Multiple-fragmentation stochastic processes driven by a spatial flow

[Processus de fragmentation multiple conduit par un courant spatial]

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The stochastic model for the one dimensional fragmentation

• The model describes the fragmentation phenomenon for an infinite particles system.

• Each particle is characterized by its size and, at some random times, it can split into two particles by conserving the total mass.

• If $c(t, x)$ the concentration of particles of size x at time t in the system then the evolution in time of $c(t, x)$ is governed by the **fragmentation equation:**

$$\begin{cases} \dfrac{\partial}{\partial t}c(t,x) = \displaystyle\int_x^1 F(x,y-x)c(t,y)\mathrm{d}y - \frac{1}{2}c(t,x)\int_0^x F(y,x-y)\mathrm{d}y, \\ \hfill \text{for all } t \geqslant 0 \text{ and } x \in [0,1], \\ c(0,x) = c_0(x) \text{ for all } x \in [0,1]. \end{cases}$$

• The **fragmentation kernel** $F : [0,1]^2 \longrightarrow \overline{\mathbb{R}}_+$ is a symmetric function, $F(x,y)$ represents the rate of fragmentation of a particle of size $x+y$ into two particles of size x and y.

• In the first line of the fragmentation equation, the first term on the right hand side is counting the creation of particles of size x, due to the fragmentation of particles of larger size, say y, with $y > x$, into two parts x and $y-x$.

• The second term counts for the particles of size x which disappears after splitting into two smaller particles of size y and $x-y$, for $y < x$.

• We emphasize a pure jump Markov process on $\mathbb{R}_+$, denoted by $(X_t)_{t\geqslant 0}$ whose law is the solution, in some sense, to the fragmentation equation. This process will describe the evolution of the size of a typical particle in the system.

• We study fragmentation equations for which the total mass will be preserved; for all $t\geqslant 0$

$$\int_0^1 xc(t,x)\mathrm{d}x = \int_0^1 xc_0(x)\mathrm{d}x$$

and from normalization reason $\int_0^1 xc_0(x)\mathrm{d}x = 1$, thus if we denote by $p(t,x) = xc(t,x)$; $x \in [0,1]$, then $p(t,x)\mathrm{d}x$ is a probability measure for each t.

Stochastic approach of the coagulation/fragmentation models

[Deaconu, Fournier, *Stoch. Process. Appl.*, 2002]

[Deaconu, Fournier, Tanré, *Ann. Probab.,* 2002]

[Fournier, Giet, *J. Stat. Physis*, 2003]

[Bertoin, *Random Fragmentation and Coagulation Processes*. Cambridge Univ. Press, 2006]

[L.B., Deaconu, Lupaşcu, *Stoch. Proc. Appl.* 2015]

[L.B., Deaconu, Lupaşcu, *J. Stat. Physics* 2016]

Weak solution of the fragmentation equation

A family $(Q_t)_{t\geqslant 0}$ of probability measures on $[0,1]$ is **solution in the weak sense** of the fragmentation equation if:

$$\langle Q_t, \phi\rangle = \langle Q_0, \phi\rangle + \int_0^t \langle Q_s, \mathcal{F}\phi\rangle \mathrm{d}s,\ \phi \in \mathcal{C}^1([0,1]), t\geqslant 0, \qquad (0.1)$$

where $\langle Q_t, \phi\rangle = \int_0^1 \phi(y) Q_t(\mathrm{d}y)$ and for any $x \in [0,1]$

$$\boxed{\mathcal{F}\phi(x) = \int_0^x [\phi(x-y) - \phi(x)] \frac{x-y}{x} F(y, x-y)\mathrm{d}y}.$$

- We describe the process having the distribution $(Q_t)_{t\geqslant 0}$.

Weak solution of the stochastic differential equation of fragmentation (SDEF)

Let Q_0 be a probability measure on $[0,1]$.

X is a **weak solution of the stochastic differential equation of fragmentation** (abbreviated (SDEF)) if:

• $X = (X_t)_{t \geqslant 0}$ is an adapted process on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geqslant 0}, \mathbb{P})$ whose paths belong to $\mathbb{D}([0,+\infty),[0,1])$.

• $\mathcal{L}(X_0) = Q_0$.

• There exists a Poisson measure $N(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u)$ adapted to $(\mathcal{G}_t)_{t\geqslant 0}$ on $[0, +\infty) \times [0, 1) \times [0, 1)$ respectively with intensity measure $\mathrm{d}s\mathrm{d}y\mathrm{d}u$ such that the following **stochastic differential equation** holds:

$$X_t = X_0 - \int_0^t \int_0^1 \int_0^1 y \mathbb{1}_{\{y\in(0,X_{s-})\}} \mathbb{1}_{\{u\leqslant \frac{X_{s-}-y}{X_{s-}} F(y,X_{s-}-y)\}} N(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u)$$

• The process X can be seen as the size of a sort of typical particle. This means that at some random instants the typical particle breaks into two smaller particles: we thus subtract y from X for some $y \in (0, X_{s-})$ at rate $F(y, X_{s-} - y)\frac{X_{s-}-y}{X_{s-}}$.

• *There exists a weak solution $X = (X_t)_{t\geqslant 0}$ of the (SDEF)*; [Fournier, Giet, 2003].

Let $Q_t := \mathcal{L}(X_t)$, $t\geqslant 0$. Then the family $\{Q_t\}_{t\geqslant 0}$ is a solution in the weak sense of the fragmentation equation.

• If Q_t, $t\geqslant 0$, has a density with respect to the Lebesgue measure on $[0, 1]$ and if we set $c(t, x) := \frac{\mathrm{d}Q_t}{x\mathrm{d}x}$, then $c(t, x)$ is a solution of the fragmentation equation.

Multiple-fragmentation: Introduction

A fragmentation model closer to the real life should also take into account the spatial position and movement of the fragments.

For coagulation of particles with position and spatial diffusion see [Deaconu, Fournier, *Stochastic Proc. Appl.*, 2002].

Aim: To study stochastic multiple-fragmentation processes driven by a spatial flow.

• The final goal is actually to make a numerical simulation of the time evolution of a system of particles located on an Euclidean surface.

• We take into account not only the fragmentation of the mass of a particle, but also of the kinetic energy and of the velocity.

• There is a loss of energy which occurs.

Multiple-fragmentation processes and their SDEs

We consider n first order integral operators $\mathcal{F}^k$, $k = \overline{1, n}$, $n \in \mathbb{N}^*$, $\mathcal{F}^k : C_l(E_k) \longrightarrow \mathcal{B}_b(E_k)$,

$$\mathcal{F}^k f(x) = \int_{E_k} [f(z) - f(x)] N_x^k(dz), \ \ f \in C_l(E_k), x \in E_k,$$

where N^k is a kernel on $E_k := [0, l_k]$, $l_k \in \mathbb{R}^*$, with $\sup_x \int_{E_k} |x - z| N_x^k(dz) < \infty$.

In our application to fragmentation processes the kernels N^k will be either of the form

$$N_x^k(dz) = \frac{z}{x} F^k(x - z, z) 1_{(0,x)}(z) dz,$$

where F^k are continuous fragmentation kernels or bounded kernels on E_k of the form

$$N_x^k = \lambda_k (\beta_k x \delta_{\beta_k x} + (1 - \beta_k) x \delta_{(1-\beta_k)x}),$$

where $\beta_k \in (0, 1)$ is a rupture factor and λ_k is a constant which depends on β_k, $\lambda_k := \frac{\theta_k}{4}(\beta_k^2 + (1 - \beta_k)^2)$, with $0 < \theta_k \leqslant 1$. The factor θ_k is a "rate of loss of fragmentation sizes".

n-dimensional fragmentation process

• Assume that each $\mathcal{F}^k$, $1 \leqslant k \leqslant n$, is the generator of a right Markov process $X^k = (X_t^k)_{t \geqslant 0}$ with state space E_k.
X_k is actually a one-dimensional fragmentation process.

• We consider the *n-dimensional fragmentation process* $X = (X_t)_{t \geqslant 0}$ with state space $E := \prod_{k=1}^n E_k$, defined as $X_t := (X_t^1, \ldots, X_t^n)$.
It is also a right Markov process and its generator is the operator $\mathcal{F} : C_l(E) \longrightarrow \mathcal{B}_b(E)$, defined as

$$\mathcal{F}g(x) = \int_E [g(z) - g(x)] N_x(dz), \ \ g \in C_l(E), x \in E,$$

where N is the kernel on E defined by

$$Ng(x) = \sum_{k=1}^n \int_{E_k} g(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_n) N_{x_k}^k(dz).$$

Let further $n_c \in \mathbb{N}$, $0 \leqslant n_c \leqslant n$, it will represent the number of continuous fragmentation kernels we shall take into account.

Assume that each $\mathcal{F}^k$ is given by a continuous fragmentation kernel F^k on E_k if $1 \leqslant k \leqslant n_c$ and by a bounded discontinuous kernel N^k if $n_c < k \leqslant n$.

Stochastic differential equation of n-dimensional fragmentation

$$\begin{cases} X_t^k = X_0^k - \\ \int_0^t \int_0^{l_k} \int_0^1 y_k 1_{[0<y_k<X_s^k]} 1_{[u \leqslant \frac{X_{s-}^k - y_k}{X_{s-}^k} F^k(y_k, X_{s-}^k - y_k)]} p(\sum_{i=1}^n dsdy_i du), \\ \qquad \text{if } \; 1 \leqslant k \leqslant n_c, \\ \\ X_t^k = X_0^k - \\ \int_0^t \int_0^{l_k} \left((1-\beta) X_{s-}^k \mathbb{1}_{[\frac{y_k}{\beta\lambda_o} < X_{s-}^k \leqslant 1]} + \beta X_{s-}^k \mathbb{1}_{[\frac{y_k}{\lambda_o} < X_{s-}^k \leqslant \frac{y_k}{\beta\lambda_o}]} \right) p(\sum_{i=1}^n dsdy_i) \\ \qquad \text{if } n_c < k \leqslant n, \end{cases}$$

where $p(\sum_{i=1}^{n_c} dsdy_i du)$ is a Poisson measure with intensity $q = \sum_{i=1}^{n_c} dsdy_i du$.

Proposition

The stochastic differential equation of n-dimensional fragmentation with the initial distribution δ_x, $x \in E$, has a weak solution which is equal in distribution with the n-dimensional fragmentation process $(X, \mathbb{P}^x)$.

Multiple-fragmentation driven by a spatial Markov process

Let F be a Lusin topological space and consider a right Markov process $Y = (Y_t, \Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}^z)$ with state space F.
Let $(T_t)_{t \geqslant 0}$ be the transition function of the process Y, $T_t f(z) = \mathbb{E}^z(f(Y_t); t < \zeta),\ t \geqslant 0, z \in F$.

Let $d : F \longrightarrow (0, \infty)$ be a (finely) continuous function and consider the continuous additive functional $A^d = (A^d_t)_{t \geqslant 0}$ induced by d: $A^d_t = \int_0^t d(Y_s)ds$, $t \geqslant 0$.
Assume that $0 < A^d_t(\omega) < \infty$ for all $\omega \in \Omega'$ and $t > 0$. Consider the *inverse* $(\tau^d_t)_{t \geqslant 0}$ of A^d, $(\tau^d_t)(\omega) := \inf\{s > 0 : A^d_s(\omega) > t\}$.

Let $Y^d = (Y^d_t, \Omega', \mathbb{P}^x)$ be the Markov process obtained from Y by time change with the inverse of A^d,

$$Y^d_t = Y_{\tau^d_t},\ t \geqslant 0$$

and let $(L^d, \mathcal{D}(L^d))$ be the generator of Y^d, $L^d = \frac{1}{d}L$.

Let $X = (X_t, \Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^x)$ be a right Markov process with state space E, a Lusin topological space.

Let $c : E \longrightarrow (0, \infty)$ be a finely continuous function $c \geqslant \alpha > 0$.
Consider $X^c = (X_t^c, \Omega, \mathbb{P}^x)$ the right Markov process with state space E, obtained from X by time change with the inverse τ_t^c, $t \geq 0$, of the additive functional $A^c = (A_t^c)_{t \geqslant 0}$ induced by c:
$A_t^c = \int_0^t c(X_s) \mathrm{d}s$, $\tau_t^c := \inf\{s \geqslant 0 : A_s^c > t\}$, $X_t^c := X_{\tau_t^c},\ t \geqslant 0$.

Let further $(D, \mathcal{D}(D))$ be the weak generator of X and $(S_t)_{t \geqslant 0}$ (resp. $(S_t^c)_{t \geqslant 0}$) the transition function on E of the process X (resp. of the process X^c).
If $\varphi : F \times E \longrightarrow \mathbb{R}$ we write $\varphi_x(z) := \varphi(z, x) =_z \varphi(x)$ for all $(z, x) \in F \times E$.

Proposition. Let $Z' = (Z'_t, \Omega' \times \Omega, \mathbb{P}^z \times \mathbb{P}^x)$ be the cartesian product of the processes Y^d and X^c, $Z'_t = (Y^d_t, X^c_t)$.

Let further $a(z,x) := \dfrac{1}{d(z)c(x)}$ and $Z = (Z_t, \Omega' \times \Omega, \mathbb{P}^z \times \mathbb{P}^x)$ be the right Markov process obtained from Z' by time change with the continuous additive functional $A^a = (A^a_t)_{t \geqslant 0}$ of Z', induced by a: $A^a_t = \int_0^t \dfrac{1}{c(X^c_s)d(Y^d_s)} \mathrm{d}s,\ t \geq 0$.

Assume that c and d are bounded functions and let $(\bar{N}, \mathcal{D}(\bar{N})$ be the generator of Z.

Let $\mathcal{D}_c(L)$ (resp. $\mathcal{D}_c(D)$) be the space of those elements from $\mathcal{D}(L)$ (resp. $\mathcal{D}(D)$) which are finely continuous on F (resp. on E).

Then the following assertions hold.

(i) $\mathcal{D}_c(L) \times \mathcal{D}_c(D)$ is a subspace of $\mathcal{D}_c(\bar{N})$ and for every $\varphi \in \mathcal{D}_c(L) \times \mathcal{D}_c(D)$

$$\bar{N}\varphi(z,x) = c(x)L\varphi_x(z) + d(z)D_z\varphi(x) \quad \text{for all } (z,x) \in F \times E.$$

(ii) For every $\varphi \in \mathcal{D}_c(L) \times \mathcal{D}_c(D)$ the process

$$\left(\varphi(Z_t) - \int_0^t [c(X_u'')L\varphi_{X_u''}(Y_u'') + d(Y_u'')D(_{Y_u''}\varphi)(X_u'')]du\right)_{t \geqslant 0}$$

is an $(\mathcal{F'}_t \times \mathcal{F}_t)_{t \geqslant 0}$-martingale under $\mathbb{P}^z \times \mathbb{P}^x$ for all $(z,x) \in F \times E$, where $Z_t = (Y_t'', X_t'') \in F \times E$.

Random fragmentation and flow of particles on a surface

Geometrical description of the surface. We describe the surface $\mathcal{S}_b$, given through a general parametric representation by $\boldsymbol{r}_b(x_1, x_2) = B_1(x)\boldsymbol{c}_1 + B_2(x)\boldsymbol{c}_2 + B_3(x)\boldsymbol{c}_3$, where $x = (x_1, x_2)$ are the parametric coordinates belonging to a two dimensional domain $\Omega \subset \mathbb{R}^2$ and $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ which is the cartesian basis with the vertical in the $\boldsymbol{c}_3$ direction.
Let $\Pi_b = \Pi_b(x)$ be the two dimensional vectorial space tangent to the bottom surface $\mathcal{S}_b$ (i.e., $\Pi_b(x) := Sp\{\boldsymbol{b}_1(x), \boldsymbol{b}_2(x)\}$).
We denote by $\boldsymbol{b}_1, \boldsymbol{b}_2$ the covariant physical basis and by β_3 the unit normal vector on $\mathcal{S}_b$ and by $\boldsymbol{k}$ the curvature tensor. We denote by
$\boldsymbol{b}_1(x) = \frac{\partial \boldsymbol{r}_b}{\partial x_1}(x), \quad \boldsymbol{b}_2(x) = \frac{\partial \boldsymbol{r}_b}{\partial x_2}(x)$
the covariant basic vectors and by
$g_{11} = |\boldsymbol{b}_1|^2, g_{22} = |\boldsymbol{b}_2|^2, g_{12} = \boldsymbol{b}_1 \cdot \boldsymbol{b}_2$, the covariant fundamental magnitudes of the first order.

We denote by g the element of area in the tangent plane

$$g(x) = \sqrt{g_{11}g_{22} - g_{12}^2}.$$

We denote also by β_1, β_2 the covariant physical basis and by β_3 the unit normal vector on $\mathcal{S}_b$:

$$\beta_1 = \frac{1}{L_1}\boldsymbol{b}_1, \quad \beta_2 = \frac{1}{L_2}\boldsymbol{b}_2, \quad \beta_3 = \frac{\boldsymbol{b}_1 \wedge \boldsymbol{b}_2}{g},$$

where L_1, L_2 are the Lamé coefficients defined as $L_1(x) = \sqrt{g_{11}}, \quad L_2(x) = \sqrt{g_{22}}$.

To introduce the contra-variant tangent basis, denoted by $\boldsymbol{b}^1, \boldsymbol{b}^2$, and the contra-variant fundamental magnitudes of the first order

$$g^{11} = |\boldsymbol{b}^1|^2 = \frac{g_{22}}{g^2}, \quad g^{22} = |\boldsymbol{b}^2|^2 = \frac{g_{11}}{g^2}, \quad g^{12} = \boldsymbol{b}^1 \cdot \boldsymbol{b}^2 = -\frac{g_{12}}{g^2}.$$

The fundamental magnitudes of the second order are given by

$$k_{11} = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_1^2} \cdot \beta_3, \quad k_{22} = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_2^2} \cdot \beta_3, \quad k_{12} = k_{21} = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_1 \partial x_2} \cdot \beta_3.$$

defining the curvature tensor $\boldsymbol{k} := k_{ij} \boldsymbol{b}^i \otimes \boldsymbol{b}^j = k_j^i \boldsymbol{b}_i \otimes \boldsymbol{b}^j = k^{ij} \boldsymbol{b}_i \otimes \boldsymbol{b}_j$, with summation on i and j from 1 to 2, while the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_i \partial x_j} \cdot \boldsymbol{b}^k.$$

Motion equations between two fragmentation moments

We shall consider a motion of N particles on the surface $\mathcal{S}_b$ during the time interval $[t_0, t_1]$.
During this time interval the number of particles N will be constant.
We denote by $\boldsymbol{r}^p(t) = \boldsymbol{r}_b(x^p(t))$, $x^p(t) = (x_1^p(t), x_2^p(t))$ the position of each particle p at $t \in [t_0, t_1]$.
We can compute the velocity and the acceleration of each particle to be:

$$\boldsymbol{v}^p = \frac{d}{dt}\boldsymbol{r}^p = \dot{x}_1^p \boldsymbol{b}_1(x^p) + \dot{x}_2^p \boldsymbol{b}_2(x^p),$$

$$\boldsymbol{a}^p = \frac{d}{dt}\boldsymbol{v}^p = \ddot{x}_1^p \boldsymbol{b}_1(x^p) + \dot{x}_1 \frac{\partial \boldsymbol{b}_1(x^p)}{\partial x_1}\dot{x}_1 + \dot{x}_1 \frac{\partial \boldsymbol{b}_1(x^p)}{\partial x_2}\dot{x}_2 + \ddot{x}_2 \boldsymbol{b}_2(x^p) +$$

$$\dot{x}_2 \frac{\partial \boldsymbol{b}_2(x^p)}{\partial x_1}\dot{x}_1 + \dot{x}_2 \frac{\partial \boldsymbol{b}_2(x^p)}{\partial x_2}\dot{x}_2.$$

Let $\mathbf{F}^p = \mathbf{F}^p(\boldsymbol{r}^1, ..., \boldsymbol{r}^N, \boldsymbol{v}^1, ..., \boldsymbol{v}^N)$ be the force acting on the particle p.
The movement of a particle is described by the Newton evolution equation

$$m^p \boldsymbol{a}^p = \mathbf{F}^p(\boldsymbol{r}^1, ..., \boldsymbol{r}^N, \boldsymbol{v}^1, ..., \boldsymbol{v}^N) + M^p(t)\beta_3, \quad \text{for all} \quad p = 1, ..., N, \tag{0.2}$$

where M^p is the reaction force of the surface $\mathcal{S}_b$.

Multiplying now by $\boldsymbol{b}^1$, respectively $\boldsymbol{b}^2$, we obtain the system:

$$\begin{cases} \ddot{x}_1^p + \Gamma^1_{11}(\dot{x}_1^p)^2 + \Gamma^1_{22}(\dot{x}_2^p)^2 + \dot{x}_1^p \dot{x}_2^p(\Gamma^1_{21} + \Gamma^1_{12}) = \\ \qquad \mathbf{F}^p(\boldsymbol{r}^1, ..., \boldsymbol{r}^N, \boldsymbol{v}^1, ..., \boldsymbol{v}^N) \cdot \boldsymbol{b}^1(x^p) \\ \\ \ddot{x}_2^p + \Gamma^2_{11}(\dot{x}_1^p)^2 + \Gamma^2_{22}(\dot{x}_2^p)^2 + \dot{x}_1^p \dot{x}_2^p(\Gamma^1_{21} + \Gamma^1_{12}) = \\ \qquad \mathbf{F}^p(\boldsymbol{r}^1, ..., \boldsymbol{r}^N, \boldsymbol{v}^1, ..., \boldsymbol{v}^N) \cdot \boldsymbol{b}^2(x^p) \end{cases} \tag{0.3}$$

for all $p = 1, ..., N$.
We have to complete the above second order nonlinear system equations with the initial conditions

$$\boldsymbol{r}^p(t_{0+}) = \boldsymbol{r}_0^p \quad \text{and} \quad \boldsymbol{v}^p(t_{0+}) = \boldsymbol{v}_0^p, \quad \text{for all} \quad p = 1, ..., N. \tag{0.4}$$

Random binary fragmentation

Our aim is to introduce a random binary fragmentation process for N particles characterized by their mass $m^1, ..., m^N$, their positions $\boldsymbol{r}^1, ..., \boldsymbol{r}^N$, and their velocities $\boldsymbol{v}^1, ..., \boldsymbol{v}^N$.
Let define t_1 the first random fragmentation time greater then t_0, having a Poisson distribution. The position $\boldsymbol{r}_1(t_1-)$ and the velocity $\boldsymbol{v}_1(t_1-)$ are computed solving the nonlinear system (0.3), described in the previous section. Let us describe what we mean by a fragmentation process. A fragmentation process $\mathcal{F}^p$ of a particle p is the function which associates at each $(m^p, \boldsymbol{r}^p, \boldsymbol{v}^p)$ the couple $[(m_1^p, \boldsymbol{r}_1^p, \boldsymbol{v}_1^p),\ (m_2^p, \boldsymbol{r}_2^p, \boldsymbol{v}_2^p)]$,

$$\mathcal{F}^p(m^p, \boldsymbol{r}^p, \boldsymbol{v}^p) = [(m_1^p, \boldsymbol{r}_1^p, \boldsymbol{v}_1^p),\ (m_2^p, \boldsymbol{r}_2^p, \boldsymbol{v}_2^p)],$$

which represents the masses, the positions, and the velocities of the resulting two particles at $t = t_1+$.

Let us describe our choice of fragmentation process.
First of all, we suppose that the position at $t = t_1+$ of the resulting two particles coincide with the position of the mother particle, i.e.

$$\boldsymbol{r}_1^p(t_1+) = \boldsymbol{r}_2^p(t_1+) = \boldsymbol{r}^p(t_1). \tag{0.5}$$

For the mass and the velocity fragmentation we choose a random procedure. We take the mass of the fragments to be

$$m_1^p = \xi m^p, \qquad m_2^p = (1-\xi)m^p, \tag{0.6}$$

where ξ is a fixed uniform random variable. Note that we have the mass conservation property, i.e.

$$m_1^p + m_2^p = m^p.$$

Concerning the velocity fragmentation the process needs more physichal restrictions. We assume that a part $(1-\theta)\mathcal{E}^p$ of the particle kinetic energy $\mathcal{E}^p = \frac{1}{2}m^p|\boldsymbol{v}^p(t_1-)|^2$ is lost in the fragmentation process, where $\theta \in (0,1)$ is a fixed rupture parameter.

The resulting two fragments will have the kinetic energy $\mathcal{E}_1^p = \theta\gamma\mathcal{E}^p$ and $\mathcal{E}_2^p = \theta(1-\gamma)\mathcal{E}^p$ with $\mathcal{E}_1^p + \mathcal{E}_2^p = \theta\mathcal{E}^p$, where γ is a fixed uniform random variable. If the first energy corresponds to the first choice of the mass fragmentation then we have

$$\begin{cases} \mathcal{E}_1^p = \frac{1}{2} m_1^p |\boldsymbol{v}_1^p(t_1+)|^2 = \theta\gamma\mathcal{E}^p \\ \\ \mathcal{E}_2^p = \frac{1}{2} m_2^p |\boldsymbol{v}_2^p(t_1+)|^2 = \theta(1-\gamma)\mathcal{E}^p, \end{cases} \tag{0.7}$$

The other choice of the corresponding energy is similar with the last one by replacing γ with $1-\gamma$.

Finally, as for the mass distribution we will suppose that we have an additive law:

$$\boldsymbol{v}_1^p(t_1+) + \boldsymbol{v}_2^p(t_1+) = \boldsymbol{v}^p(t_1+). \tag{0.8}$$

Let prove now that if the fragmentation parameters θ, ξ, γ satisfy the following inequality

$$\theta\gamma\xi(1-\xi)^2 - \theta^2\gamma^2(1-\xi)^2 - \theta\gamma(1-\xi)b - \frac{1}{4}b^2 \geq 0, \qquad (0.9)$$

$$\text{with } b := (\xi - \xi^2 - \gamma(\xi + \theta - 2\xi\theta)),$$

there exist exactly two possibilities for the resulting fragmentation process.
We also give a method to compute the resulting velocity $\boldsymbol{v}_1^p(t_1+)$ and $\boldsymbol{v}_2^p(t_1+)$ from the fragmentations laws (0.6),(0.7), and (0.8).

In that follows we will write $\boldsymbol{v}^p, \boldsymbol{v}_1^p, \boldsymbol{v}_2^p$ instead of $\boldsymbol{v}^p(t_1+), \boldsymbol{v}_1^p(t_1+), \boldsymbol{v}_2^p(t_1+)$.
Replacing the masses m_1^p and m_2^p in (0.7) we have the system

$$\begin{cases} \xi|\boldsymbol{v}_1^p|^2 = \theta\gamma|\boldsymbol{v}^p|^2 \\ \\ (1-\xi)|\boldsymbol{v}_2^p|^2 = (1-\theta)\gamma|\boldsymbol{v}^p|^2, \end{cases} \tag{0.10}$$

and we can compute the velocities norms to be

$$\begin{cases} |\boldsymbol{v}_1^p| = \sqrt{\frac{\theta\gamma}{\xi}}|\boldsymbol{v}^p| \\ \\ |\boldsymbol{v}_2^p| = \sqrt{\frac{1-\theta}{1-\xi}}\gamma|\boldsymbol{v}^p|, \end{cases} \tag{0.11}$$

From
$|\boldsymbol{v}^p|^2 = |\boldsymbol{v}_1^p|^2 + |\boldsymbol{v}_2^p|^2 + 2\boldsymbol{v}_1^p \cdot \boldsymbol{v}_2^p = \frac{\theta\gamma}{\xi}|\boldsymbol{v}^p|^2 + \frac{1-\theta}{1-\xi}\gamma|\boldsymbol{v}^p|^2 + 2\boldsymbol{v}_1^p \cdot \boldsymbol{v}_2^p$,
we get

$$2\boldsymbol{v}_1^p \cdot \boldsymbol{v}_2^p = |\boldsymbol{v}^p|^2 \left(1 - \gamma \frac{\xi + \theta - 2\xi\theta}{\xi(1-\xi)}\right),$$

and since

$$\begin{cases} \boldsymbol{v}^p \cdot \boldsymbol{v}_1^p = |\boldsymbol{v}_1^p|^2 + \boldsymbol{v}_2^p \cdot \boldsymbol{v}_1^p \\ \\ \boldsymbol{v}^p \cdot \boldsymbol{v}_2^p = |\boldsymbol{v}_2^p|^2 + \boldsymbol{v}_2^p \cdot \boldsymbol{v}_1^p, \end{cases} \tag{0.12}$$

we can compute $\boldsymbol{v}^p \cdot \boldsymbol{v}_1^p$ and $\boldsymbol{v}^p \cdot \boldsymbol{v}_2^p$.

Let decompose the velocities $\boldsymbol{v}^p$ and $\boldsymbol{v}_1^p$ in the local basis $\boldsymbol{b}_1, \boldsymbol{b}_2$ as $\boldsymbol{v}^p = \dot{x}^p \boldsymbol{b}_1 + \dot{y}^p \boldsymbol{b}_2$ and $\boldsymbol{v}_1^p = \dot{x}_1^p \boldsymbol{b}_1 + \dot{y}_1^p \boldsymbol{b}_2$.
We can decompose the two velocities in the basis $\boldsymbol{v}^p$ and $(\boldsymbol{v}^p)^\perp = -\dot{y}^p \boldsymbol{b}^1 + \dot{x}^p \boldsymbol{b}^2$ as

$$\begin{cases} \boldsymbol{v}_1^p = \alpha_1 \boldsymbol{v}^p + \beta_1 (\boldsymbol{v}^p)^T \\ \\ \boldsymbol{v}_2^p = \alpha_2 \boldsymbol{v}^p + \beta_2 (\boldsymbol{v}^p)^T, \end{cases} \tag{0.13}$$

where the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ have to be computed. From (0.8) we have $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 0$ and using (0.12), we get $\boldsymbol{v}_1^p \cdot \boldsymbol{v}^p = \alpha_1 |\boldsymbol{v}^p|^2$, so

$$\alpha_1 = \frac{\boldsymbol{v}_1^p \cdot \boldsymbol{v}^p}{|\boldsymbol{v}^p|^2} \quad \text{and} \quad \alpha_2 = 1 - \alpha_1.$$

Multiplying the first equation from the system (0.13) by $(\boldsymbol{v}^p)^T$, we obtain

$$\boldsymbol{v}_1^p \cdot (\boldsymbol{v}^p)^T = \beta_1 ||(\boldsymbol{v}^p)^T||^2 = \beta_1 g^{11} (\dot{x}^p)^2 + \dot{y}^p g^{22} + 2g^{12} \dot{x}^p \dot{y}^p.$$

The parameter β_1 can be computed from $\boldsymbol{v}_1^p \cdot \boldsymbol{v}_2^p = \alpha_1\alpha_2|\boldsymbol{v}^p|^2 + \beta_1\beta_2|(\boldsymbol{v}^p)^T|^2 = -\beta_1^2|(\boldsymbol{v}^p)^T|^2 + \frac{\boldsymbol{v}_1^p \cdot \boldsymbol{v}^p}{|\boldsymbol{v}^p|^2} + \boldsymbol{v}_2^p \cdot \boldsymbol{v}^p$ if and only if (0.9) holds. In this case we deal with two solutions given by

$$\beta_1 = \pm\beta, \text{ with } \beta = \frac{\sqrt{\alpha_1(1-\alpha_1)|\boldsymbol{v}^p|^2 - \boldsymbol{v}_1^p \cdot \boldsymbol{v}_2^p}}{|(\boldsymbol{v}^p)^T|^2}. \qquad (0.14)$$

Replacing $|(\boldsymbol{v}^p)^T|^2 = (\dot{x}^p)^2 g^{22} + (\dot{y}^p)^2 g^{11} + 2\dot{x}^p\dot{y}^p g^{12}$ in (0.14) we get β_1.
The velocity for a first one fragment $\boldsymbol{v}_1^p$ satisfies

$$\boldsymbol{v}_1^p = \alpha_1(\dot{x}^p\boldsymbol{b}_1 + \dot{y}^p\boldsymbol{b}_2) \pm \beta(-\dot{y}^p\boldsymbol{b}^1 + \dot{x}^p\boldsymbol{b}^2) = \dot{x}_1^p\boldsymbol{b}_1 + \dot{y}_1^p\boldsymbol{b}_2. \quad (0.15)$$

Multiplying the last equality by $\boldsymbol{b}_1$ we obtain

$$\dot{x}_1^p\boldsymbol{b}_1 \cdot \boldsymbol{b}_1 + \dot{y}_1^p\boldsymbol{b}_2 \cdot \boldsymbol{b}_2 = \alpha_1(\dot{x}^p\boldsymbol{b}_1 \cdot \boldsymbol{b}_1 + \dot{y}^p\boldsymbol{b}_2 \cdot \boldsymbol{b}_1) \mp \beta\dot{y}^p.$$

That means

$$g_{11}\dot{x}_1^p + g_{12}\dot{y}_1^p = \alpha_1(\dot{x}^p g_{11} + \dot{y}^p g_{12}) \mp \beta\dot{y}^p.$$

Multiplying the equation (0.15) by $\boldsymbol{b}_2$ we obtain

$$g_{12}\dot{x}_1^p + g_{22}\dot{y}_1^p = \alpha_1(\dot{x}^p g_{12} + \dot{y}^p g_{22}) \pm \beta\dot{x}^p.$$

From the last two equations we can compute the components $\dot{x}_1^p, \dot{y}_1^p$ of the velocity $\boldsymbol{v}_1^p$ for the first fragment:

$$\begin{cases} \dot{x}_1^p = \dfrac{g_{11}[\alpha_1(\dot{x}^p g_{11} + \dot{y}^p g_{12}) - \beta \dot{y}^p] - g_{12}[\alpha_1(\dot{x}^p g_{12} + \dot{y}^p g_{22}) \pm \beta \dot{x}^p]}{g_{11}g_{22} - g_{12}^2} \\ \dot{y}_1^p = \dfrac{g_{11}[\alpha_1(\dot{x}^p g_{12} + \dot{y}^p g_{22}) + \beta \dot{x}^p] - g_{12}[\alpha_1(\dot{x}^p g_{11} + \dot{y}^p g_{12}) \mp \beta \dot{y}^p]}{g_{11}g_{22} - g_{12}^2}. \end{cases} \tag{0.16}$$

Following the same procedure for $\boldsymbol{v}_2^p$ we get the corresponding expressions of the components of second fragment

$$\begin{cases} \dot{x}_2^p = \dfrac{g_{11}[\alpha_2(\dot{x}^p g_{11} + \dot{y}^p g_{12}) - \beta \dot{y}^p] - g_{12}[\alpha_2(\dot{x}^p g_{12} + \dot{y}^p g_{22}) \mp \beta \dot{x}^p]}{g_{11} g_{22} - g_{12}^2} \\ \dot{y}_2^p = \dfrac{g_{11}[\alpha_2(\dot{x}^p g_{12} + \dot{y}^p g_{22}) + \beta \dot{x}^p] - g_{12}[\alpha_2(\dot{x}^p g_{11} + \dot{y}^p g_{12}) \pm \beta \dot{y}^p]}{g_{11} g_{22} - g_{12}^2}. \end{cases} \tag{0.17}$$

Algorithm for the random binary fragmentation

Problem setting

• set the bottom surface $\mathcal{S}_b$ as in Section 1
• set $\xi, \gamma \sim \mathcal{U}([0,1])$, and $\theta \in (0,1)$, satisfying the inequaliy (0.9)
• set T_{final} and N_{max}

Iterative algorithm

Input at t_k+

• N^k the number of particles
• the mass of each particle
• the position of each particle
• the velocity of each particle

The transition from t_k to $t_{k+1}+$

• compute the fragmentation time t_{k+1}, having a Poisson distribution

• solve the differential equations on the interval time (t_k, t_k+) with the previous initial conditions, by using a numerical approach

• *fragmentation process:* *i*) $N^{k+1} = 2N^k$

ii) set the mass of each particle according to (0.6)

iii) choose the position of each particle according to (0.5)

iv) set the energy of each particle according to (0.7)

v) compute the velocities of each particle at $t_{k+1}+$ following the formula (0.16) and (0.17)

• update the input at $t_{k+1}+$

STOP: $t_k > T_{final}$ or $N^k > N_{max}$