Stochastic equation of fragmentation and branching processes related to avalanches

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Aim:

- A unifying approach for constructing branching processes in continuous time, on the space of all fragmentation sizes induced by:
  – continuous fragmentation kernels;
  – a discontinuous one, related to the avalanches.
- A stochastic model for the fragmentation phase of an avalanche.
- To establish a specific stochastic equation of fragmentation.
- Numerical approach.

Branching processes on the space of all fragmentation sizes

Consider $E := [0,1]$ and fix a sequence $(d_n)_{n \geqslant 1} \subset (0,1)$ of thresholds for the fragmentation dimensions, strictly decreasing to zero.

Let $E_n := [d_n, 1]$ and $E'_n := [d_{n+1}, d_n)$, $E'_0 := E_1$.

Hypotheses:

- For each $n \geqslant 1$ there exists a right Markov process X^n with state space E_n and transition function $(P^n_t)_{t \geqslant 0}$ such that $P^{n+1}_{t,x} = P^n_{t,x}$ for all $n \geqslant 1$, $t \geqslant 0$, and $x \in E_n$.

- For every $n \geqslant 0$ the set E'_n is absorbing in E_{n+1} with respect to process X^{n+1}.

The space $\widehat{E}$ of finite configurations of $E \subset [0,1]$:

$$\widehat{E} := \left\{ \mu \text{ positive measure on } E \;:\; \mu = \sum_{k=1}^{m} \delta_{x_k},\; x_1, \ldots, x_m \in E \right\} \cup \{\mathbf{0}\}.$$

Branching process: a Markov process X on $\widehat{E}$ is a *branching process* if for each two measures $\mu_1, \mu_2 \in \widehat{E}$:

$$X^{\mu_1 + \mu_2} \overset{(\mathrm{d})}{=} X^{\mu_1} + X^{\mu_2}.$$

For all $n \geqslant 1$ it is given a Markovian kernel B^n from $\widehat{E_n}$ to E_n such that

$$\sup_{x \in E_n} B^n l_1(x) < \infty \text{ and } B_x^{n+1} = B_x^n \text{ for all } x \in E_n, n \geqslant 1,$$

where for a function $f \in p\mathcal{B}(E_n)$ we consider the mapping $l_f : \widehat{E_n} \longrightarrow \mathbb{R}_+$ defined as $l_f(\mu) := \int f \mathrm{d}\mu, \ \mu \in \widehat{E_n}$.

- The kernel B^n will control the non-local branching of a forthcoming process with state space the finite configurations of E_n:

Theorem 1. *For each $n \geqslant 1$, there exists a branching process $\widehat{X}^n$ on $\widehat{E_n}$, induced by the base process X^n on E_n and by the kernel B^n.*

[L. Beznea & O. Lupașcu, *Trans. Amer. Math. Soc.*, 2016]

The space $S^{\downarrow}$ of all fragmentation sizes (J. Bertoin):

$$S^{\downarrow} := \{\mathbf{x} = (x_k)_{k\geqslant 1} \subseteq [0,1] : (x_k)_{k\geqslant 1} \text{ decreasing}, \lim_k x_k = 0\}.$$

• $\mathbf{x} \in S^{\downarrow}$: "the sizes of the fragments resulting from the split of some block with unit size" .

• We identify a sequence $\mathbf{x} = (x_k)_{k\geqslant 1}$ from $S^{\downarrow}$ with the σ-finite measure $\mu_{\mathbf{x}}$ on $[0,1]$, defined as

$$\mu_{\mathbf{x}} := \begin{cases} \sum_k \delta_{x_k} & , \text{ if } \mathbf{x} \neq \mathbf{0}, \\ \mathbf{0} & , \text{ if } \mathbf{x} = \mathbf{0}, \end{cases}$$

• The mapping $\mathbf{x} \longmapsto \mu_{\mathbf{x}}$ identifies each $\widehat{E_n}$ with a subset of $S^{\downarrow}$.

Define the mapping $\alpha_n : S^{\downarrow} \longmapsto \widehat{E_n}$ as

$$\alpha_n(\mathbf{x}) := \mu_{\mathbf{x}}|_{E_n}, \ \mathbf{x} = \mu_{\mathbf{x}} \in S^{\downarrow}.$$

and

$$S_\infty := \{(\mathbf{x}^n)_{n\geqslant 1} \in \prod_{n\geqslant 1} \widehat{E_n} : \ \mathbf{x}^n = \alpha_n(\mathbf{x}^m) \text{ for all } m > n\geqslant 1\}.$$

- The mapping $i : S^{\downarrow} \longmapsto S_\infty$, defined as

$$i(\mathbf{x}) := (\alpha_n(\mathbf{x}))_{n\geqslant 1}, \ \mathbf{x} \in S^{\downarrow},$$

is a bijection.

Construction of fragmentation-branching processes on the space of all finite configurations

Theorem 2. *(i) There exists a branching semigroup $(\widehat{P_t})_{t \geqslant 0}$ on $S^{\downarrow}$, obtain as the projective limit of the sequence $(\widehat{P_t^n})_{n \geqslant 1}$, i.e.,*

- the sequence of probability measures $(\widehat{P_{t,\mathbf{x}_n}^n})_{n \geqslant 1}$ is projective with respect to $(\widehat{E_n}, \alpha_n)_{n \geqslant 1}$ for every $\mathbf{x} \in S^{\downarrow}$, $\mathbf{x}_n := \alpha_n(\mathbf{x}) \in \widehat{E_n}$, $n \geqslant 1$, and $t > 0$;

- its limit is $\widehat{P_{t,\mathbf{x}}}$, that is

$$\widehat{P_{t,\mathbf{x}_{n+1}}^{n+1}} \circ \alpha_n^{-1} = \widehat{P_{t,\mathbf{x}_n}^n} \text{ and } \widehat{P_{t,\mathbf{x}}} \circ \alpha_n^{-1} = \widehat{P_{t,\mathbf{x}_n}^n} \text{ for all } n \geqslant 1.$$

(ii) Suppose that for every $n \geqslant 1$ $(\widehat{P_t^n})_{t \geqslant 0}$ is the transition function of a Markov process with state space $\widehat{E_n}$ and $P_t^n(1_{(x,1]})(x) = 0$ for all $t \geqslant 0$ and $x \in E_n$.

Then $(\widehat{P_t})_{t \geqslant 0}$ is the transition function of a branching process $\widehat{X} = (\widehat{X}_t)_{t \geqslant 0}$ with state space $S^{\downarrow}$, and the following **fragmentation property** *holds: if $\mathbf{x} \in \widehat{E}$ and $\mathbf{y} \in S^{\downarrow}$, $\mathbf{y} \leqslant \mathbf{x}$, then $\mathbb{P}^{\mathbf{y}}$–a.s. $\widehat{X}_t \leqslant \mathbf{x}$.*

Sketch of the proof.

- For each $t \geqslant 0$ the sequence of probability measures $(\widehat{P^n_{t,\mathbf{x}_n}})_{n\geqslant 1}$ is projective.

- Since $S^{\downarrow}$ is identified with S_∞, by Bochner-Kolmogorov Theorem there exists a transition function $(\widehat{P_t})_{t\geqslant 0}$ on $S^{\downarrow}$, as the limit of the $(\widehat{P^n_t})_{t\geqslant 0}$.

- $(\widehat{P_t})_{t\geqslant 0}$ is the transition function of a branching process with state space $S^{\downarrow}$.

[L. Beznea, M. Deaconu & O. Lupașcu, *Stochastic Process. Appl.*, 2015]

The case of a continuous fragmentation kernel

Fragmentation kernel: $F : (0,1]^2 \longrightarrow \overline{\mathbb{R}}_+$, symmetric

- $F(x,y)$: the rate of fragmentation of a particle of size $x+y$ into two particles of size x and y.

Hypothesis

- F is continuous from $[0,1]^2$ to $\mathbb{R}_+ \cup \{+\infty\}$.

The rate of loss of mass of particles of mass x:

$$\psi(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x y(x-y)F(y,x-y)\mathrm{d}y & \text{if } x>0, \\ 0 & \text{if } x=0. \end{cases}$$

- F is such that ψ is continuous on $[0,1]$.

- **Example:** $F(x,y) := x+y$

[Fournier, Giet, *J. Stat. Physis*, 2003].

Let

$$\mathcal{F}f(x) = \int_0^x [f(x-y) - f(x)] \frac{x-y}{x} F(y, x-y) \mathrm{d}y, \;\; x \in [0,1].$$

We have $\mathcal{F} = \widetilde{N^F}$, where

$$\widetilde{N}f(x) := \int_E [f(y) - f(x)] N_x(\mathrm{d}y)$$

for all $f \in bp\mathcal{B}(E)$ and $x \in E$, and

$$N^F f(x) := \int_0^x f(z) \frac{z}{x} F(x-z, z) \mathrm{d}z.$$

Truncated fragmentation kernels:

$$F_n(x,y) := 1_{(d_n,1]}(x \wedge y) F(x,y), \;\; x, y \in E := [0,1], n \geq 1.$$

- $\widetilde{N^{F^n}}$ is the generator of a jump Markov process with state space E (since N^{F^n} is bounded kernel).

Proposition 3. *If $k \geqslant 1$ then $\underset{\sim}{E_k}$ is an absorbing set with respect to the Markov process generated by $\widetilde{N^{F^k}}$ and let $X^k = (X^k_t)_{t \geqslant 0}$ be the restriction of this process to E'_k.*

(i) *Let L^n be the infinitesimal generator of $(P^n_t)_{t \geqslant 0}$, regarded as a C_0-semigroup of contractions on $b\mathcal{B}(E_n)$.*
Then for every $\phi \in b\mathcal{B}(E_n)$ we have

$$\mathsf{L}^n \phi = \sum_{k=1}^n 1_{E'_{k-1}} \mathcal{F}_k \overline{\phi},$$

where $\mathcal{F}_k$ is the operator $\mathcal{F}$ with F_k instead of F. In particular, for each probability ν on E'_n, the process

$$\phi(X^n_t) - \sum_{k=1}^n \int_0^t (1_{E'_{k-1}} \mathcal{F}_k \overline{\phi})(X^n_s) \mathrm{d}s, \ \ t \geqslant 0,$$

is a martingale under $\mathbb{P}^\nu = \int \mathbb{P}^x \nu(\mathrm{d}x)$, with respect to the filtration of X^n.

(ii) *For every $x \in E_n$ the* **stochastic equation of fragmentation** *with the initial distribution δ_x,*

$$\boxed{X_t = X_0 - \int_0^t \int_0^1 \int_0^1 y 1_{\{y \in (0, X_{s-})\}} 1_{\{u \leqslant \frac{X_{s-} - y}{X_{s-}} F_n(y, X_{s-} - y)\}} p(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u), \ t \geqslant 0,}$$

has a solution which is equal in distribution with $(X^n, \mathbb{P}^x)$, where $p(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u)$ is an adapted Poisson measure on $[0, +\infty) \times [0, 1) \times [0, 1)$ with intensity measure $\mathrm{d}s\mathrm{d}y\mathrm{d}u$.

The case of (discontinuous) fragmentation kernels for avalanches

- $F : (0,1]^2 \longrightarrow \overline{\mathbb{R}}_+$, a fragmentation kernel
- Assume that there exists a function $\Phi : (0,\infty) \longrightarrow (0,\infty)$ such that

$$F(y,\alpha) = \Phi\left(\frac{y}{\alpha}\right),\ \forall\, y,\alpha > 0, \qquad \text{and} \qquad \Phi(x) = \Phi\left(\frac{1}{x}\right),\ \forall\, x > 0.$$

- **Example:** Let r, $0 < r < 1$, and consider the fragmentation kernel $F^r : [0,1]^2 \longrightarrow \mathbb{R}_+$, defined as

$$F^r(x,y) := \begin{cases} \frac{1}{2}(\delta_r(\frac{x}{y}) + \delta_{1/r}(\frac{x}{y})), \text{ if } & x,y > 0, \\ 0 \qquad\qquad\qquad\qquad\quad , \text{ if } & xy = 0. \end{cases}$$

We have $F^r(x,y) = \Phi^r(\frac{x}{y})$ for all $x,y > 0$, where $\Phi^r : (0,\infty) \longrightarrow (0,\infty)$ is defined as

$$\Phi^r(z) := \frac{1}{2}(\delta_r(z) + \delta_{1/r}(z)), \quad z > 0.$$

- Φ^r is not continuous.

- By approximating the function Φ^r with a sequence of continuous functions, one can see that the kernel N^{F^r} associated with F^r is:

$$N_x^{F^r} := \lambda_o(\beta x \delta_{\beta x} + (1-\beta) x \delta_{(1-\beta)x}),$$

where $\lambda_o := \frac{\beta^2+(1-\beta)^2}{4}$ with $\beta := \frac{r}{1+r}$.

- Assume that $d_1 < \beta \leqslant 1/2$ and $d_{n+1}/d_n < \beta$ for all $n \geqslant 1$. Then $E_n = \bigcup_{k=1}^n E'_{k-1}$.

- Define the kernel N_n^r on E_n as $N_n^r f := \sum_{k=1}^n 1_{E'_{k-1}} N^{F^r}(f 1_{E'_{k-1}})$ for all $f \in bp\mathcal{B}(E_n)$.

- The integral operator $\mathcal{F}_n^r f(x) := \widetilde{N_n^r} f(x) = \int_{E_n} [f(y) - f(x)] N_n^r(\mathrm{d}y)$ for all $f \in bp\mathcal{B}(E_n)$, $x \in E_n$. is the generator of a jump Markov process $X^{r,n} = (X_t^{r,n})_{t \geqslant 0}$.

- Its transition function is $P_t^{r,n} := e^{\mathcal{F}^r{}_n t}$, $t \geqslant 0$.

- For every $x \in [0,1]$ let $E_{\beta,x} := \{\, \beta^i (1-\beta)^j x : \ i,j \in \mathbb{N}\} \cup \{0\}$ and $E_{\beta,x,n} := E_{\beta,x} \cap E_n$ for $n \geqslant 1$.

Theorem 4. *E_n is an absorbing set with respect to the Markov process $X^{r,n}$ with state space E_n and the following assertions hold.*

(i) For every $\phi \in bp\mathcal{B}(E_n)$ and each probability ν on E_n, the process $\phi(X_t^{r,n}) - \int_0^t \mathcal{F}_n^r \phi(X_s^{r,n})\mathrm{d}s$, $t \geqslant 0$, is a martingale under $\mathbb{P}^\nu$.

(ii) If $x \in E_n$, $n \geqslant 1$, then the following **stochastic equation of fragmentation for avalanches** *with the initial distribution δ_x has a solution which is equal in distribution with $(X^{r,n}, \mathbb{P}^x)$:*

$$X_t = X_0 - \int_0^t \int_0^\infty p(\mathrm{d}\alpha, \mathrm{d}s) X_{\alpha-} \sum_{k=1}^n ((1-\beta) 1_{[\frac{d_k}{\beta} \leqslant X_{\alpha-} < d_{k-1},\ \frac{s}{\lambda_o \beta} < X_{\alpha-}]}$$

$$+ \beta 1_{[\frac{d_k}{1-\beta} \leqslant X_{\alpha-} < \frac{d_k}{\beta},\ \frac{s}{1-\beta} < X_{\alpha-} \leqslant \frac{s}{\beta}] \cup [\frac{d_k}{\beta} \leqslant X_{\alpha-} < d_{k-1},\ s < X_{\alpha-} \leqslant \frac{s}{\beta}]}),\ t \geqslant 0,$$

where $p(\mathrm{d}\alpha, \mathrm{d}s)$ is a Poisson measure with intensity $q := \mathrm{d}\alpha\mathrm{d}s$.

(iii) If $x \in E_n$ then $\mathbb{P}^x$-a.s. $X_t^{r,n} \in E_{\beta,x,n}$ for all $t \geqslant 0$.

Sketch of the proof.

- Define the bounded kernel K^n on $\mathbb{R}$;
$K^n_x := \lambda_o x[\beta 1_{E'_{k-1}}(\beta x)\delta_{(\beta-1)x} + (1-\beta)1_{E'_{k-1}}((1-\beta)x)\delta_{-\beta x}]$ if $x \in E'_{k-1}$, $1 \leqslant k \leqslant n$, and $K^n_x := 0$ else.

- $\mathcal{F}^r_n f(x) = \int_{\mathbb{R}}[f(x+y) - f(x)]K^n_x(\mathrm{d}y)$ for all $f \in bp\mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$.

- To prove the existence of the corresponding stochastic differential equation we use the existence of the solution of the martingale problem associated to the operator $\mathcal{F}^r_n$.

[L. Beznea, M. Deaconu and O. Lupașcu, Stochastic equation of fragmentation and branching processes related to avalanches, J. Stat. Phys., 162 (2016), 824-841]

Fragmentation-branching processes related to avalanches

- Let $d > 0$, define further the kernel $B_d : p\mathcal{B}(\widehat{[d,1]}) \longrightarrow p\mathcal{B}([d,1])$ as

$$B_d h(x) := \frac{6}{x^3} \int_0^x y(x-y)_d (h^{(2)})(y) \mathrm{d}y, \; x \in [d,1].$$

- The kernel B_d is Markovian and consider the kernel B^n from $\widehat{E_n}$ to E_n defined as

$$B^n h := \sum_{k=1}^{n} 1_{E'_{k-\mathbf{1}}} B_{d_k} h, \; h \in bp\mathcal{B}(\widehat{E_n}).$$

- Define the Markovian kernel $B^{r,n}$ from $\widehat{E_n}$ to E_n as

$$B^{r,n} h(x) := \frac{1}{a(x)} \sum_{1 \leqslant k \leqslant n} \sum_{E_{\beta,x} \ni y \leq x} 1_{E'_{k-\mathbf{1}}}(x)_{d_k} h(y,y) y(x-y), \; h \in bp\mathcal{B}(\widehat{E_n}), \; x \in E_n$$

where $a(x) := \sum_{E_{\beta,x} \ni y \leq x} y(x-y) < \infty$ for all $x \in E_n$.

- If $x_1, \ldots, x_k \in E$ and $\mathbf{x} = \delta_{x_1} + \ldots + \delta_{x_k} \in \widehat{E}$, we put $E_{\beta,\mathbf{x}} := \bigcup_{j=1}^{k} E_{\beta,x_j}$ and $E_{\beta,\mathbf{x},n} := \bigcup_{j=1}^{k} E_{\beta,x_j,n}$.

Corollary 5. (i) *There exists a branching process with state space* $\widehat{E_n}$, *induced by the transition function* $(P^n_t)_{t\geqslant 0}$ *and by the kernel* B^n.

(ii) *There exists a branching process* $\widehat{X^{r,n}} = (\widehat{X^{r,n}_t})_{t\geqslant 0}$ *with state space* $\widehat{E_n}$, *induced by the transition function* $(P^{r,n}_t)_{t\geqslant 0}$ *and by the kernel* $B^{r,n}$. *For every* $\mathbf{x} \in \widehat{E_n}$, $\mathbf{y} \in \widehat{E_{\beta,\mathbf{x},n}}$, *and* $t \geqslant 0$ *we have* $\mathbb{P}^{\mathbf{y}}$*–a.s.* $\widehat{X^{r,n}_t} \in \widehat{E_{\beta,\mathbf{x},n}}$.

(iii) *There exists a branching right (Markov) process with state space* $S^{\downarrow}$, *having càdlàg trajectories, as stated in Theorem 2, associated to a continuous fragmentation kernel* F.

(iv) *There exists a branching right process* $\widehat{X^r} = (\widehat{X^r_t})_{t\geqslant 0}$ *with state space* $S^{\downarrow}$, *having càdlàg trajectories, as stated in Theorem 2, associated to the discontinuous fragmentation kernel* F^r *for avalanches. For each* $\mathbf{x} \in \widehat{E}$, *the set* $S^{\downarrow}_{\beta,\mathbf{x}} := \{\mathbf{y} = (y_k)_{k\geqslant 1} \in S^{\downarrow} : y_k \in E_{\beta,\mathbf{x}}$ *for all* $k \geqslant 1\}$ *is absorbing in* $S^{\downarrow}$, *that is, if* $\mathbf{y} \in S^{\downarrow}_{\beta,\mathbf{x}}$ *then* $\mathbb{P}^{\mathbf{y}}$*–a.s.* $\widehat{X^r_t} \in S^{\downarrow}_{\beta,\mathbf{x}}$ *for all* $t \geqslant 0$.

Remark

- The last part of assertion (iv) emphasizes a **fractal property of an avalanche**, closed to its real physical properties: if we regard the fragmentation–branching process $\widehat{X^r}$ on the set $S^{\downarrow}_{\beta,\mathbf{x}}$, then independent to the sequence of sizes $\mathbf{x}$ of the initial fragments, from the moment when the avalanche started, and remaining constant in time, the ratio between the resulting fragments are all powers of β.

- A fractal model for grain size distribution of a snow avalanche is developed in the paper
[J. Faillettaz, F. Louchet, J.R. Grasso, Two-threshold model for scaling laws of noninteracting snow avalanches, Phys. Rev. Lett. (2004)]

- **The fractal character of the snow** has been studied in the paper
[V., De Biagi, B., Chiaia, B., Frigo, Fractal Grain Distribution in Snow Avalanche Deposits, *J.of Glaciology*, 2012].

1. Continuous fragmentation kernel

• approximate the process according to the probabilistic interpretation for the solution of the stochastic differential equation of fragmentation.

$$X_t = X_0 - \int_0^t \int_0^1 \int_0^1 y 1_{\{y \in (0, X_{s-})\}} 1_{\{u \leqslant \frac{X_{s-} - y}{X_{s-}} F(y, X_{s-} - y)\}} p(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u),\ t \geqslant 0,$$

X: at some random instants a particle breaks into two smaller particles, we thus subtract y from X_{s-}, $y \in (0, X_{s-})$, at rate $F(y, X_{s-} - y)\frac{X_{s-} - y}{X_{s-}}$.

The fragmentations occur at some Poisson random instants smaller than a final time T

Algorithm

Initialization: Sample the initial particle $X_0 \sim Q_0$. Set $T_0 = 0$.

Step p: Sample a random variable $y \sim \mathcal{U}([0, X_{p-1}])$.

Compute $m_p = \dfrac{X_{p-1} - y}{X_{p-1}} F(y, X_{p-1} - y)$.

Sample a random variable $S_p \sim \mathrm{Exp}(m_p)$.

Set $T_p = T_{p-1} + S_p$.

Sample a random variable $u \sim \mathcal{U}([0, 1])$.

If $u \leq m_p$, then a fragmentation occurs and set $X_p = X_{p-1} - y$.

Else set $X_p = X_{p-1}$.

Stop: When $T_p > T$.

Outcome: The approximated particle mass at T, X_{p-1}.

Numerical results

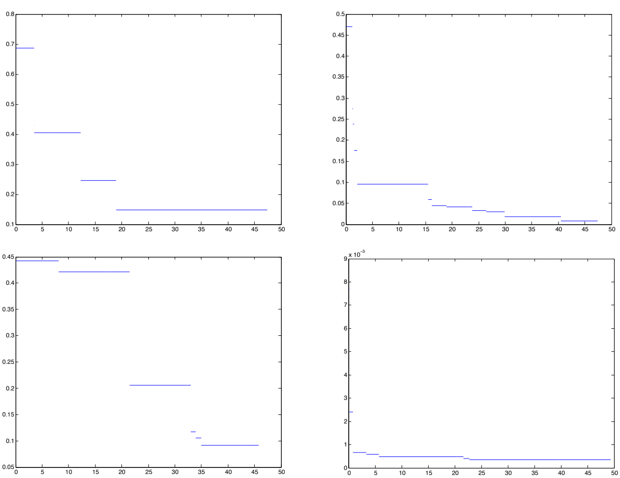

Figure : **The paths of the fragmentation process** for $F(x,y) = x + y, F(x,y) = 2, F(x,y) = 1/(x+y+1), F(x,y) = 2/(x+y)^3$, with $Q_0 \sim \mathcal{U}([0,1])$ and $T = 50$.

The evolution in time of $\mathbb{E}(X_T)$, $T = 50$

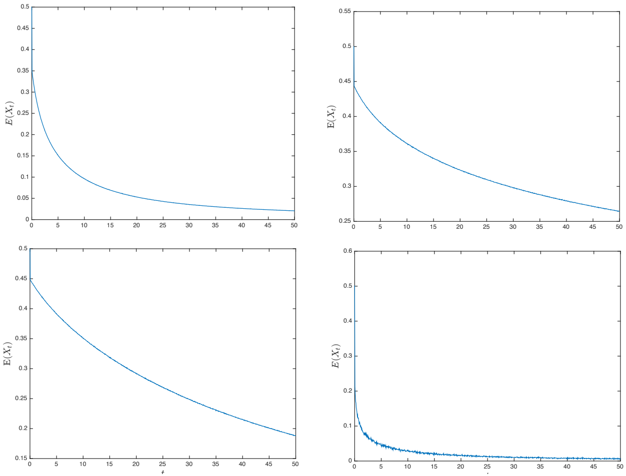

Figure : **The path of the Monte Carlo estimator** $t \mapsto \hat{I}_M(t)$ for $F(x,y) = x + y$, $F(x,y) = 2$, $F(x,y) = 1/(x+y+1)$, $F(x,y) = 2/(x+y)^3$ for $t \in [0,50]$, the Monte Carlo parameter is 10^4.

Comparison with the exact solution of the fragmentation equation

- $F(x, y) = 2$ for $x, y \in [0, \infty)$. The deterministic fragmentation equation

$$\begin{cases} \frac{\partial}{\partial t} c(t,x) & = 2\int_x^{\infty} c(t,y)\mathrm{d}y - xc(t,x) \quad \text{for all } x \geqslant 0, \\ c(0,x) & = c_0(x) \quad \text{for all } x \geqslant 0. \end{cases} \tag{1}$$

- For the initial condition $c(0, x) = e^{-x}$, the **exact solution** is

$$c(t,x) = (1+t)^2 e^{-x(1+t)} \quad \text{for all } t \geqslant 0 \text{ and } x \geqslant 0.$$

- The theoretical mean of the exact solution, which equals $\frac{2}{1+t}$, and the Monte Carlo mean for $Q_0(\mathrm{d}x) = xe^{-x}$, if $x \geqslant 0$.

t	Mean $\widehat{I_M}$	Exact solution
50	0.0415	0.0392
70	0.0286	0.0282
150	0.0126	0.0132

1. Approximate the process by using the stochastic differential equation of fragmentation, with the discontinuous kernel F^r and $\beta < \frac{1}{2}$.

Algorithm

Step 0: `Sampling the initial particle` $X_0 \sim Q_0$
Step p: `Sampling a random variable` $S_p \sim \mathrm{Exp}(\lambda_0)$
`Set` $T_p = T_{p-1} + S_p$
`Set` $X_t = X_{p-1}$ `for each` $t \in [T_{p-1}, T_p)$
`Set` $X_p = \beta X_{p-1}$ `with probability` βX_{p-1},
$X_p = (1-\beta) X_{p-1}$ `with probability` $(1-\beta) X_{p-1}$,
`or` $X_p = X_{p-1}$ `with probability` $1 - X_{p-1}$
Stop: `When` $T_p > T$.
Outcome: The approximated particle mass at time T, X_{p-1}.

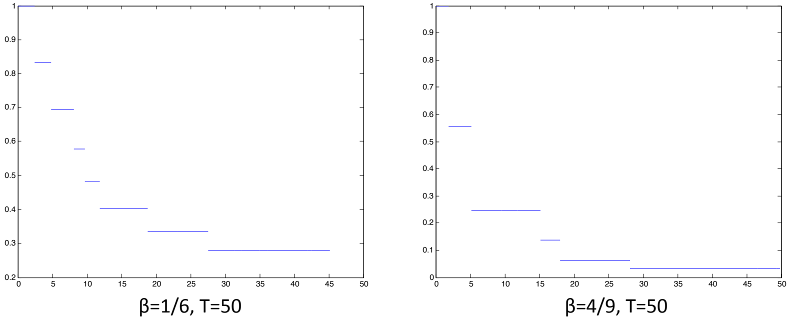

Figure : **The paths of the fragmentation process with discontinuous kernel F^r** and the size of the initial particle 1.

Compute the distribution of the branching process, $\widehat{P^n_t}$: approximate $h_t = \widehat{P^n_t}\widehat{\varphi}|_{E_n}$ by Picard iterations h^k_t, $k \geqslant 0$, $t \leqslant T$, $\varphi \leq 1$ and $P^n_t\varphi$ with iterations $p^k_t(\varphi)$, $k \geqslant 0$.

- **Initialization step:** Set $h^0_t = e^{-t}P_t\varphi$, $p^0_t(\varphi) = \varphi$

- **Step $k, k \geq 1$:**

$\mathrm{h}^k_t = e^{-t}P_t\varphi + \int_0^t e^{-(t-u)}P_{t-u}B\widehat{h^{k-1}_u}\,\mathrm{d}u\,,\;\; p^k_t(\varphi) := \sum_{i=0}^k \frac{t^i}{i!}\mathcal{F}^i\varphi, (2)$

$$h'^k_t := e^{-t}p^k_t(\varphi) + \int_0^t e^{-(t-u)}p^{k-1}_{t-u}(B\widehat{h^{k-1}_u})\,\mathrm{d}u.$$

- Fix $m \geqslant 1$ and compute $\widehat{h^k_t}|_{E^{(m)}}$, $k \geqslant 0$, which will approximate the distribution $\widehat{P_t}$ of the branching process. The connection between the transition functions of X and of the branching process:

$$\widehat{P_t}\widehat{\varphi}|_{E^{(m)}} = \widehat{h}_t|_{E^{(m)}}.$$

- **Stop** the algorithm at k and t_0, such that $\sum_{i\geqslant 0}\frac{(2t_0)^{k+i-1}+(\mu_o t_0)^{k+i-1}}{(k+i-1)!} < \frac{\varepsilon}{2m}$.

The following proposition shows that the above algorithm approximates indeed the distribution of the branching process.

Proposition *Let k having a convenient value, take $\varphi = 1_A$ with $A \in \mathcal{B}(E)$, $\mathbf{x} \in E^{(j)}$, and $j \leqslant m$. Then $\widehat{h_t'^k}(\mathbf{x})$ approximates with error less than ε the probability that the branching process $\widehat{X}$ starting from $\mathbf{x}$ lies at the time moment t in the set A^j, i.e.,*

$$|\mathbb{E}^{\mathbf{x}}(\widehat{X}_t \in A^j) - \widehat{h_t'^k}(\mathbf{x})| < \varepsilon.$$

[L. Beznea, M. Deaconu, O. Lupașcu, *Numerical approach for stochastic differential equations of fragmentation; application to avalanches*, preprint, 2017]

Numerical results.

- $m = 2,\ n = 1,\ E = E_1 = [\frac{1}{4}, 1]$,
$\beta = \frac{4}{9}, \mu_o = 2\lambda_o = \frac{6}{25}, \mathbf{x} = (1, 1, \ldots)$, and
$A := E_{\frac{4}{9},1} \cap E_1 = \{(\frac{5}{9})^2, \frac{4}{9}, \frac{5}{9}, 1\}$.

- We know $\mathbb{E}^{\mathbf{x}}(\widehat{X}_t \in \widehat{A}) = \widehat{P}_t(1_{\widehat{A}})(\mathbf{x}) = 1$.

- Using the algorithm for $k = 1, \varepsilon = 0.5, t_0 = 0.05$, we obtain $h'^{1}_{t_0} = 0.9998$.

- the approximate value of $\mathbb{E}^{\mathbf{x}}(\widehat{X}_{t_0} \in \widehat{A})$ (= the probability that the branching process $\widehat{X}$ starting from $\mathbf{x}$ lies in the set $\widehat{A}$ at the time moment t_0) is $\widehat{h'^{1}_{t_0}}(\mathbf{x}) = 0.9996$, which is indeed a value from the error interval $(1 - \varepsilon, 1]$.