Recent results on porous medium equations with nonlocal pressure

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joint work with

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Preliminaries on the PME

Derivation of the Porous Medium Equation

- Physical Model: a continuum (fluid or population) with density distribution $u(x,t) \geq 0$ and velocity field $\mathbf{v}(x,t)$.
- Continuity equation $u_t = \nabla(u \cdot \mathbf{v})$.
- Darcy's law: $\mathbf{v}$ derives from a potential (fluids in porous media): $\mathbf{v} = -\nabla p$.
- The relation between p and u: for gasses in porous media, Leibenzon and Muskat (1930) derived a relation in the form of the state law

  $$p = f(u),$$

  where f is a nondecreasing scalar function. $f(u)$ is linear when the flow is isothermal and is a higher power of u when the flow is adiabatic, i.e. $f(u) = cu^{m-1}$ with $c > 0$ and $m > 1$.
- The linear dependence $f(u) = cu \longrightarrow$ Boussinesq (1903) modelling water infiltration in an almost horizontal soil layer $\longrightarrow u_t = c\Delta u^2$.
- The model $u_t = (c/m)\Delta u^m$.
- The Porous Medium Equation $u_t = \Delta u^m$.

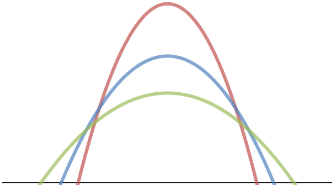
Porous Medium Equation / Fast Diffusion Equation

PME/FDE $u_t(x,t) = \Delta u^m(x,t) \quad x \in \mathbb{R}^N,\ t > 0$

Self Similar solutions: $\mathcal{U}(x,t) = t^{-\frac{N}{N(m-1)+2}} F\left(|x| t^{-\frac{1}{N(m-1)+2}}\right)$

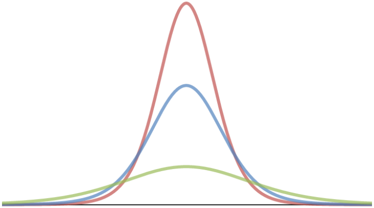
Slow Diffusion $m > 1$

$F(y) \sim (R^2 - |y|^2)_+^{1/(m-1)}$



Fast Diffusion $m < 1$

$F \sim (R^2 + |y|^2)^{-1/(1-m)}$



The Fractional Version

Definition of the Fractional Laplacian

Several equivalent definitions of the nonlocal operator $(-\Delta)^s$ (Laplacian of order $2s$):

1. Fourier transform $\widehat{(-\Delta)^s g}(\xi) = (2\pi|\xi|)^{2s}\hat{g}(\xi)$.

   [can be used for positive and negative values of s]

2. Singular Kernel $(-\Delta)^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\, dz$

   [can be used for $0 < s < 1$, where $c_{N,s}$ is a normalization constant.]

3. Heat semigroup

$$(-\Delta)^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta} g(x) - g(x)\right) \frac{dt}{t^{1+s}}\ .$$

4. Generator of the $2s$-stable Levy process:

$$(-\Delta)^s g(x) = \lim_{h\to 0} \frac{1}{h}\mathbb{E}[g(x) - g(x+X_h)].$$

Porous medium with nonlocal pressure

- The pressure $p = (-\Delta)^{-s}(u)$, $0 < s < 1$:

$$(-\Delta)^{-s}(u) = K_s \star u = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-2s}} dy, \quad K_s(x) = C_{N,s}|x|^{-(N-2s)}.$$

- The model:

$$\partial_t u = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-s}(u).$$

Difficulties: no maximum principle, no uniqueness.

References:

- 1D: crystal dislocations model, Biler, Karch and Monneau, Comm.Math.Phys. 2010.
- Existence and finite speed of propagation: Caffarelli and Vázquez, ARMA 2011.
- Asymptotic behavior: Caffarelli and Vázquez, DCDS 2011.
- Regularity: Caffarelli, Soria and Vázquez, JEMS 2013.
- Exponential convergence towards stationary states in 1D: Carrillo, Huang, Santos and Vázquez, JDE 2015.

Porous Medium with nonlocal pressure

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla p), \quad p = (-\Delta)^{-s}(u). \qquad \text{(M1)}$$

for $x \in \mathbb{R}^N$, $t > 0$, $N \geq 1$. We take $m > 1$, $0 < s < 1$ and $u(x,t) \geq 0$.

The initial data $u(x,0) = u_0(x)$ for $x \in \mathbb{R}^N$, $u_0 : \mathbb{R}^N \to [0,\infty)$ is assumed to be a bounded integrable function.

- D. Stan, F. del Teso and J.L. Vázquez, *Finite and infinite speed of propagation for porous medium equations with fractional pressure*, Comptes Rendus Mathematique, 352 (2014), Issue 2, 123—128.
- D. Stan, F. del Teso and J.L. Vázquez, *Transformations of Self-Similar Solutions for porous medium equations of fractional type*, Nonlinear Analysis Theory Meth.Appl., 119 (2015) 62–73.
- D. Stan, F. del Teso and J.L. Vázquez, *Finite and infinite speed of propagation for porous medium equations with nonlocal pressure*, Journal of Differential Equations 260 (2016), Issue 2, 1154–1199.
- D. Stan, F. del Teso and J.L. Vázquez, *Existence of weak solutions for a general porous medium equation with nonlocal pressure*, arXiv:1609.05139.
- D. Stan, F. del Teso and J.L. Vázquez, *Porous medium equation with nonlocal pressure*, preprint 2017.

Existence of solutions

New idea: existence for all $m > 1$ when $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

Based on suitable energy methods.

Formally:

$$\int_{\mathbb{R}^N} u_0^p(x)dx - \int_{\mathbb{R}^N} u(x,t)^p dx = C_1 \int_0^t \int_{\mathbb{R}^N} u^{m+p-2}(-\Delta)^{1-s} u \, dx\, dt$$

$$\geq C_2 \int_0^t \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-2}{2}}\right|^2 dxdt$$

by the Stroock-Varoupolos Inequality.

Here $C_1 = (p-1)/(m+p-2)$.

New approximation method

$$u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-1} (-\Delta)^{1-s} u) \qquad (P)$$

Then we approximate the operator $\mathcal{L} = (-\Delta)^{1-s}$ by

$$\mathcal{L}_\epsilon^{1-s}(u)(x) = C_{N,1-s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{\left(|x-y|^2 + \epsilon^2\right)^{\frac{N+2-2s}{2}}} dy.$$

- Convergence: $\mathcal{L}_\epsilon^{1-s}[u] \to (-\Delta)^{1-s} u$ pointwise in $\mathbb{R}^N$ as $\epsilon \to 0$
- Generalized Stroock-Varopoulos Inequality for $\mathcal{L}_\epsilon^s$: Let $u \in H_\epsilon^s(\mathbb{R}^N)$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi \in C^1(\mathbb{R})$ and $\psi' \geq 0$. Then

$$\int_{\mathbb{R}^N} \psi(u) \mathcal{L}_\epsilon^s[u] dx \geq \int_{\mathbb{R}^N} \left| (\mathcal{L}_\epsilon^s)^{\frac{1}{2}} [\Psi(u)] \right|^2 dx,$$

where $\psi' = (\Psi')^2$.

Approximating problem

We consider the approximating problem $(P_{\epsilon\delta\mu R})$

$$\begin{cases} (U_1)_t = \delta\Delta U_1 + \nabla\cdot((U_1+\mu)^{m-1}\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1]) & \text{for } (x,t)\in B_R\times(0,T) \\ U_1(x,0)=\widehat{u_0}(x) & \text{for } x\in B_R, \\ U_1(x,t)=0 & \text{for } x\in\partial B_R,\ t\in(0,T) \end{cases}$$

with parameters $\epsilon, \delta, \mu, R > 0$.

• Existence of solutions of $(P_{\epsilon\delta\mu R})$ → fixed points of the following map given by the Duhamel's formula

$$\mathcal{T}(v)(x,t) = e^{\delta t\Delta}u_0(x) + \int_0^t \nabla e^{\delta(t-\tau)\Delta}\cdot G(v)(x,\tau)d\tau,$$

where $G(v) = (v+\mu)^{m-1}\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^s[v]$ and $e^{t\Delta}$ is the Heat Semigroup.

• Existence of solutions of (P)

$$(P_{\epsilon\delta\mu R}) \xrightarrow{\epsilon\to 0} (P_{\delta\mu R}) \xrightarrow{R\to\infty} (P_{\delta\mu}) \xrightarrow{\mu\to 0} (P_\delta) \xrightarrow{\epsilon\to 0} (P).$$

Existence of weak solutions for $m > 1$

Theorem. Let $1 < m < \infty$, $N \geq 1$, and let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and nonnegative. Then we prove:

- Existence of a weak solution $u \geq 0$ of Problem (M1) with initial data u_0.
- Conservation of mass: For all $0 < t < T$: $\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx$.
- L^∞ estimate: $\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty$, $\forall 0 < t < T$
- L^p energy estimate: For all $1 < p < \infty$ and $0 < t < T$ we have

$$\int_{\mathbb{R}^N} u^p(x,t)dx + C(m,p)\int_0^t \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-1}{2}}\right|^2 dxdt \leq \int_{\mathbb{R}^N} u_0^p(x)dx.$$

- Second energy estimate: For all $0 < t < T$ we have

$$\frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{-\frac{s}{2}} u(t)\right|^2 dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} \left|\nabla(-\Delta)^{-s} u(t)\right|^2 \leq \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{-\frac{s}{2}} u_0\right|^2 dx.$$

Smoothing effect

Theorem

Let $u \geq 0$ be a weak solution of Problem (M1) with $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, as constructed before. Then

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m,p}\, t^{-\gamma_p} \|u_0\|_{L^p(\mathbb{R}^N)}^{\delta_p} \quad \text{for all} \quad t > 0,$$

where $\gamma_p = \frac{N}{(m-1)N+2p(1-s)}$, $\delta_p = \frac{2p(1-s)}{(m-1)N+2p(1-s)}$.

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$\Rightarrow$ Existence of weak solutions for only $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$.

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Existence for measure data

Let $1 < m < \infty$, $N \geq 1$ and $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Then there exists a weak solution $u \geq 0$ of Problem (M1) s.t. the smoothing effect holds for $p = 1$:

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m}\, t^{-\gamma} \mu(\mathbb{R}^N)^{\delta} \quad \text{for all} \quad t > 0,$$

Moreover:

- Regularity: $u \in L^\infty((\tau,\infty) : L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (\tau,\infty)) \cap L^\infty((0,\infty) : \mathcal{M}^+(\mathbb{R}^N))$ for all $\tau > 0$
- Conservation of mass: For all $0 < t < T$, $\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} d\mu(x)$.
- L^p energy estimate: For all $1 < p < \infty$ and $0 < \tau < t < T$ we have

$$\int_{\mathbb{R}^N} u^p(x,t)dx + C(m,p)\int_\tau^t \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-1}{2}}\right|^2 dxdt \leq \int_{\mathbb{R}^N} u^p(x,\tau)dx.$$

- Second energy estimate: For all $0 < \tau < t < T$ we have

$$\frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{-\frac{s}{2}} u(t)\right|^2 dx + \int_\tau^t \int_{\mathbb{R}^N} u^{m-1} \left|\nabla(-\Delta)^{-s} u(t)\right|^2 \leq \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{-\frac{s}{2}} u(\tau)\right|^2 d$$

Positivity results

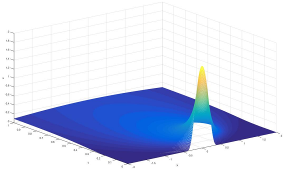


Figure: $m = 1.5,\ s = 0.25$

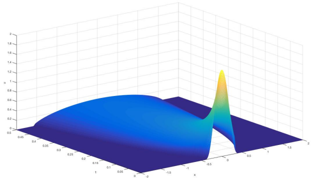


Figure: $m = 2,\ s = 0.25$

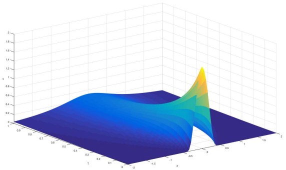
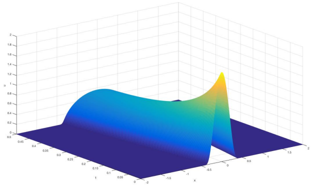


Figure: $m = 1.5,\ s = 0.75$



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Infinite vs. finite speed of propagation

Finite speed of propagation for $m \geq 2$

Theorem

Assume that u_0 has compact support and $u(x,t)$ is bounded for all x, t. Then $u(\cdot, t)$ is compactly supported for all $t > 0$.

If $0 < s < 1/2$ and

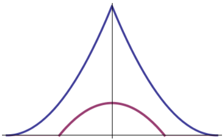
$$u_0(x) \leq U_0(x) := a(|x| - b)^2,$$

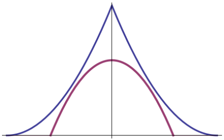
then there is a constant C large enough s.t.

$$u(x,t) \leq \mathcal{U}(x,t) := a(Ct - (|x| - b))^2.$$

For $1/2 \leq s < 2 \Rightarrow C = C(t)$ is an increasing function of t.

Consequence: Free Boundaries!





Infinite speed of propagation for $m \in (1, 2)$ and $N = 1$

Theorem. Let $m \in (1, 2)$, $s \in (0, 1)$ and $N = 1$. Let u be the solution of Problem (PMFP) with initial data $u_0 \geq 0$ radially symmetric and monotone decreasing in $|x|$. Then $u(x, t) > 0$ for all $t > 0$, $x \in \mathbb{R}$.

Idea of the proof: Prove that
$v(x, t) = \int_{-\infty}^{x} u(y, t)dy > 0 \quad$ for $t > 0$, $x \in \mathbb{R}$.

The integrated problem

$$\partial_t v = -|v_x|^{m-1}(-\Delta)^{1-s} v \quad (IP)$$

The initial data is given by
$v_0(x) = \int_{-\infty}^{x} u_0(y)dy$.

Initial data $v_0(x)$ satisfies:
$v_0(x) = 0$ for $x < -\eta$,
$v_0(x) = M$ for $x > \eta$,
$v_0'(x) \geq 0$ for $x \in (-\eta, \eta)$.

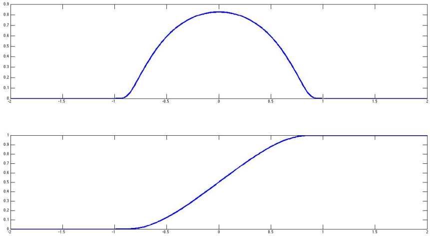


Figure: Typical initial data for models (P) and (IP).

Transformations

Porous Medium with Fractional Pressure

$$V_t = \nabla \cdot (V^{m-1} \nabla (-\Delta)^{-s} V)$$

Fractional Porous Medium Equation

$$U_t + (-\Delta)^{\sigma} U^q = 0$$

Self Similar Solutions

$$V(x,t) = t^{-\alpha_2} F_2(t^{-\beta_2} x) \quad \text{with}$$
$$\alpha_2 = N\beta_2, \quad \beta_2 = \frac{1}{N(m-1)+2-2s},$$

$$\nabla \cdot (F_2^{m-1} \nabla (-\Delta)^{-s} F_2) = -\beta_2 \nabla \cdot (y F_2).$$

Self Similar Solutions

$$U(x,t) = t^{-\alpha_1} F_1(t^{-\beta_1} x) \quad \text{with}$$
$$\alpha_1 = N\beta_1, \quad \beta_1 = \frac{1}{N(q-1)+2\sigma},$$

$$(-\Delta)^s F_1^q = \beta_1 \nabla \cdot (y F_1).$$

D. Stan, F. del Teso and J.L. Vázquez, Non.Analysis, 2015.

J. L. Vázquez. JEMS 2014.

Theorem. Transformation of self similar solutions

If $q > N/(N+2\sigma)$, $\sigma \in (0,1)$ then

$$F_2(x) = (\beta_1/\beta_2)^{\frac{q}{1-q}} (F_1(x))^q$$

is a solution to the profile equation (PF2) if we put $m = (2q-1)/q$ and $s = 1-\sigma$.

- FPME: The profile $F_1(y)$ is a smooth, positive and radial function in $\mathbb{R}^N$, $F'(r) < 0$ and for $q > N/(N+2\sigma)$, $F_1(y) \sim |y|^{-(N+2\sigma)}$ for large $|y|$.

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- Consequences for (M1):

  $F_2 > 0$ and

$$F_2(x) \sim C|x|^{-(N+2-2s)/(2-m)} \quad \text{if } m \in ((N-2+2s)/N, 2)\,.$$

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$\Longrightarrow$ Infinite Propagation for Self-Similar Solutions of the PMFP in $\mathbb{R}^N$, $N \geq 1$, $m < 2$.

Similar results are proved for smaller values of parameters.

Related models for $m > 2$

$$v_t + v^2(-\Delta)^{1-s} v^{\overline{m}} = 0, \quad x \in \mathbb{R}^N, \; t > 0. \qquad \text{(M3)}$$

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Self-similar solutions:

$$v(x,t) = t^{-a}\psi(y), \quad y = x\,t^{b}, \quad a = bN, \ b = \frac{1}{N(\overline{m}+1) + 2(1-s)},$$

$$b(N\psi - y\nabla\psi) = \psi^2(-\Delta)^{1-s}\psi^{\overline{m}}.$$

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$$b(N\psi - y\nabla\psi) = \psi^2(-\Delta)^{1-s}\psi^{\overline{m}}.$$

Let V is a self Similar Solution to (M1): $V(x,t) = t^{-\alpha_2}F_2(t^{-\beta_2}x)$

Transformation. From (M1) we obtain self-similar solutions to (M3)

$$\psi := \frac{1}{c}F_2^{m-2}, \qquad \overline{m} = \frac{1}{m-2},\; c = \left(\frac{\beta_2}{b}\right)^{1/(m-1)}$$

Asymptotic Behavior

Asymptotic Behavior

Uniqueness of weak solutions is proved in the one-dimensional case.

Theorem

Let $m \in (1, +\infty)$, $s \in (0, 1)$, $N = 1$ and $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Then there exists a unique weak solution to Problem (M1).

The proof is done via the integrated problem.

Theorem

Let $m \in (1, \infty)$, $s \in (0,1)$ and $N = 1$. Assume that $u_0 \in L^1(\mathbb{R})$, $\|u_0\|_{L^1(\mathbb{R})} = M$ and let u be the corresponding weak solution of (M1). Then

$$t^{\frac{N(1-\frac{1}{p})}{(m-1)N+2-2s}} \|u(\cdot,t) - U_M(\cdot,t)\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to \infty$$

for any $p > 1$, where U_M is the unique self-similar solution of (M1) with initial data $\mu = M\delta_0$.

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Notice that U_M can be transformed into a self-similar solution of (M2) (for $m < 2$) or (M3) (for $m > 2$).

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Proof. Rescale $u_\lambda(x,t) = \lambda^N u(\lambda x, \lambda^b t)$, for $\lambda > 0$. Then use the four step method:
(I) compactness estimates + convergence in $L^2(B_R)$,
(II) tail control in $\mathbb{R}^N \setminus B_R$,
(III) convergence in $L^p(\mathbb{R})$,
(IV) Put $t = 1$ and then λ is the new time.

Conclusions

• We proved existence of suitable weak solutions of problem (M1) and finite vs infinite speed of propagation depending on m. Asymptotic behavior in $1D$ by means of an integrated version of the problem.

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• Another pending issue is continuity of weak solutions. In the case $m = 2$ Hölder continuity is proved in [CSV, CV2].

• We proved existence of suitable weak solutions of problem (M1) and finite vs infinite speed of propagation depending on m. Asymptotic behavior in $1D$ by means of an integrated version of the problem.

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• Recently, the problem posed in a bounded domain was considered in [NguyenVaz 2017] for dimension $N \geq 1$. Further work is to be done on that issue.

MULTUMESC!