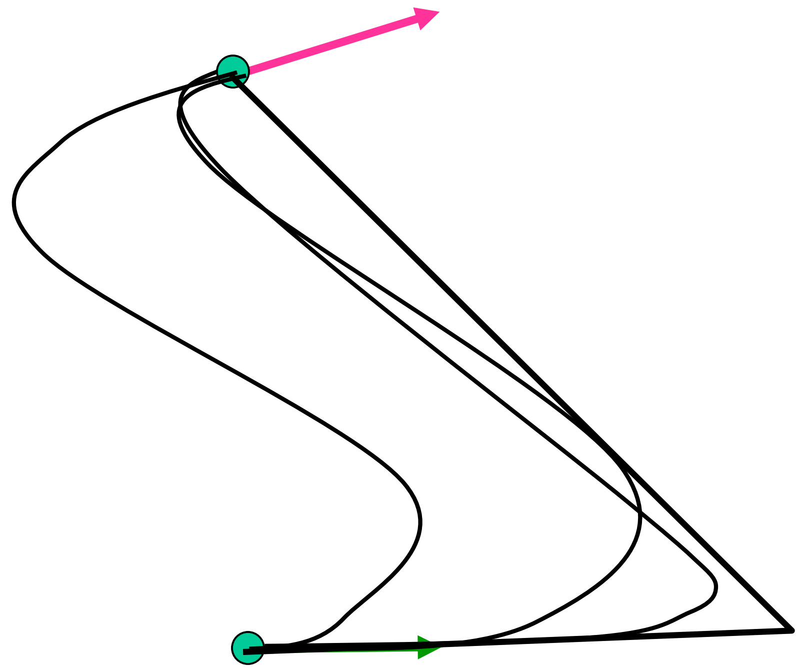
$$\inf\{\int_\Sigma \kappa : \Sigma \in C(p,q,T_p,T_q,L)\}$$

where

$C(p,q,T_p,T_q,L)$ is the set of all curves such as



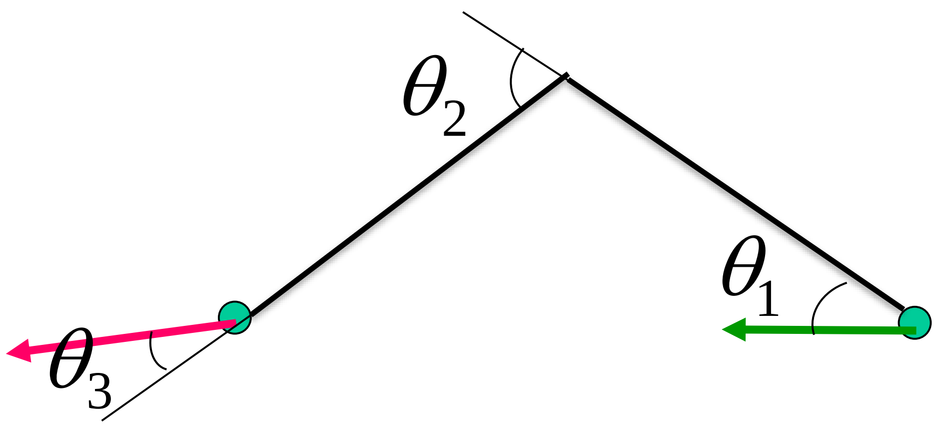
p,q
T_p,T_q } fixed
length L



Enomoto-Itoh-Sinclair (2008)

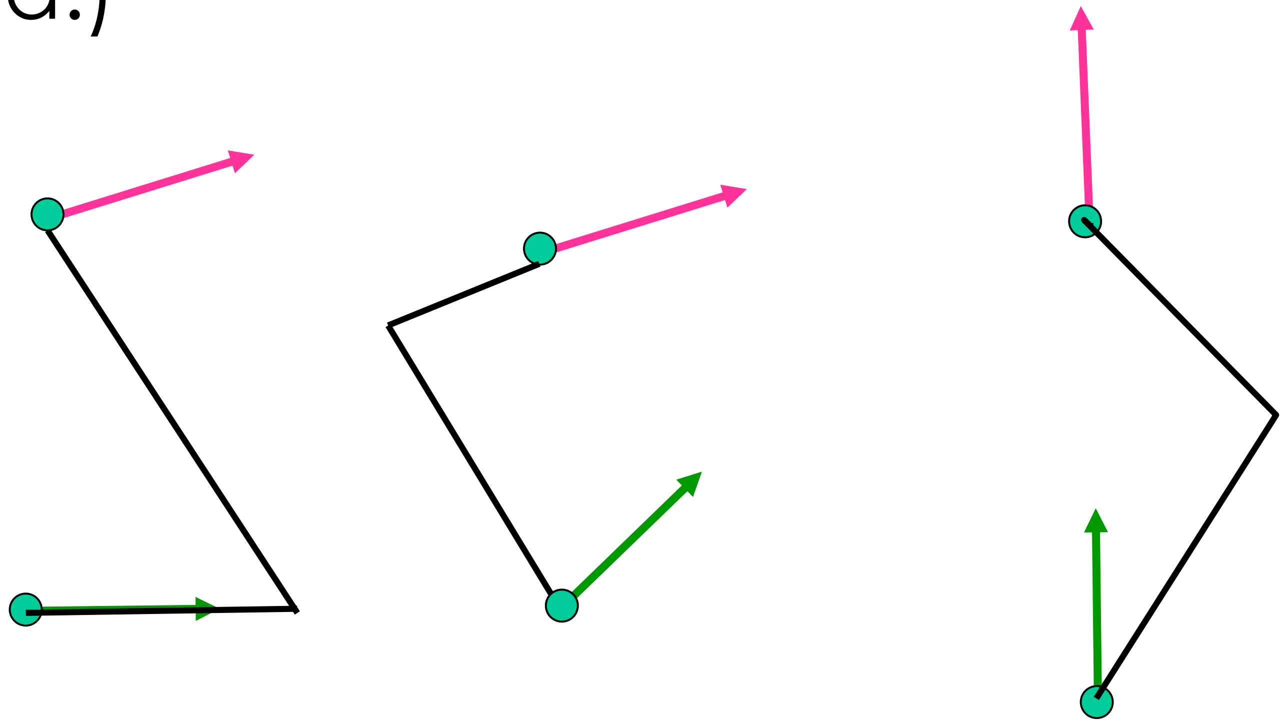
A piecewise linear curve with 2 edges

gives $\inf\{\int_\Sigma \kappa : \Sigma \in C(p,q,T_p,T_q,L)\}$ (as the limit) .



$$\inf\{\int_\Sigma \kappa : \Sigma \in C(p,q,T_p,T_q,L)\} = \theta_1 + \theta_2 + \theta_3$$

In most cases, it is the only curve that gives the infimum.
(most= except for the case when there exists a plane convex arc tangent to T_p at p, to T_q at q, including the case when the curve is closed.)

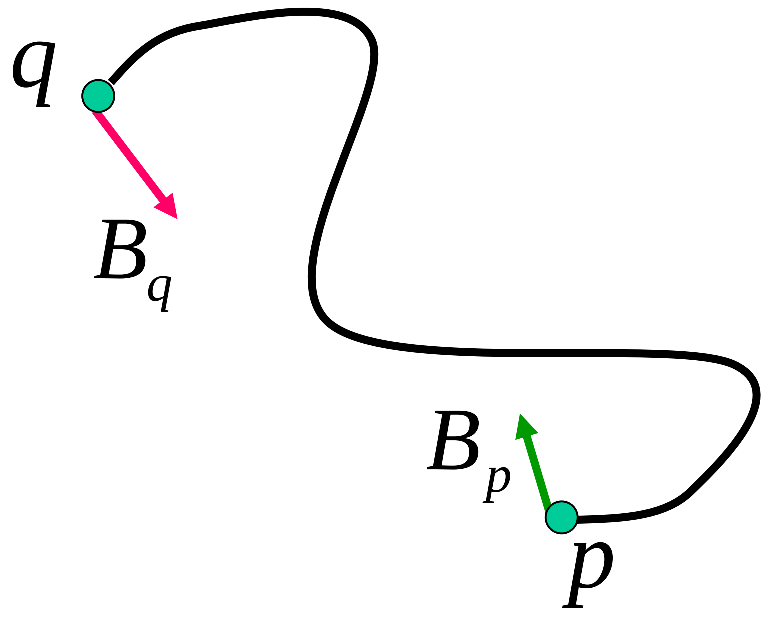


Enomoto-Itoh (2013) determines

$$\inf\{\int_\Sigma |\tau| : \Sigma \in C(p,q,B_p,B_q,L)\}$$

where

$C(p,q,B_p,B_q,L)$ is the set of all curves such as

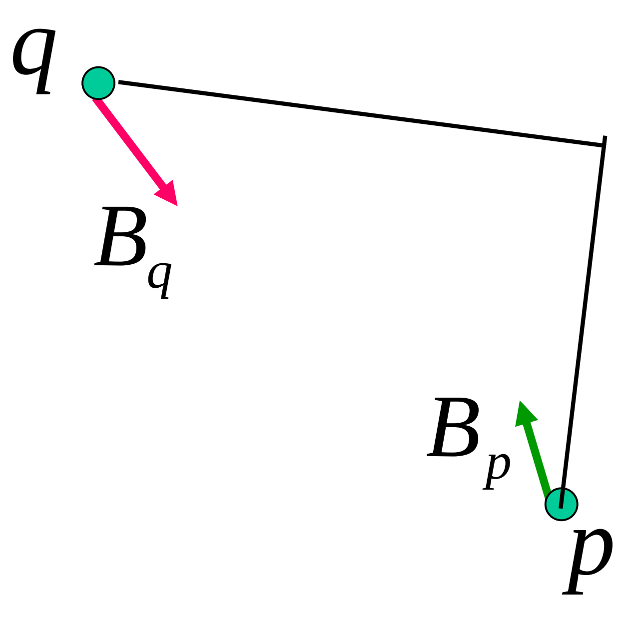


p, q
B_p, B_q
length L
} fixed

Enomoto-Itoh (2013)

$$\inf\{\int_\Sigma |\tau| : \Sigma \in C(p, q, B_p, B_q, L)\}$$

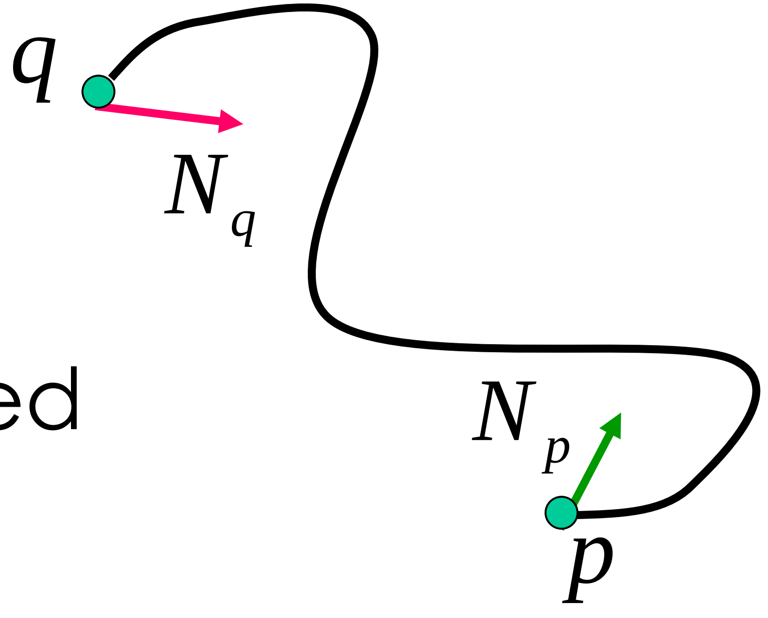
is attained by a curve (as the limit) shown below.



Enomoto-Itoh (2017)

$$\int_{\Sigma} \sqrt{\kappa^2 + \tau^2} \geq \angle(N_p, N_q)$$

for $\Sigma \in C(p, q, N_p, N_q)$



p, q
N_p, N_q } fixed

Enomoto-Itoh (2017)

If $\angle(\overrightarrow{pq}, N_p) < \frac{\pi}{2}$ and $\angle(\overrightarrow{pq}, N_q) > \frac{\pi}{2}$,

there always exists $\Sigma \in C(p, q, N_p, N_q)$

such that

$$\int_{\Sigma} \sqrt{\kappa^2 + \tau^2} = \angle(N_p, N_q).$$

Such Σ is a subarc of a generalized helix.

Enomoto-Itoh (2017)

$$\inf\{\int_{\Sigma}\sqrt{\kappa^2+\tau^2} : \Sigma \in C(p,q,N_p,N_q,L)\}$$

is attained by a curve which is a union of plane curves and generalized helices.

For more details, please see

Illinois J. Math. 52 (2008).

for “total absolute curvature”

Illinois J. Math. 57 (2013)

for “total absolute torsion”

Geom. Dedicata (to appear)

for “total mixed curvature”

Homomorphisms of abelian p-groups produce p-automatic recurrent sequences

Mihai Prunescu

Bucharest, September 4 - 7, 2017

Definition

$(A, f, 1)$: A finite, $f : A^3 \to A$, $1 \in A$

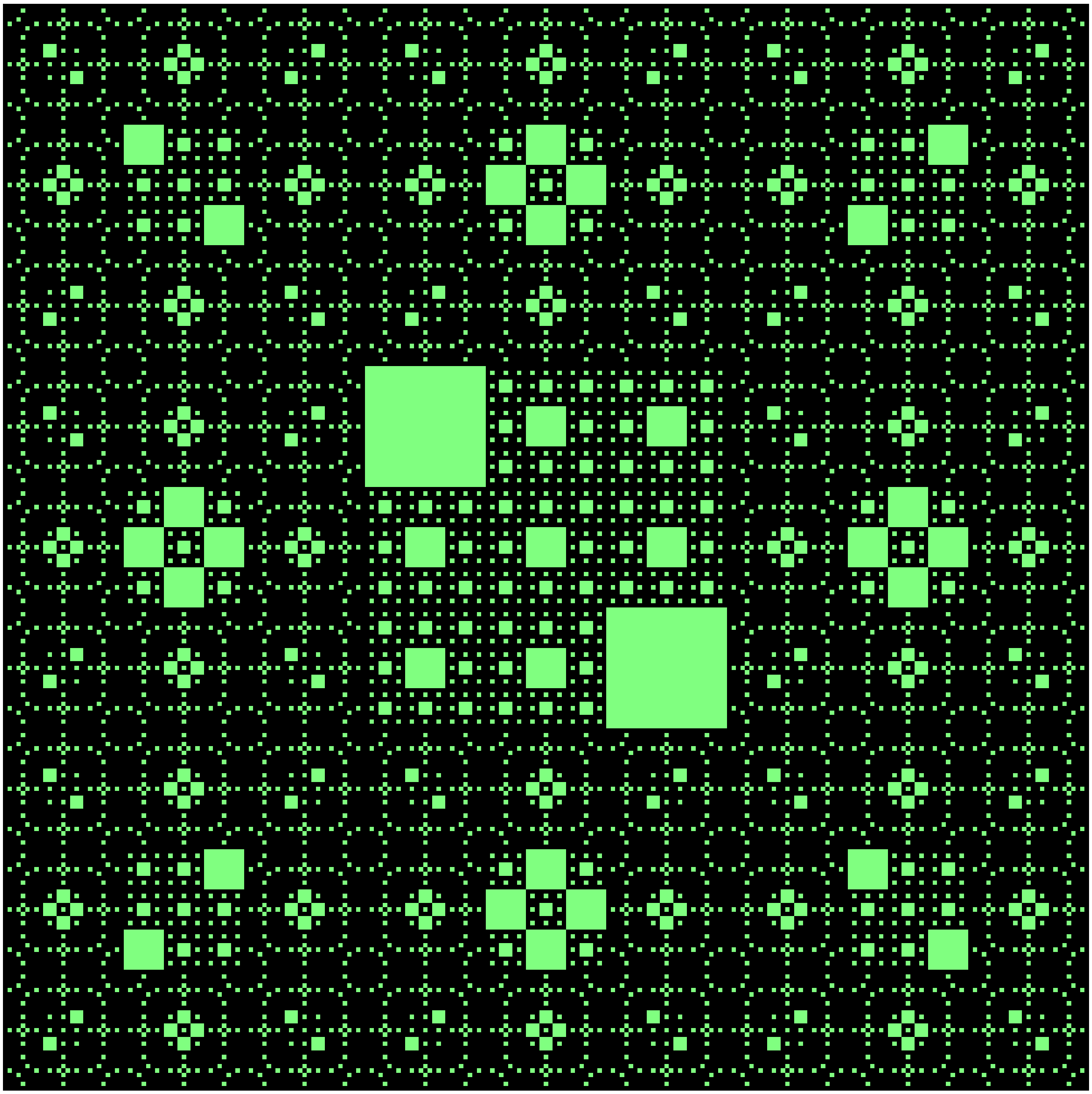
Recurrent double sequence $(a(i,j))$:

- $\forall i \ \ \forall j \ \ a(i,0) = a(0,j) = 1$

- $i > 0 \ \wedge \ j > 0$:

$$a(i,j) = f(a(i-1,j), a(i-1,j-1), a(i,j-1))$$

Passoja-Lakhtakia Carpet modulo 9



$(\mathbb{Z}/9\mathbb{Z}, x + y + z, 1)$

Christol-Salon Theorem

Theorem 1 $a : \mathbb{N}^n \to A$, A *finite,* p *prime. Then the following are equivalent:*

1. a *is* p*-automatic.*

2. $\exists\ (B, \mathcal{E}, b_1, \Theta)$, $\Theta(b_i) \in B^{p^n}$, $\Theta(b_1)(\vec{0}) = b_1$, $b = \lim_{i\to\infty} \Theta^i(b_1)$, $g : B \to A$, $a = g(b)$.

3. $\forall$ *embedding* $\iota : A \to K$ *in a sufficiently large finite field* K *of characteristic* p*,* $S = \sum \iota(a(\vec{x}))\vec{X}^{\vec{x}}$ *algebraic /* $K(\vec{X})$*.*

4. $\exists$ *embedding* $\iota : A \to K$ *in a sufficiently large finite field* K *of characteristic* p*;* $S = \sum \iota(a(\vec{x}))\vec{X}^{\vec{x}}$ *algebraic /* $K(\vec{X})$*.*

Denef-Lipshitz Theorem

Theorem 2 *p prime, $k > 0$, $a : \mathbb{N}^n \to \mathbb{Z}_p$, $\sum a(\vec{x})\vec{X}^{\vec{x}}$ algebraic / $\mathbb{Z}_p(\vec{X})$.
Then $(a(\vec{x}) \mod p^k)$ p-automatic.*

$\forall\ b : \mathbb{N}^n \to \mathbb{Z}/p^k\mathbb{Z}$ p-*automatic* $\exists\ a : \mathbb{N}^n \to \mathbb{Z}_p$
$\forall\ \vec{x} \in \mathbb{N}^n$, $a(\vec{x}) \equiv b(\vec{x}) \mod p^k$ *and*
$\sum a(\vec{x})\vec{X}^{\vec{x}}$ *algebraic* / $\mathbb{Z}_p(\vec{X})$.

Main Result

Theorem 3 *p prime, $m \geq 1$,*

$H = \mathbb{Z}/p^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{d_s}\mathbb{Z}$,

$f : H^m \to H$ a shifted homomorphism.

$(H, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ n-dimensional recurrence,

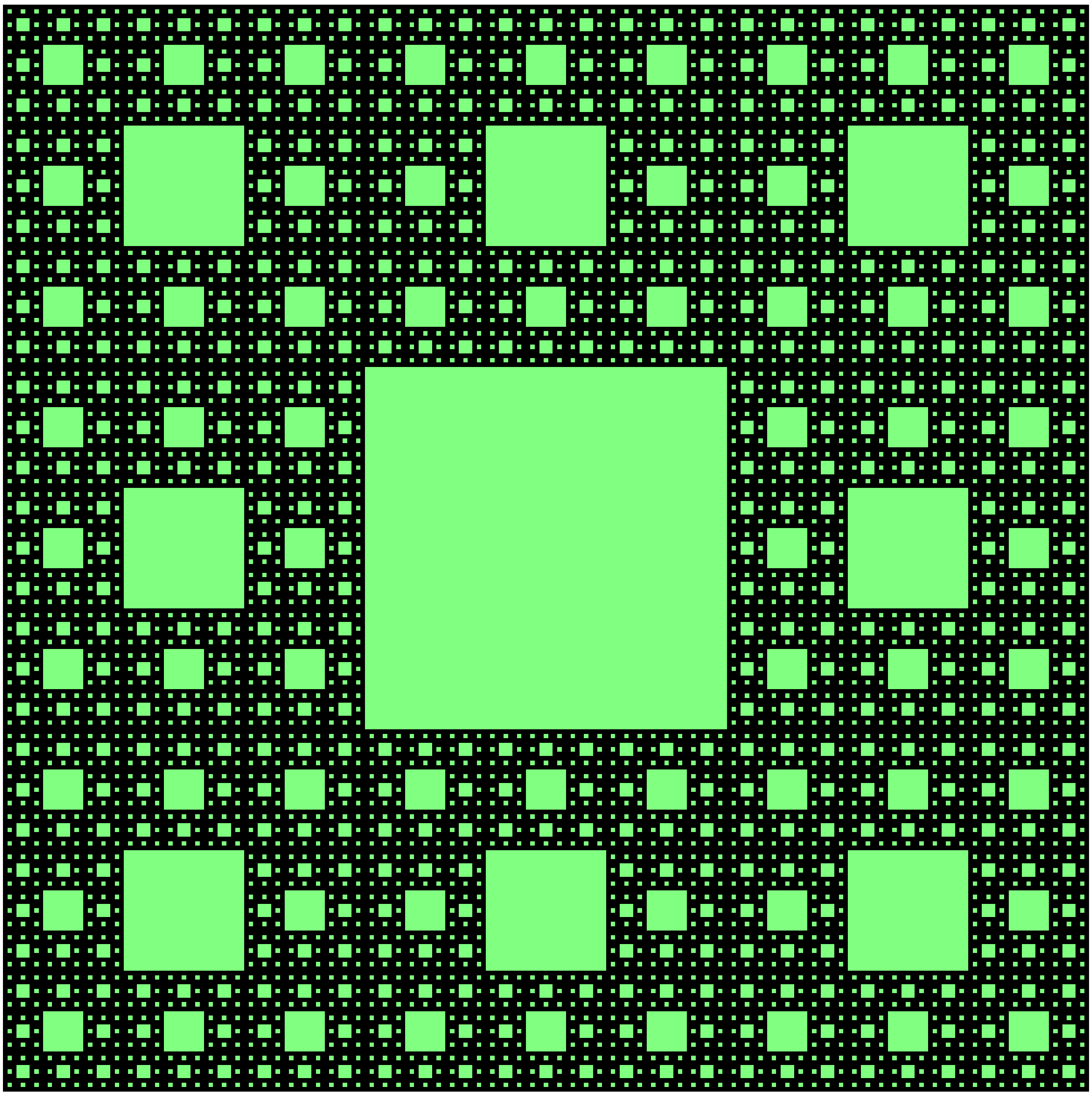
$c : C_P \to H$ satisfies $\forall\ i = 1, \ldots, n$, $\forall\ a \in \mathbb{N}$, if $(x_i = a) \cap \mathbb{N}^n \subset C_P$, $c\,|\,(x_i = a) \cap \mathbb{N}^n$ is p-automatic.

Then $(H, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ produces a p-automatic n-dimensional sequence.

Corollary 4 *The sequence can be defined by a substitution of type $p^a \to p^b$, $a < b$.*

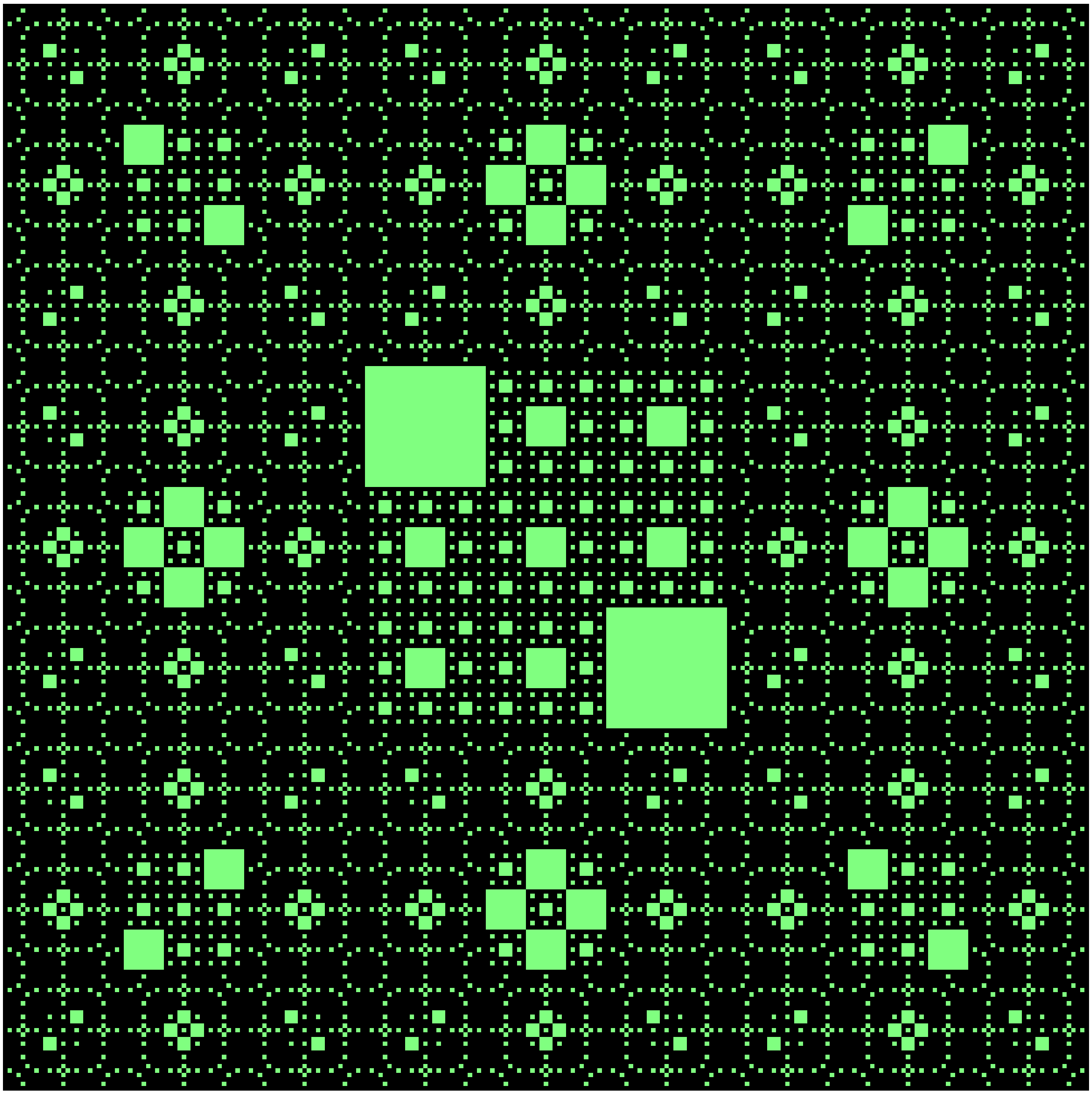
There is an algorithm able to find it.

Sierpinski's Carpet



$(\mathbb{Z}/3\mathbb{Z}, x + y + z, 1)$

Passoja-Lakhtakia Carpet modulo 9



$$(\mathbb{Z}/9\mathbb{Z}, x + y + z, 1)$$

Sierpinski's Carpet

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$F_n = F \otimes F \cdots \otimes F$$

$$s(F) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$S_n = s(F_n) = s(F) \otimes s(F) \cdots \otimes s(F)$$

$$L = \{(x, y) \in \mathbb{N}^2 \,|\, s(x, y) = 0\}$$

$$L = A^* \binom{1}{1} A^*$$

$$A = \{\binom{0}{0}, \binom{0}{1}, \binom{0}{2}, \binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{2}{0}, \binom{2}{1}, \binom{2}{2}\}$$

Passoja-Lakhtakia Carpet modulo 9

In this case the algorithm finds out a square substitution of type $3 \to 9$ with 57 rules.

$s : \mathbb{Z}/9\mathbb{Z} \to \{0,1\}$ given by $s(0) = 0$ and $\forall x \neq 0$, $s(x) = 1$.

$(s(a(m,n)))$ is also 3-automatic.

There exist 3×3 matrices with $A \neq B$ and $\Sigma(A) \neq \Sigma(B)$ such that $s(A) = s(B)$ but still $s(\Sigma(A)) \neq s(\Sigma(B))$.

Happily $(a(m,n))$ is also given by another system of substitutions of type $9 \to 27$ which has also 57 rules.

By application of s, this system of substitutions collapses successfully on a consistent system of substitutions of type $9 \to 27$ with 8 rules.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\Sigma(A) = \begin{pmatrix} A & B & C \\ B & D & B \\ C & B & A \end{pmatrix} \quad \Sigma(B) = \begin{pmatrix} A & B & C \\ B & E & B \\ C & B & A \end{pmatrix}$$

$$\Sigma(C) = \begin{pmatrix} A & B & C \\ B & F & B \\ C & B & A \end{pmatrix}$$

$$\Sigma(D) = \begin{pmatrix} G & H & H \\ H & H & H \\ H & H & G \end{pmatrix} \quad \Sigma(E) = \begin{pmatrix} H & G & H \\ G & H & G \\ H & G & H \end{pmatrix}$$

$$\Sigma(F) = \begin{pmatrix} H & H & G \\ H & H & H \\ G & H & H \end{pmatrix}$$

$$\Sigma(G) = \begin{pmatrix} G & G & G \\ G & G & G \\ G & G & G \end{pmatrix} \quad \Sigma(H) = \begin{pmatrix} H & H & H \\ H & G & H \\ H & H & H \end{pmatrix}$$

About endpoints of convex surfaces. The 13^{th} International Conference on Discrete Mathematics : Discrete Geometrie and Convex Bodies. Bucharest

Alain Rivière[1]

[1]Université de Picardie Jules Verne
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
CNRS-UMR 7352

September, 2017.

notations, definitions

- d is an integer ≥ 2 (often think $d = 2!$).

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- The boundary $\Sigma = \partial C$ of $C \in \mathcal{B}$ is a *convex surface*, it is endowed with its inner geodesic metric.
- $\mathcal{E}_C$ or $\mathcal{E}_\Sigma$ denotes the set of all *endpoints* of Σ, that is the points which are not in the interior of some shorter path in Σ.

Some Examples

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- If $\mathbb{E}^{d+1} = E \oplus F$ with $\dim F = 2$, $r > 0$ and $C = \mathrm{conv}(S_E \cup rS_F)$, then $S_E \subset \mathcal{E}_\Sigma$.
  Here $\dim_{\mathrm{H}} \mathcal{E}_\Sigma = d-2$ and $\mathcal{H}^{d-2}(\mathcal{E}_\Sigma) > 0$.

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  Here $\dim_{\mathrm{H}} \mathcal{E}_\Sigma = d-2$ and $\mathcal{H}^{d-2}(\mathcal{E}_\Sigma) > 0$.
- $a \in \Sigma$ is said *regular* when $\mathrm{T}_a\Sigma$ is isometric to $\mathbb{E}^d$.
  The set $\mathcal{R}_\Sigma$ of regular points is always strongly convex in Σ.
  Petrunyn 1998 (Milka 1983 enough for us).
  This is used to check that $S_E \subset \mathcal{E}_\Sigma$ in the above example.

Typical case

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- Remarks
  - $d-2 = d/3 \Leftrightarrow d=3$ and $d-2 > d/3 \Leftrightarrow d \geq 4$.
  - Proof of $\dim_{\mathrm{H}} \mathcal{E}_C \geq d-2$ uses exple $\mathrm{conv}(S_E \cup rSF)$.
  - Proof of $\dim_{\mathrm{H}} \mathcal{E}_C \geq d/3$ uses classical conical points.

Some questions

- Does it exist a convex body $C \in \mathcal{B}$ satisfying $\dim_{\mathrm{H}} \mathcal{E}_C > \max(d-2, d/3)$
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- Are there convex bodies $C_0, C \in \mathcal{B}$ and $a \in \partial C \cap \partial C_0$ such that $C_0 \subset C$, $a \in \mathcal{E}_C$ but $a \notin \mathcal{E}_{C_0}$. (yes known ?)

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- Can we also ask that $d = 2$ and C to be of revolution around an axe containing a ? (no known ?)

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $C \in \mathcal{B}$ and $\varepsilon > 0$, let $\mathcal{M}_{C,\varepsilon}$ be the set of all the points of ∂C which are the middle of some shorter path of ∂C with length 2ε, and $\mathcal{E}_{C,\varepsilon} = \partial C \setminus \mathcal{M}_{C,\varepsilon}$.
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- For $\|x\| = 1$ we set $\{\Phi_C(x)\} = \mathbb{R}_+ x \cap \partial C$.
- We look for a compact set K of the unit sphere, with $\mathrm{dim_H}\, K = d-2$, and such that the following G_δ is dense in $\mathcal{B}_0$:

$$G_K = \{C \in \mathcal{B}_0 \mid \Phi_C(K) \subset \mathcal{E}_C\} = \bigcap_{\varepsilon>0} G_{K,\varepsilon}$$

  where $G_{K,\varepsilon} = \{C \in \mathcal{B}_0 \mid \Phi_C(K) \subset \mathcal{E}_{C,\varepsilon}\}$.

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $\mathbb{E}^{d+1} = E \oplus F$ like in the exemple, we can find $K \subset S_E$ with $\mathrm{dim_H}\, K = d-2$, and with K *strongly radially porous*. This means that for each $\varepsilon > 0$ and $n \geq 1$, K has a finite covering by pairwise disjointed balls $B_i(c_i, r_i)$ and such that $K \cap B(a_i, r_i) \subset B(a_i, r_i/n)$

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- Using our exemple and the strong porosity, we can check the wanted density.
  In this point, Riv2015 is rather clumsy !

About the proof of density of G_K

- First we have a dense set of smooth $C \in \mathcal{B}_0$ such that for some $0 < r < R$ and all $a \in \partial C$, there are closed balls containing a in their boundary spheres, and such that $B(c, r) \subset C \subset B(c', R)$.

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- Given such a C and $\varepsilon > 0$, we find a finite family of half spaces H_i such that if $C' = C \cap \bigcap H_i$, then $d_H(C, C') < \varepsilon$ and $C' \cap \Phi_C(K) = \emptyset$, because of the strong porousity of K.

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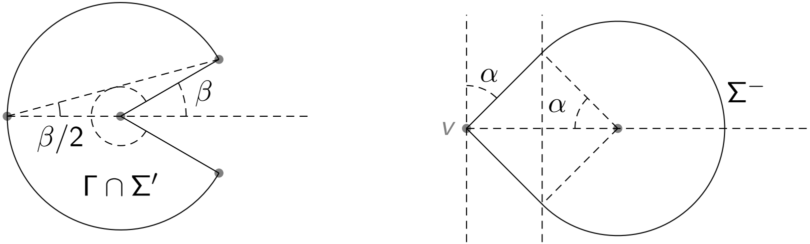
- Then we get $C'' \in G_K$ with $d_H(C, C'') < \varepsilon$ by substituting to each H_i some C_i congruent to our exemple associated with $\mathbb{E}^{d+1} = E \oplus F_i$.

Why $d/3$?



- Our modified sphere is the boundary Σ' of the largest convex set among all those whose boundaries contain the truncated sphere Σ^- (of radius 1).

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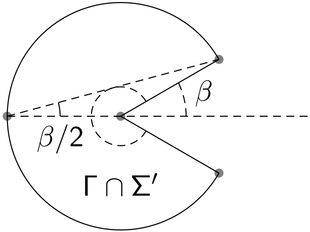

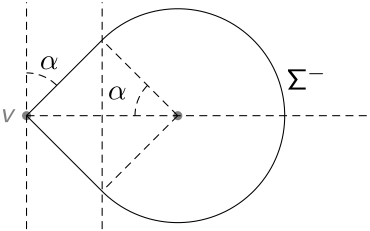

- Our modified sphere is the boundary Σ' of the largest convex set among all those whose boundaries contain the truncated sphere Σ^- (of radius 1).
- For a small α, the smallest possible distance r_α from the vertex v to a shorter path γ, in Σ' and between points of Σ^- satisfies $r_\alpha \sim \frac{\pi}{4}\alpha^3$ and $\varepsilon_a = 4\tan\alpha \sim 4\alpha$

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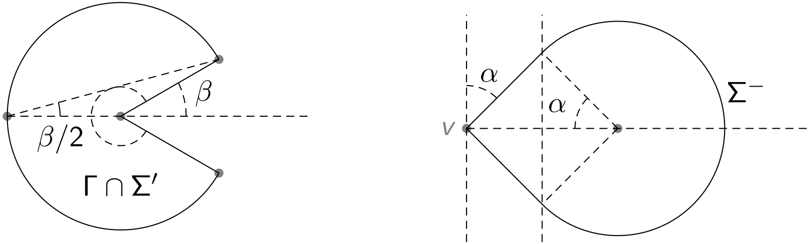

- Because of this we choose a function $h > 0$ with $h(t) = \mathrm{o}(t^3)$ (like $t^3/|\ln t|$ near zero), and then K *h-radially porous*, that is: for all $x \in K$, there is a sequence of balls such that for each n we have $\mathrm{B}_K(x, r_n) \subset \mathrm{B}_K(x, h(r_n))$, and with the radius sequence (r_n) decreasing of null limit.

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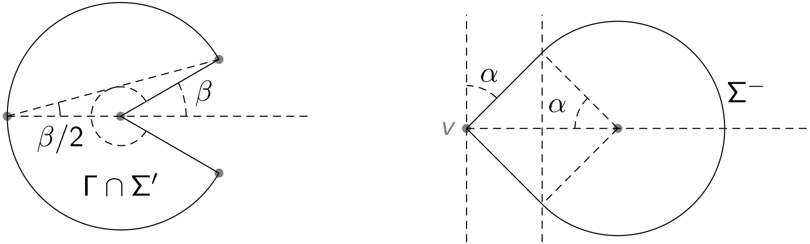

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- We can also ask $\dim_{\mathrm{H}} K = d/3$.

Generic Properties of Lengths Spaces

Work in progress

Joël Rouyer

September 2017

Imbedded in $\mathbb{R}^n$

Convex (hyper)surfaces

Abstract

- 1959: V. Klee, A generic C.S. is C^1 and strictly convex.
- 1977: P. Gruber, ... and not C^2.
- 1979: R. Schneider, 1980,1988: T. Zamfirescu, 2012: K. Adiprasito and T. Zamfirescu, 2015: Schneider 2015. Study of directional curvature. (extrinsic property)
- 1982: T. Zamfirescu, A generic point is an *endpoint*.
- 1995: T. Zamfirescu, A generic point has a single farthest point, to whom it is joined by exactly 3 segments.
- 1988,91: P. Gruber, A generic C.S. has no (simple) closed geodesic.

Baire Categories in Geometry

- 1988: J. A. Wieaker, Most compacta are homeomorphic to a cantor set.
- 1989: P. Gruber, generic dimension of compacta and continua.
- 1997: A. V. Kuz'minykh, Most compacta are totally anisometric : $d(a, b) = d(a', b') > 0 \Rightarrow \{a, b\} = \{a', b'\}$
- 1989–2005: Results on the embedding: E.S. De Blasi, P. Gruber, J. Myjak & R. Rudnick, J. A. Wieaker, T. Zamfirescu, N.V. Zhivkov.

Baire Categories in Geometry

Imbedded in $\mathbb{R}^n$	Convex (hyper)surfaces	Continua	Compacta
Abstract			compact metric spaces

- 2011: J. Rouyer.

Imbedded in $\mathbb{R}^n$	Convex (hyper)surfaces	Continua	Compacta
Abstract	Alexandrov Surface		compact metric spaces

- 2012: K. Adiprazito and T. Zamfirecu, Most points are endpoints.
- 2015: J.-I. Itoh, J. R. , C. Vîlcu, No conical points, but no Gaussian curvature.
- 2016: J. R. , C. Vîlcu, No/infinitely many simple closed geodesic, depending on the curvature bound and the connected component of the space.

Imbedded in $\mathbb{R}^n$	Convex (hyper)surfaces	Continua	Compacta
Abstract	Alexandrov Surface	Length spaces	compact metric spaces

Work in progress.

Contents

Baire Categories

Let X be a topological space.

- $R \subset X$ is rare or nowhere dense iff $\mathrm{int}\,(\mathrm{cl}\,(R)) = \emptyset$.
- $M \subset X$ is meager or of first category iff it is included in a countable union of rare sets.
- X is a Baire space iff any meager set have empty interior.
- The Baire's theorem states that any complete metric space is a Baire space.

Convention

We say that

- most $x \in X$ are ...
- or that a generic $x \in X$ is ...

to express that the set of those $x \in X$ which are not ... form a meager set in X.

Length spaces I

Preliminary Remark

During this talk, a *length space* is supposed to be compact. (unlike most authors)

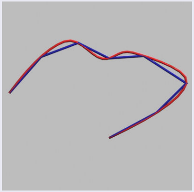
Definition

Let $\gamma : [a, b] \to X$*. The length of* γ *is*

$$L(\gamma) = \sup_{(t_0,\dots,t_n)\in S} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where

$S = \{(t_0, \dots, t_n) \in \mathbb{R}^n \,|\, n \in \mathbb{N}, a = t_0 < t_1 < \dots < t_n = b\}$



Length spaces II

Theorem

Let X be a compact metric space. Denote by $\Gamma(x,y)$ the set of curves from x to y. The following statements are equivalent:

- **existence of segments:** $\forall x, y \in X \exists \gamma \in \Gamma(x,y)$ *s.t.* $d(x,y) = L(\gamma)$.
- **existence of midpoints:** $\forall x, y \in X \exists z \in X$ *s.t.*

$$d(x,z) = d(z,y) = \frac{1}{2} d(x,y).$$

- **intrinsic metric:** $\forall x, y \in X, d(x,y) = \inf_{\gamma \in \Gamma(x,y)} L(\gamma)$.

Definition

A compact metric space satisfying these properties is called a (compact) length space.
The set of length spaces is denoted by $\mathcal{L}$.

Length spaces III

Examples

(Counter)example

$\left\{x \in \mathbb{R}^2 \,\middle|\, \|x\| = 1\right\}$ endowed with the metric $d_0(x, y) = \|x - y\|$ is not a length space, but endowed with $d_1(x, y) = \arccos\langle x, y\rangle$ is a length space.

Example

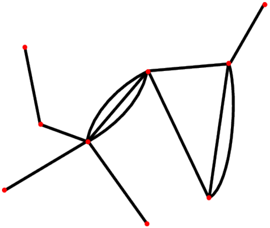
$\mathbb{R}^2/\mathbb{Z}^2$ *endowed with*

$$d\left((x_1, y_1), (x_2, y_2)\right) = \min\left(|x_1 - x_2|, 1 - |x_1 - x_2|\right) + \min\left(|y_1 - y_2|, 1 - |y_1 - Y_2|\right).$$

Example

More generally, any reversible (compact) Finsler manifold, and so any (compact) Riemannian manifold.

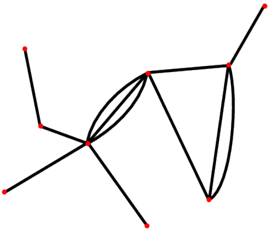
Length space IV
Finite metric graphs



- Start with a finite combinatorial graph: (V, E), $E \subset \mathcal{P}_2(V) \times \mathbb{N}$
- Assign lengths to edges: choose $\lambda : E \to]0, +\infty[$

Length space IV

Finite metric graphs



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Length space IV

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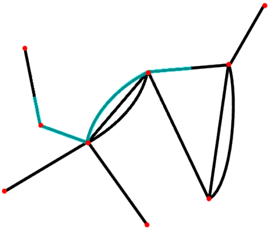

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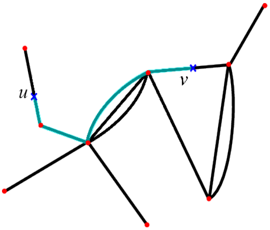
- Define the length of a simple path: $\ell_G(\gamma) = \sum_{\Delta \in E} \lambda(\delta) \ell_{]0,1[}(\gamma_E)$, where $\gamma_E = \gamma \cap E$.

Length space IV

Finite metric graphs



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- Define $d(u, v) = inf_{\gamma} \ell(\gamma)$, where the infimum is taken over all the simple paths γ from u to v.

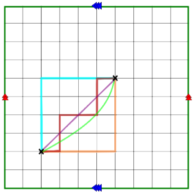
Any finite metric graph is a length space,
We denote by $\mathcal{G}$ the set of finite metric graphs.

Length space V

Geodesics in length spaces

Definition

A geodesic is a path which is locally a segment.



Finsler torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with $\| \ \|_1$.

- Geodesics may branch.
- No injectivity radius.

Length space V

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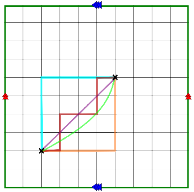
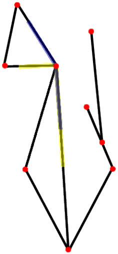


Finsler torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with $\|\ \|_1$.

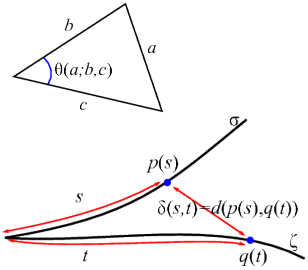
A metric graph.



- Geodesics may branch.
- No injectivity radius.
- Geodesics may stop.
- Existence of *endpoints*.

Length space VI

Angles in length spaces 1



One can define *lower* and *upper angles* between segments.

$$\underline{\measuredangle}(\sigma,\zeta)=\liminf_{s,t\to0}\theta(\delta(s,t);s,t)$$

$$\overline{\measuredangle}(\sigma,\zeta)=\limsup_{s,t\to0}\theta(\delta(s,t);s,t)$$

- When the two angles agree, we say that the segments make a well-defined angle.
- In Alexandrov spaces, all angle are well-defined.
- In Riemannian manifold, this notion of angles is equivalent to the usual one.

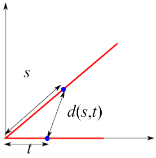
Length space VII

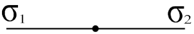
Angles in length spaces 2

For instance, in $(\mathbb{R}^2/\mathbb{Z}^2, \|\ \|_1)$, the angles between $\sigma : Y = 0$ and $\zeta : Y = aX$ are

$$\measuredangle(\sigma, \zeta) = 0,$$

$$\overline{\measuredangle}(\sigma, \zeta) = \arccos\left(\frac{1-a}{1+a}\right).$$





In any length space, if σ_1 and σ_2 are two parts of a same segment then

$$\measuredangle(\sigma_1, \sigma_1) = \overline{\measuredangle}(\sigma_1, \sigma_1) \quad = 0,$$

$$\measuredangle(\sigma_1, \sigma_2) = \overline{\measuredangle}(\sigma_1, \sigma_2) \quad = \pi.$$

Space of length spaces I

The Gromov-Hausdorff metric

Notation. Let Z be a metric space.

- $\mathcal{K}(Z)$ denotes the set of nonempty compact subsets of Z
- for $A \in \mathcal{K}(Z)$ and $\rho \in \mathbb{R}_+$,
  $A + \rho \stackrel{\text{def}}{=} \{y \in Z \,|\, \exists x \in A \text{ s.t. } d(x,y) \leq \rho\}$

Definition

- *for $A, B \in \mathcal{K}(Z)$, the Pompeiu-Hausdorff distance is:*

$$d^Z_{PH}(A,B) = \inf\{\varepsilon \,|\, A \subset B + \varepsilon, B \subset A + \varepsilon\}$$

- *for X, Y compact metric spaces, the Gromov-Hausdorff distance is:* $d_{GH}(X,Y) = \inf_{Z,f,g} d^Z_{PH}(f(X), g(Y))$,
  where the infimum is taken over all metric spaces Z and all isometric embeddings $f : X \to Z$, $g : Y \to Z$.

Space of Length spaces II

Theorem

The set $\mathcal{GH}$ of all compact metric spaces, up to isometries, endowed with d_{GH} is a complete metric space.

Theorem

$\mathcal{L}$ is closed in $\mathcal{GH}$ and so, is a complete metric space.

Theorem

$\mathcal{G}$ is dense in $\mathcal{L}$.

Theorem

The set of Riemannian surfaces is dense in $\mathcal{L}$.

Definition of f-tangency I

motivation

- No differential structure
- No angle

Definition

A comparison function *is smooth increasing function* $f :]0, \infty[\to]0, \infty[$ *s.t.* $f(x) = o(x)$ *when* x *goes to* 0 .

Notation

The set of segments emanating from a point x *will be denoted by* Σ_x.

Definition of f-tangency II

Definition

Let f be a comparison function, $\sigma, \gamma \in \Sigma_x$

1. *σ, γ are said to be weakly f-tangent if there exists a sequence of positive numbers t_n tending to 0 such that $\sigma(t_n)\gamma(t_n) < f(t_n)$.*
2. *σ, γ are said to be f-tangent if there exists $\tau > 0$ such that for any $t \in [0, \tau]$ $\sigma(t)\gamma(t) \leq f(t)$.*
3. *σ, γ are said to be strongly f-tangent if there exists $\tau > 0$ such that for any $s, t \in [0, \tau]$ $\sigma(t)\gamma(s) \leq |s - t| + f(\min(s, t))$.*

f-tangency and angles

Proposition

For any comparison function f,

1. *If $\sigma,\gamma \in \Sigma_x$ are strongly f-tangent then $\overline{\measuredangle}(\sigma,\gamma) = 0$.*
2. *If $\sigma,\gamma \in \Sigma_x$ are weakly f-tangent then $\underline{\measuredangle}(\sigma,\gamma) = 0$.*

Theorem

Let f be a comparison function. For most $X \in \mathcal{L}$, if $\sigma, \gamma \in \Sigma_x$ are f-tangent, then either $\sigma \subset \gamma$ or $\gamma \subset \sigma$.

Corollary

In a generic length space geodesics do not bifurcate.

A generic result about angles

Theorem

In a generic length space, at any point x, any two segments $\sigma,\gamma \in \Sigma_x$ satisfy $\underline{\measuredangle}(\sigma,\gamma) = 0$ or $\overline{\measuredangle}(\sigma,\gamma) = \pi$.

Problem

- *How common/rare are the pairs $(\sigma,\gamma) \in \Sigma_x^2$ such that $\underline{\measuredangle}(\sigma,\gamma) = 0$* **and** *$\underline{\measuredangle}(\sigma,\gamma) = \pi$?*
- *How common/rare are the pairs of segments with a well-defined angle ?*

Definition of f-cusp

Definition

Let f be a comparison function.
If $x \in X \in \mathcal{L}$ is such that any two segments σ,γ emanating from x are (resp. weakly, resp. strongly) tangent we call x a (resp. weak, resp. strong) f-cusp.

Example

If $X \in G$, its (weak/strong) f-cusp are exactly its endpoints.

Cusp properties

Proposition

1. *A strong f-cusp is a f-cusp.*
2. *A f-cusp is a weak λf-cusp for any $\lambda > 1$.*

Proposition

A weak f-cusp is interior to no segment.

Theorem

Let f be a comparison function. In a generic length space,

1. *there is no f-cusp,*
2. *a generic point $x \in X$ is a weak f-cusps.*

Dimensions

The many names dimension

Names are: *box dimension*, *box-counting* $\sim$, *capacity* $\sim$, *fractal* $\sim$, *Kolmogorov* $\sim$, *Minkowski* $\sim$, *Minkowski-Bouligand*, ...

Notation

- $N(X,\varepsilon) = \min\{\operatorname{card}(F) \mid F \subset X \ \forall x \in X \ d(x,F) \leq \varepsilon\}$
- $M(X,\varepsilon) = \max\left\{\operatorname{card}(F) \,\middle|\, \begin{array}{l} F \subset X \text{ and} \\ \forall x,y \in F \ x \neq y \Rightarrow xy \geq \varepsilon \end{array}\right\},$

Theorem and definition

The *upper* and *lower box dimension* of a compact metric space X are defined as

$$\dim^B(X) = \limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \limsup_{\varepsilon\to 0} \frac{\log M(X,\varepsilon)}{-\log\varepsilon}$$

$$\dim_B(X) = \liminf_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \liminf_{\varepsilon\to 0} \frac{\log M(X,\varepsilon)}{-\log\varepsilon}.$$

Generic dimension

$$\dim_T \leq \dim_H \leq \dim_B \leq \dim^B$$

Theorem

Let X be a generic length space.

- $\dim_B(X) = 1$ *and* $\dim^B(X) = \infty$.
- $\mathcal{H}^1(X) = \infty$

Theorem

In a generic compact length space, $\forall x \in X, \forall \rho > 0, \dim_B(S_x(\rho)) = 0$.

Question

What can one say (generically) of $\dim^B S_x(\rho)$ *?*

Notation

For $x \in X \in \mathcal{L}$*,* $S_x(\rho)$ *is the sphere centred at* x *with radius* ρ*, that is* $\{y \in X \,|\, d(x,y) = \rho\}$

Farthest points

Notation

$\rho_x = max_{y \in X} d(x, y)$

$F_x = S_x(\rho_x)$

Theorem

For a generic $X \in \mathcal{L}$ and a generic $x \in X$, $\mathrm{card}(F_x) = 1$.

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Embedded in $\mathbb{R}^n$	**Convex surfaces** T.Z. (1995)	**Continua** ?	**Compacta** Kuz'minykh (1997)
Abstract	**Alex. Surfaces** J.R & C.V. (2018?)	**Length spaces** J.R.(2019?)	**compact metric spaces** J.R. (2011)

Proof I

Preliminary

Theorem

On a compact manifold endowed with a generic Riemannian structure, a generic point has a single farthest point.

J. Rouyer (2003).

Lemma

The function F is upper semi-continuous, that is

$$\lim_{x \to x_0} F_x \subset F_{x_0}.$$

Proof II

Denote by $\delta(A)$ the diameter of A.

$$\begin{aligned}\mathcal{M} &\stackrel{\text{def}}{=} \{X \,|\{x \in X \,|\#F_x > 1\} \text{ non meager}\} \\ &= \bigcup_p \{X \,|\text{int}\{x \in X \,|\delta(F_x) \geq 1/p\} \neq \varnothing\} \\ &= \bigcup_p \bigcup_q \left\{X \,\middle|\exists y \in X \; \bar{B}\left(y, \frac{1}{q}\right) \subset \{x \in X \,|\delta(F_x) \geq 1/p\}\right\} \\ &\stackrel{\text{def}}{=} \bigcup_p \bigcup_q \mathcal{M}_{pq}.\end{aligned}$$

- $\mathcal{M}_{pq}$ has empty interior.
- It remains to prove that it is closed.

Proof III

$$\mathcal{M}_{pq} = \left\{ X \,\middle|\, \exists y \in X \;\bar{B}\left(y, \frac{1}{q}\right) \subset \{x \in X \,|\delta\left(F_x\right) \geq 1/p\} \right\}$$

- $\mathcal{M}_{pq} \ni X_n \in \mathcal{M}_{pq} \underset{GH}{\to} X \in \mathcal{L}$.
- W.l.g., we can assume that $X_n, X \subset Z$ and $X_n \underset{PH}{\to} X$.
- Take $y_n \in X_n$ s.t. $\bar{B}\left(y_n, \frac{1}{q}\right) \subset \{x \in X_n | \delta\left(F_x\right) \geq 1/p\}$
- Take a converging sub-sequence; let $y \in X$ be the limit.
- We claim that $\bar{B}\left(y, \frac{1}{q}\right) \subset \{x \in X_n | \delta\left(F_x\right) \geq 1/p\}$, and so, $X \in \mathcal{M}_{pq}$.
  - $z \in \bar{B}\left(y, \frac{1}{q}\right) \leftarrow z_n \in \bar{B}\left(y_n, \frac{1}{q}\right) \subset \left\{x \in X_n \,\middle|\, \delta(F_x) \geq \frac{1}{p}\right\}$
  - $\delta(F_{z_n}) \geq \frac{1}{p}$.
  - By semi-continuity of F, $\delta(F_z) \geq \frac{1}{q}$
  - $z \in \left\{x \in X \,\middle|\, \delta(F_x) \geq \frac{1}{p}\right\}$

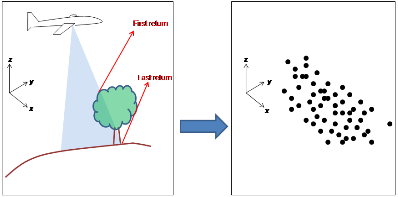
Thank you very much for your attention !

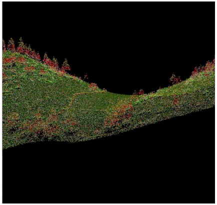
A comparison of discrete curvature schemes applied for triangle meshes derived from geo-spatial data

Mihai-Sorin Stupariu
University of Bucharest

International Conference on Discrete Mathematics:
Discrete Geometry and Convex Bodies
Bucharest, September 2017

Motivation





Airborne laser scanning provides a cloud of points situated in the 3D-space (LiDAR data).

Such data sets contain a lot of information useful in practical problems.

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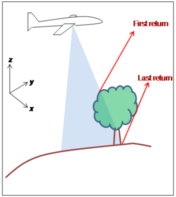

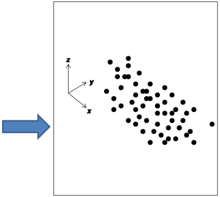

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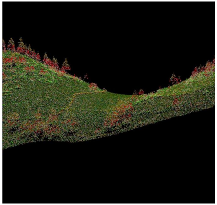

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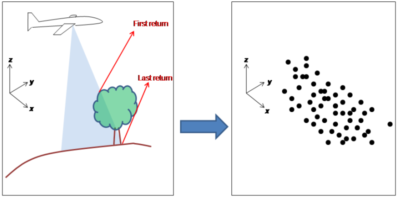

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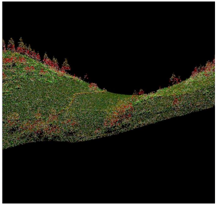
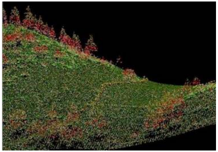


Such data sets contain a lot of information useful in practical problems.

- *Challenge:* explore the opportunity of using tools from Discrete Differential Geometry.
- *Aim:* perform numerical experiments based on true terrain data.

Geo-spatial data - format and representation

Point clouds (LiDAR data)
- rich in information (+)
- appropriate algorithms (+)
- lack of 2D correspondent (–)



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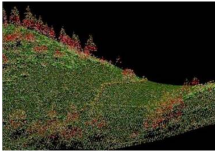
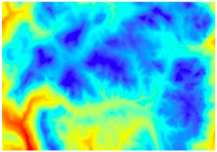


Regularly spaced grids
- easy to handle (+)
- standard patch-corridor model (+)
- lack of details (–)



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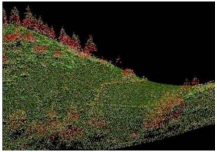
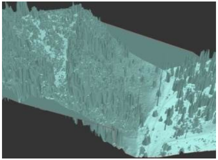


Triangulated terrains (TIN)
- still carry a lot of information (+)
- 2D correspondent possible (+)
- high computational costs (–)



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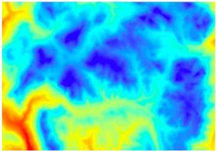
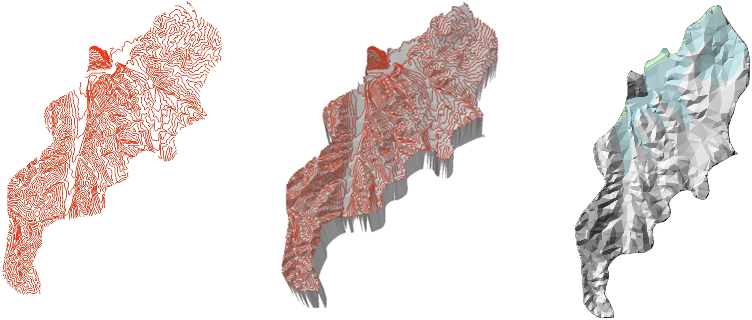


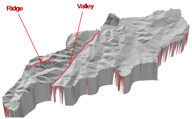
Digital models of the terrain



Three representations of the same study site (contour lines, combined, TIN), as provided by GIS-software

TIN representations and terrain variability

- Triangulated terrains in GIS [e.g. de Floriani et al., 1997]



Ridges or valleys are visible in a TIN model

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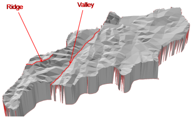

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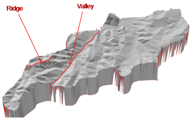

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- Main research question: *to what extent is it possible to extract relevant information from geo-spatial data when triangle meshes are used?* Specifically: *how can one measure the lack of flatness?*

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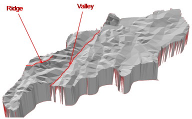

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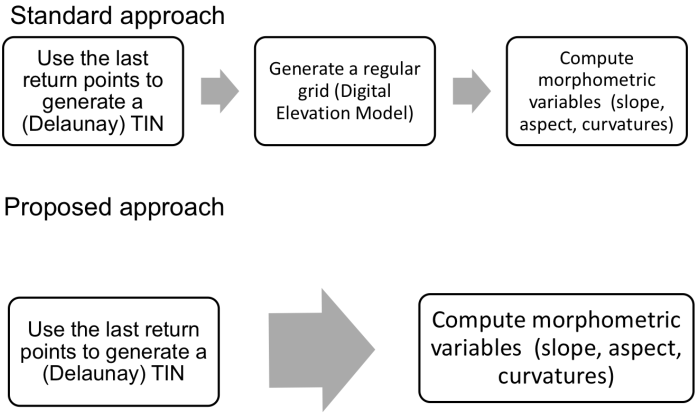
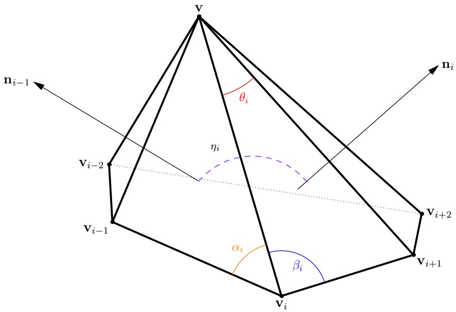
- Main research question: *to what extent is it possible to extract relevant information from geo-spatial data when triangle meshes are used?* Specifically: *how can one measure the lack of flatness?*
- Main hypothesis: *discrete curvatures for triangle meshes could provide relevant numerical descriptors (morphometric variables, e.g. slope, curvatures) quantifying terrain features.* Two tracks: (i) comparisons for various methods; (ii) identification of specfic structures.

Morphometric variables — the discrete approach



Notation: 1-ring $\mathcal{N}_\mathbf{v}$



Geometric elements around a vertex $\mathbf{v}$:

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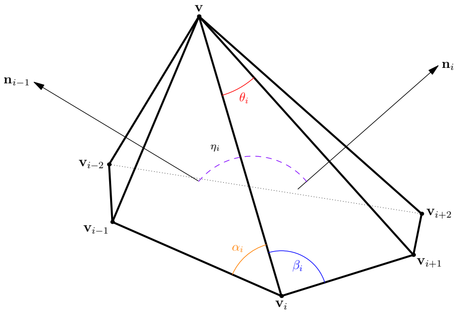

Geometric elements around a vertex $\mathbf{v}$:

- Edges / faces incident to $\mathbf{v}$ (or associated measures – lengths, areas).

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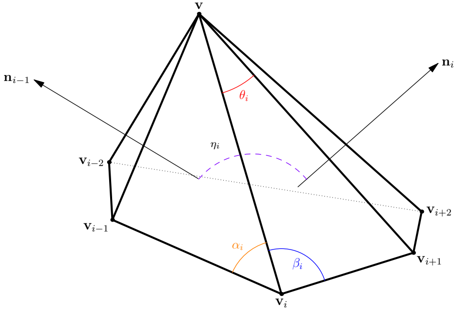

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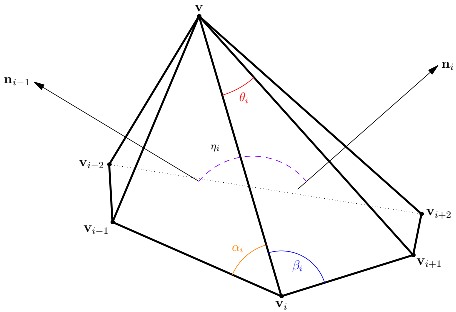

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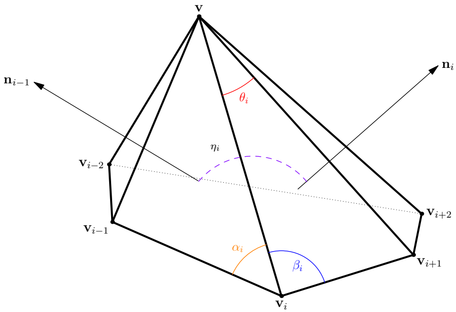

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- Angles $(\theta_i)_i$ between edges incident to $\mathbf{v}$.
- Angles $(\eta_i)_i$ between normals of faces incident to $\mathbf{v}$.
- Angles $(\alpha_i)_i, (\beta_i)_i$ between edges of the 1-ring that are not incident to $\mathbf{v}$.

Method 1: Gauss-Bonnet scheme (1) GB1

- **Gaussian curvature at v**

$$K_{\mathbf{v}} = \frac{2\pi - \sum_{\mathbf{v}_i \in \mathcal{N}_{\mathbf{v}}} \theta_i}{\frac{1}{3} A}, \tag{1}$$

where $2\pi - \sum_{\mathbf{v}_i \in \mathcal{N}_{\mathbf{v}}} \theta_i$ is the angular defect at **v**, and A is the total area of the triangles in the 1-ring neighborhood of v

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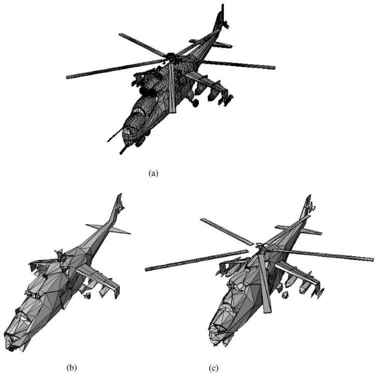
- **Mean curvature** at **v**

$$H_{\mathbf{v}} = \frac{\frac{1}{4} \sum_{\mathbf{v}_i \in \mathcal{N}_{\mathbf{v}}} \|\overrightarrow{\mathbf{v}\mathbf{v}_i}\| \eta_i}{\frac{1}{3}A} \quad (2)$$

(measures the variation of the normals along the edges incident to **v**)

Method 1: Gauss-Bonnet scheme (1) GB1

Used by [Dyn et al., 2001]; [Kim et al., 2002] for simplifying triangle meshes



Helicopter model. (a) Original. (b), (c) Simplified versions. In (c) the discrete curvatures were used.
Source: [S.J. Kim, C.H. Kim, D. Levin, *Computers & Graphics*, 2002]

Method 2: Gauss-Bonnet scheme (2) GB2

- Proposed by [Meyer et al., 2003]; considers for averaging A_{mixed} − area of a region determined by circumcenters instead of barycenters (adapted for obtuse triangulations).
- **Gaussian curvature** at **v**

$$K_{\mathbf{v}} = \frac{2\pi - \sum_{\mathbf{v}_i \in \mathcal{N}_{\mathbf{v}}} \theta_i}{A_{\text{mixed}}}. \tag{3}$$

Each triangle of $\mathcal{N}_{\mathbf{v}}$ "contributes" to A_{mixed}. If $\Delta \mathbf{v}\mathbf{v}_{i-1}\mathbf{v}_i$, is non-obtuse, its contribution is $\frac{1}{8}(\|\overrightarrow{\mathbf{v}\mathbf{v}_i}\|^2 \cot(\widehat{\mathbf{v}\mathbf{v}_{i-1}\mathbf{v}_i}) + \|\overrightarrow{\mathbf{v}\mathbf{v}_{i-1}}\|^2 \cot(\widehat{\mathbf{v}\mathbf{v}_i\mathbf{v}_{i-1}}))$. If $\Delta$is obtuse: (i) at **v**: $\frac{1}{2}A(\Delta)$, (ii) at a vertex different of **v**: $\frac{1}{4}A(\Delta)$.

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- **The mean curvature** $H_{\mathbf{v}} = \frac{1}{2}\|\mathbf{H}_{\mathbf{v}}\|$ is the norm of the mean curvature operator

$$\mathbf{H}_{\mathbf{v}} = \frac{1}{2A_{\text{mixed}}} \sum_{\mathbf{v}_i \in \mathcal{N}_{\mathbf{v}}} (\cot(\widehat{\mathbf{v}\mathbf{v}_{i-1}\mathbf{v}_i}) + \cot(\widehat{\mathbf{v}\mathbf{v}_{i+1}\mathbf{v}_i}))\overrightarrow{\mathbf{v}_i\mathbf{v}}. \quad (4)$$

Method 3: approach based on Euler's theorem ET

- Proposed by [Watanabe & Belyaev, 2001], based on integral formulas derived from Euler's theorem

$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_\nu(\varphi)d\varphi; \qquad K = 3H^2 - \frac{1}{\pi}\int_0^{2\pi} \kappa_\nu(\varphi)^2 d\varphi,$$

$\kappa_\nu(\varphi)$ is the normal curvature of the normal section curve corresponding to the angle φ.

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  $\kappa_\nu(\varphi)$ is the normal curvature of the normal section curve corresponding to the angle φ.

- Approximate $\kappa_\nu(\varphi)$ along the edges of the 1-ring by

$$\kappa_{\nu,i} \simeq \frac{2 < \mathbf{n}_v, \overrightarrow{vv_i} >}{\| \overrightarrow{vv_i} \|^2} \tag{5}$$

  ($\mathbf{n}_v$ weighted normal; weights are relative areas).

- Use approximation and put

$$H_v = \frac{1}{2\pi}\sum_{i=1}^{n} \kappa_{\nu,i}\frac{\theta_{(i-1) \bmod n} + \theta_i}{2}; \quad K_v = 3H_v^2 - \frac{1}{\pi}\sum_{i=1}^{n} \kappa_{\nu,i}^2\frac{\theta_{(i-1) \bmod n} + \theta_i}{2}$$

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- Define the matrix $M_\mathbf{v}$ as a weighted sum, $M_\mathbf{v} = \sum_{i=1}^{d_\mathbf{v}} \rho_i \kappa_{\nu,i} \overrightarrow{\mathbf{t}}_i \overrightarrow{\mathbf{t}}_i^t$, where the weight ρ_i the relative area of the faces that are adjacent to the edge $\overrightarrow{\mathbf{vv}_i}$.

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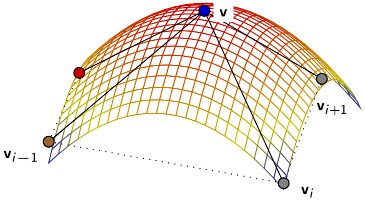
- By the construction of $M_{\mathbf{v}}$, one of its eigenvalues is 0, with associated eigenvector $\overrightarrow{\mathbf{n}}_{\mathbf{v}}$. Let λ and μ be the other eigenvalues of $M_{\mathbf{v}}$. Put

$$K_{\mathbf{v}} = (3\lambda - \mu) \cdot (3\mu - \lambda); \qquad H_{\mathbf{v}} = \frac{1}{2}\left[(3\lambda - \mu) + (3\mu - \lambda)\right].$$

Method 5: paraboloid fitting PF

Assume that $v = 0$ and $\mathbf{n_v} = (0,0,1)$; take its 1-ring neighborhood and find a paraboloid $z = ax^2 + bxy + cy^2$ that better fits this data (using least squares fitting, e.g. [Hamann, 1993]); then compute K_v, H_v by using standard formulas for the smooth paraboloid

$$K_{\mathbf{v}} = 4ac - b^2; \qquad H_{\mathbf{v}} = a + c. \tag{6}$$



Method 6: Shape Operator SO

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- The *Shape Operator* at the vertex **v**

$$S(\mathbf{v}) = \frac{1}{2} \sum_{e_i \in \mathcal{N}_{\mathbf{v}}} \omega_{e_i} H_{e_i} \; \vec{t}_{e_i} \vec{t}_{e_i}^{\,t},$$

where $\omega_e = \langle \mathbf{n}_{\mathbf{v}}, \mathbf{n}_e \rangle$, and $\vec{t}_e$ is the versor of the projection on the "tangent" plane at **v** of the vector $\vec{e} \times \vec{\mathbf{n}}_e$.

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- The Gaussian curvature and the mean curvature, respectively, are defined by

$$K_{\mathbf{v}} = \det(S(\mathbf{v})); \; H_{\mathbf{v}} = \frac{1}{2}\mathrm{tr}(S(\mathbf{v})). \tag{7}$$

Concept

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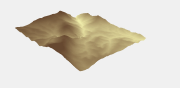
- Comparisons between the methods: realized for surfaces such as plane, sphere, cone, cylinder [Magid, Soldea, Rivlin, 2007].
- **Aim:** computation and comparisons for geo-spatial data, obtained thrhough *in situ* measurements – true terrains, with unknown geometry of the underlying surface.
- Two complementary approaches: refining and coarsening.

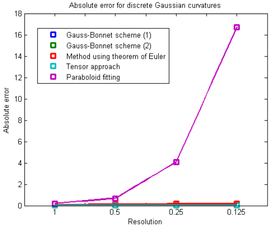
Approach 1 - approximation accuracy

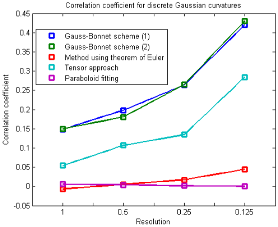
- Generate a discrete height function starting from the elevation digital model of a site situated in a mountainous region (cca. 23 km^2).



- Produce a smooth surface S by standard interpolation techniques.
- Select on S, through jittered sampling with decreasing cell size (i.e., increasing resolution), sets of random points ('pseudo-LiDAR data sets'). Four cell sizes were used throughout the experiments having a size equal to a ratio of 1, 0.5, 0.25 and 0.125 to the original cell size. These values correspond to real cell sizes of 18 m, 9 m, 4.5 m and 2.25 m, respectively.
- Generate a 2.5D triangular irregular network for each point set, obtained for each of the four levels of resolution.
- Compare the discrete Gaussian curvature and discrete mean curvature with the 'true' smooth ones. For each method, at each of the four levels of resolution, two numerical quantities were computed: (i) the absolute error (normalized L^1-norm of the vector of differences between 'discrete' and 'smooth' curvatures); (ii) the correlation coefficients between the discrete and the smooth curvatures.

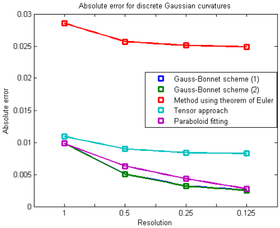
Results (1): Gaussian curvature

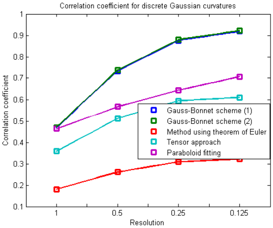




– In the computation of the absolute error and of the correlation coefficient all points are taken into account
– **Gauss-Bonnet scheme**: best approximation
– **Paraboloid fitting**: bad behavior (occurrence of outliers)
– Hierarchy is similar for spline interpolation
– The results for SO-method are not included in the diagrams

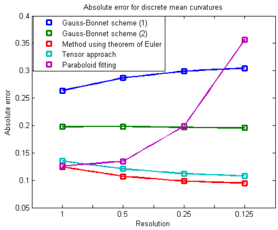
Results (1): Gaussian curvature - outliers removed

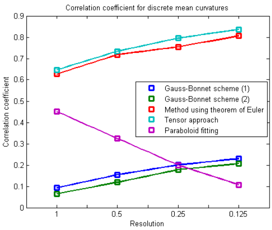




– In the computation of the absolute error and of the correlation coefficient the ‘outliers’ were removed
– **Gauss-Bonnet scheme**: best approximation
– **Paraboloid fitting**: sensitive to occurrence of outliers

Results (1): mean curvature





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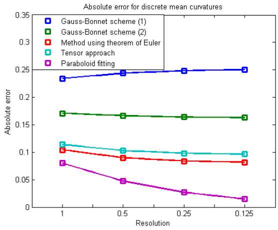
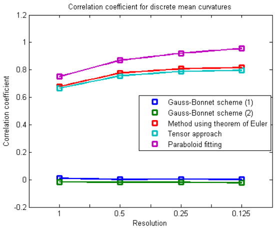




– In the computation of the absolute error and of the correlation coefficient 'outliers' are removed

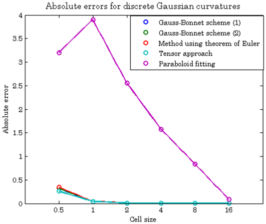
– **Method using Euler's theorem**: good approximation / not sensitive to outliers

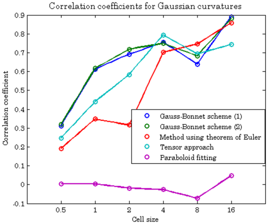
– **Paraboloid fitting**: best approximation / sensitive to occurrence of outliers

Approach 2 - smoothening

- Numerical experiments based on true terrain data: high resolution point cloud of size 427K; surface of cca. 2.5 ha.
- Preprocess data: crop and a rectangle having sizes 128 m and 160 m.
- For cell sizes equal to 0.5 m, 1 m, 2 m, 4 m, 8 m, 16 m, regularly spaced grids were generated. For each cell $\mathcal{C}$, a single point was obtained, by averaging the coordinates of the points of the original cloud situated in $\mathcal{C}$.
- For each point set, obtained for each of the six levels of resolution, a 2.5D Delaunay triangulation was generated.
- The values of the discrete Gaussian and mean curvatures for the vertices of each set and for the corresponding regularly spaced grids were computed. For each method, the discrete Gaussian curvature and discrete mean curvature were compared with the ones computed for the corresponding regular grids. The comparison was achieved by computing two numerical quantities: (i) the absolute error (normalized L^1-norm of the difference vectors), (ii) the correlation coefficients. For a better relevance, border vertices or vertices for which some of the methods could not provide any value were removed from the statistics.

Results (2): Gaussian curvature

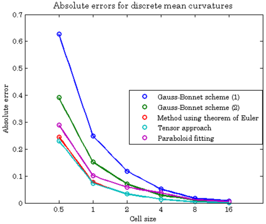


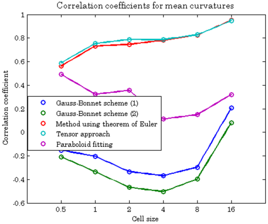


– Absolute error and correlation coefficients: GB1, GB2, ET, TA comparable results (smoothening effect).

– The results for SO-method are not included in the diagrams

Results (2): mean curvature





– **Method using Euler's theorem and the tensor approach**: best approximation

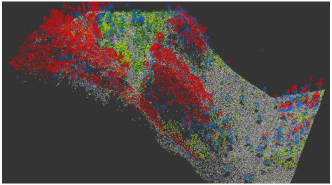
– Weak correlations for GB1, GB2 (only positive values).

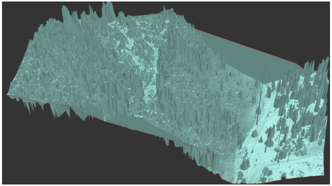
Conclusions

For the **Gaussian curvature**, the best approximation was given by the Gauss-Bonnet schemes, while in the case of the **mean curvature**, the tensor approach and the method based on Euler's theorem provided an accurate estimate. These findings are consistent for both approaches and they are consistent with previous studies conducted for smooth surfaces with known underlying geometry.

Problem statement

Vegetation structures (e.g. trees) are visible in a high density point cloud and in the associated triangulation.





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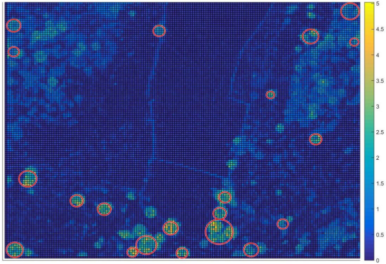
- Use pattern recognition techniques (the Hough transform, implemented in Matlab, sensitivity factor 0.85) for detecting circles: horizontal projections of tree crowns usually yield circular shapes.

Tree detection – results

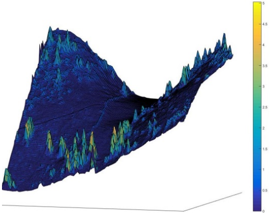
LiDAR point cloud (colours represent height above ground, in particular trees are coloured in red).



Grid generated by using the mean curvature, as provided by the shape operator method. The red circles represent

Tree detection – results

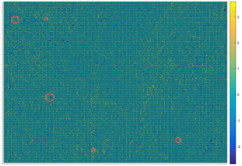
The point cloud (3D representation).



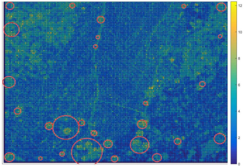
Grid of mean curvatures for SO (3D representation).

Comparisons – mean curvature grids

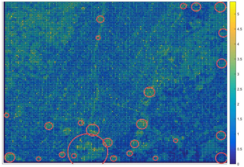
Point cloud.



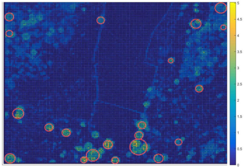
ET



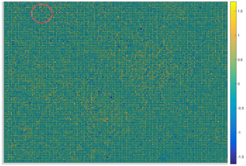
GB1



GB2



SO



TA

Comments and conclusions

- Mean curvature - makes it possible to detect trees and the size of their crowns.
- Good results for SO; similar results for GB1, GB2.
- Advantages:
  - The method presented is independent on any *a priori* knowledge, while state of the art techniques require a preliminary field survey, enabling an appropriate calibration and developing suitable regression models, (e.g. [Popescu, 2003]).
  - Independence on tree species, while other approaches are species sensitive: [Falkowski et al., 2006] an aaproach on the Mexican Hat wavelet appropriate for coniferous trees.

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Thank you!

Envelopes of α-sections

Costin Vîlcu

Simion Stoilow Institute of Mathematics of the Romanian Academy

September 2017

Collaborators

This talk is based on a joint work with

Nicolas Chevallier and **Augustin Fruchard**

Laboratoire de Mathématiques, Informatique et Applications
Faculté des Sciences et Techniques
Université de Haute Alsace

Definitions

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- $\alpha \in]0, 1[$
  α-section of K = an oriented line $\Delta \subset E$ cutting K in two parts,
  K^- (to the right) of area $|K^-| = \alpha |K|$, and
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- K_α is the *α-core* of K = the intersection of all K^+.
- m_α = the envelope of all α-sections of K.

Two examples

If K is a disk then $m_\alpha = \partial K_\alpha$ is a circle.

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If K is a polygon then m_α is made of arcs of hyperbolae, $\forall \alpha \in]0, 1[$.

Related topic: (ir)reducibility theory of convex bodies

Study secants between parallel supporting lines to K, whose distances to the corresponding lines make a ratio of $\alpha/(1-\alpha)$.

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Contributors:

- P.C. Hammer, 1951;
- V. Klee, 1953;
- T. Zamfirescu, 1967.

Related topic: common tangents and transversals

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- S.E. Cappell, J.E. Goodman, J. Pach, R. Pollack, M. Sharir, R. Wenger, 1994;
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Theorem

For any well-separated family of k strictly convex bodies in $\mathbb{R}^d$, $k \leq d$, the space of all α-sections is diffeomorphic to $\mathbb{S}^{d-k}$.

Related topic: outer billiards

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If ℓ is the envelope of α-sections of a convex set bounded by a curve κ, for some α, then κ is a caustic for the outer billiard of table $L = \mathrm{conv}\ell$.

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Theorem

$K \subset \mathbb{R}^d$ with boundary of class $\mathcal{C}^{\geq 4}$; K_δ is homothetic to K, for some sufficiently small $\delta > 0$, if and only if K is an ellipsoid.

Related topic: fair partitioning of convex bodies

Fair / balanced / equi- partions of convex bodies (measures) by use of

- *k*-fans;
- hyperplanes;
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Contributors:

- I. Bárány, J. Matousek, 2001;
- T. Sakai, 2002;
- S. Bereg, 2009;
- I. Bárány, P. Blagojević, A. Szúcs, 2010;
- P. V. M. Blagojević, G.M. Ziegler, 2014;
- R. N. Karasev, A. Hubard, B. Aronov, 2014;

Fair partitioning of convex bodies II

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For a pizza (K, L), with $L \subset K \subset E$, use a succession of double operations:

- a cut by a *full* straight line, followed by
- a Euclidean move of one of the pieces.

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Theorem

Given an integer $n \geq 2$, there exists a fair partition of any pizza (K, L) into n parts if and only if n is even.

If $\alpha > 1/2$ then $K_\alpha = \emptyset$.

Main result: m_α for symmetric K

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Theorem

If K is symmetric then

- $m_\alpha = \partial K_\alpha$ *for all* $\alpha \in \left]0, \frac{1}{2}\right[$;
- m_α *is of class* $\mathcal{C}^1$ *for all* $\alpha \in \left]0, \frac{1}{2}\right[$ *if and only if* K *is strictly convex.*

Main result: m_α for non-symmetric K

We cannot have $m_\alpha = \partial K_\alpha$ for all $\alpha \in \left]0, \frac{1}{2}\right[$, because m_α exists for all α, but $K_\alpha = \emptyset$ for α close enough to $\frac{1}{2}$.

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If K is non-symmetric then there exists $\alpha_B \in \left[0, \frac{1}{2}\right[$ s.t.

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Corollary

K is non-symmetric iff there exists a triangle containing more than half of K (in area), with one side in K and the other two disjoint from $\mathrm{int} K$.

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There exists $\alpha_K \in \left[\frac{4}{9}, \frac{1}{2}\right]$ *s.t.*

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Hint: G is the mid-point of at least three secants of K.

K non-symmetric $\Rightarrow \alpha_B < \alpha_K$, K symmetric $\Rightarrow \alpha_B = \alpha_K = \frac{1}{2}$.

Thank you for your attention!