# Equation stochastique de la chaleur avec un bruit fractionnaire

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1. Time-space white noise
2. Colored noise in space
3. Fractional noise in time
   - Sharp regularity in time
   - Sharp regularity in space
4. Quadratic variations

**The random noise of :**
Consider a centered Gaussian field
$W = \{W(t, A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance

$$\mathbf{EW(t, A)W(s, B) = (t \wedge s)\lambda(A \cap B), \quad t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)} \quad (1)$$

where $\lambda$ denotes the Lebesque measure.

The noise **W** is usually referred to as a *space-time white noise* because it behaves as a Brownian motion both with respect to the time and to the space variable.

To every Gaussian process one can associate a canonical Hilbert space

The canonical Hilbert space $\mathcal{H}$ associated to the Gaussian process $\mathbf{W} = \{\mathbf{W}(\mathbf{t}, \mathbf{A}); \mathbf{t} \geq 0, \mathbf{A} \in \mathcal{B}_{\mathbf{b}}(\mathbb{R}^{\mathbf{d}})\}$ is defined as closure of the linear span generated by the indicator functions $1_{[0,\mathbf{t}]\times\mathbf{A}}, \mathbf{t} \in [0, \mathbf{T}], \mathbf{A} \in \mathcal{B}_{\mathbf{b}}(\mathbb{R}^{\mathbf{d}})$ with respect to the inner product

$$\langle 1_{[0,\mathbf{t}]\times\mathbf{A}}, 1_{[0,\mathbf{s}]\times\mathbf{B}} \rangle_{\mathcal{H}} = (\mathbf{t} \wedge \mathbf{s})\lambda(\mathbf{A} \cap \mathbf{B}).$$

In our case the space $\mathcal{H}$ is $\mathbf{L}^2([0, \mathbf{T}] \times \mathbb{R}^{\mathbf{d}})$.

Also consider the stochastic partial differential equation

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial \mathbf{t}} &= \frac{1}{2}\Delta \mathbf{u} + \dot{\mathbf{W}}, \quad \mathbf{t} \in [0, \mathbf{T}], \; \mathbf{x} \in \mathbb{R}^{\mathbf{d}} \\ \mathbf{u}(0, \mathbf{x}) &= 0, \quad \mathbf{x} \in \mathbb{R}^{\mathbf{d}}, \end{aligned} \tag{2}$$

where the noise **W** is time-space white noise.

-this is the stochastic linear heat equation

Usually, the solution is defined through its mild form

$$\mathbf{u}(\mathbf{t},\mathbf{x}) = \int_0^{\mathbf{t}} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{G}(\mathbf{t}-\mathbf{s},\mathbf{x}-\mathbf{y}) d\mathbf{W}(\mathbf{s},\mathbf{y})$$

where $\mathbf{G}$ is the solution of $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} - \Delta \mathbf{u} = 0$ and the above integral is a Wiener integral with respect to $\mathbf{W}$.

The fundamental solution of the heat equation is :

$$\mathbf{G}(\mathbf{t},\mathbf{x}) = \begin{cases} (2\pi \mathbf{t})^{-\mathbf{d}/2} \exp\left(-\frac{|\mathbf{x}|^2}{2\mathbf{t}}\right) & \text{if } \mathbf{t} > 0, \mathbf{x} \in \mathbb{R}^{\mathbf{d}} \\ 0 & \text{if } \mathbf{t} \leq 0, \mathbf{x} \in \mathbb{R}^{\mathbf{d}}. \end{cases} \quad (3)$$

One needs to integrate $\mathbf{G}(\mathbf{t}-\cdot,\mathbf{x}-\cdot)$ with respect o $\mathbf{W}$.

Let us see if the solution to the heat equation exists as a $\mathbf{L}^2$ object. We will use the Fourier transform of the Green kernel $\mathbf{G}$
Fix $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$. For every $\mathbf{s}, \mathbf{t} \in [0, \mathbf{T}]$, then for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^{\mathbf{d}})$,

$$\int_{\mathbb{R}^{\mathbf{d}}} \int_{\mathbb{R}^{\mathbf{d}}} \varphi(\mathbf{x})\psi(\mathbf{y})\mathbf{dxdy} = (2\pi)^{-\mathbf{d}} \int_{\mathbb{R}^{\mathbf{d}}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mathbf{d}\xi$$

We also need the following expression of the Fourier transform of the Green kernel

$$\mathcal{F}\mathbf{G}(\mathbf{t}, \cdot)(\xi) = \exp\left(-\frac{\mathbf{t}|\xi|^2}{2}\right), \quad \mathbf{t} > 0, \xi \in \mathbb{R}^{\mathbf{d}} \tag{4}$$

where $\mathcal{F}\mathbf{G}(\mathbf{t}, \cdot)$ means the Fourier transform of the function $\mathbf{y} \to \mathbf{G}(\mathbf{t}, \mathbf{y})$.

Take $\mathbf{s} \leq \mathbf{t}$.

$$
\begin{aligned}
& \mathbf{E}\mathbf{u}(\mathbf{t},\mathbf{x})\mathbf{u}(\mathbf{s},\mathbf{x}) \\
= & \int_0^{\mathbf{t}\wedge\mathbf{s}} \mathbf{d}\mathbf{u} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{G}(\mathbf{t}-\mathbf{u},\mathbf{x}-\mathbf{y})\mathbf{G}(\mathbf{s}-\mathbf{u},\mathbf{x}-\mathbf{y})\mathbf{d}\mathbf{y} \\
= & (2\pi)^{-\mathbf{d}} \int_0^{\mathbf{s}} \mathbf{d}\mathbf{u} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{d}\xi \mathcal{F}\mathbf{G}(\mathbf{t}-\mathbf{u},\mathbf{x}-\cdot)(\xi)\overline{\mathcal{F}\mathbf{G}(\mathbf{s}-\mathbf{u},\mathbf{x}-\cdot)}(\xi) \\
= & (2\pi)^{-\mathbf{d}} \int_0^{\mathbf{s}} \mathbf{d}\mathbf{u} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{d}\xi \mathbf{e}^{-\frac{1}{2}(\mathbf{t}-\mathbf{u})|\xi|^2} \mathbf{e}^{-\frac{1}{2}(\mathbf{s}-\mathbf{u})|\xi|^2}
\end{aligned}
$$

The integral $\mathbf{d}\xi$ can be easily computed.

$$\mathbf{Eu(t,x)u(s,x)} = (2\pi)^{-\mathbf{d}} \int_0^{\mathbf{s}} \mathbf{du(t+s-2u)}^{-\frac{\mathbf{d}}{2}} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{d}\xi \mathbf{e}^{-\frac{1}{2}|\xi|^2}$$
$$= (2\pi)^{-\mathbf{d}/2} \int_0^{\mathbf{s}} \mathbf{du(t+s-2u)}^{-\frac{\mathbf{d}}{2}}.$$

Take $\mathbf{t} = \mathbf{s}$. Then

$$\mathbf{Eu(t,x)}^2 = (4\pi)^{-\mathbf{d}/2} \int_0^{\mathbf{t}} \mathbf{du(t-u)}^{-\frac{\mathbf{d}}{2}}$$

and it is obvious that the integral above is finite if and only if $\mathbf{d} = 1$. In that case,

$$\mathbf{Eu(t,x)u(s,x)} = (2\pi)^{-1/2} \left( (\mathbf{t}+\mathbf{s})^{\frac{1}{2}} - (\mathbf{t}-\mathbf{s})^{\frac{1}{2}} \right).$$

## Proposition

*The solution to the linear heat equation exists if and only if* $\mathbf{d} = 1$. *Moreover, the covariance of the solution satisfies the following : for every* $\mathbf{x} \in \mathbb{R}$ *we have*

$$\mathbf{E}\left(\mathbf{u}(\mathbf{t}, \mathbf{x})\mathbf{u}(\mathbf{s}, \mathbf{x})\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{\mathbf{t}+\mathbf{s}} - \sqrt{|\mathbf{t}-\mathbf{s}|}\right), \mathbf{s}, \mathbf{t} \in [0, \mathbf{T}]. \quad (5)$$

In conclusion :

- the (linear) heat equation admits a solution in the space of real-valued processes, if and only if $\mathbf{d} = 1$.
- This phenomenon can be explained intuitively by saying that, while the Laplacian smooths, the white noise roughens.
- If the spatial dimension $\mathbf{d}$ is larger than 2, then the roughness effect of the white noise overcomes the smoothness influence of the Laplacian.

This fact establishes an interesting connection between the law of the solution to the heat equation with time-space white noise and the bifractional Brownian motion.

## Definition

The **bifractional Brownian motion** $(\mathbf{B}_{\mathbf{t}}^{\mathbf{H},\mathbf{K}})_{\mathbf{t}\geq 0}$ is a centered Gaussian process, starting from zero, with covariance

$$\mathbf{R}^{\mathbf{H},\mathbf{K}}(\mathbf{t},\mathbf{s}) := \mathbf{R}(\mathbf{t},\mathbf{s}) = \frac{1}{2^{\mathbf{K}}}\left(\left(\mathbf{t}^{2\mathbf{H}} + \mathbf{s}^{2\mathbf{H}}\right)^{\mathbf{K}} - |\mathbf{t}-\mathbf{s}|^{2\mathbf{HK}}\right) \quad (6)$$

with $\mathbf{H} \in (0,1)$ and $\mathbf{K} \in (0,1]$.

Note that, if $\mathbf{K} = 1$ then $\mathbf{B}^{\mathbf{H},1}$ is a fractional Brownian motion (fBm) with Hurst parameter $\mathbf{H} \in (0,1)$. The fBm is a centered Gaussian process $(\mathbf{B}^{\mathbf{H}}_{\mathbf{t}})_{\mathbf{t}\in[0,1]}$ with covariance

$$\mathbf{R_H}(\mathbf{t},\mathbf{s}) = \mathbf{E}\mathbf{B}^{\mathbf{H}}_{\mathbf{t}}\mathbf{B}^{\mathbf{H}}_{\mathbf{s}} = \frac{1}{2}(\mathbf{t}^{2\mathbf{H}} + \mathbf{s}^{2\mathbf{H}} - |\mathbf{t}-\mathbf{s}|^{2\mathbf{H}}). \tag{7}$$

It can be also defined as the only Gaussian process which is self-similar and has stationary increments. When $\mathbf{H} > \frac{1}{2}$, the covariance (**??**) can be expressed as

$$\mathbf{R_H}(\mathbf{t},\mathbf{s}) = \alpha_{\mathbf{H}} \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} |\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\mathbf{dvdu}. \tag{8}$$

The restriction $\mathbf{d} = 1$ for the existence of the solution with space -time white noise is not convenient because we need to consider such models in higher dimension.
This leads the researchers in the last decades to consider other types of noise that would allow to consider higher dimension.

Let $\mu$ be a non-negative tempered measure on $\mathbb{R}^{\mathbf{d}}$, i.e. a non-negative measure which satisfies :

$$\int_{\mathbb{R}^{\mathbf{d}}} \left( \frac{1}{1+|\xi|^2} \right)^{\mathbf{l}} \mu(\mathbf{d}\xi) < \infty, \quad \text{for some } \mathbf{l} > 0.$$

Since the integrand is non-increasing in $\mathbf{l}$, we may assume that $\mathbf{l} \geq 1$ is an integer. Note that $1 + |\xi|^2$ behaves as a constant around 0, and as $|\xi|^2$ at $\infty$, and hence (**??**) is equivalent to :

$$\int_{|\xi|\leq 1} \mu(\mathbf{d}\xi) < \infty, \quad \text{and} \quad \int_{|\xi|\geq 1} \mu(\mathbf{d}\xi) \frac{1}{|\xi|^{2\mathbf{l}}} < \infty, \qquad (9)$$

for some integer $\mathbf{l} \geq 1$.

Let $\mathbf{f}$ be the Fourier transform of the measure $\mu$ i.e.

$$\int_{\mathbb{R}^{\mathbf{d}}} \mathbf{f}(\mathbf{x})\varphi(\mathbf{x})\mathbf{dx} = \int_{\mathbb{R}^{\mathbf{d}}} \mathcal{F}\varphi(\xi)\mu(\mathbf{d}\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^{\mathbf{d}}).$$

# Examples

## Example

The Riesz kernel of order $\alpha$ :

$$\mathbf{f}(\mathbf{x}) = \mathbf{R}_\alpha(\mathbf{x}) := \gamma_{\alpha,\mathbf{d}} |\mathbf{x}|^{-\mathbf{d}+\alpha}, \quad 0 < \alpha < \mathbf{d},$$

where $\gamma_{\alpha,\mathbf{d}} = 2^{\mathbf{d}-\alpha}\pi^{\mathbf{d}/2}\Gamma((\mathbf{d}-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(\mathbf{d}\xi) = |\xi|^{-\alpha}\mathbf{d}\xi$.

## Example

The Bessel kernel of order $\alpha$ :
$\mathbf{f}(\mathbf{x}) = \mathbf{B}_\alpha(\mathbf{x}) := \gamma'_\alpha \int_0^\infty \mathbf{w}^{(\alpha-\mathbf{d})/2-1} \mathbf{e}^{-\mathbf{w}} \mathbf{e}^{-|\mathbf{x}|^2/(4\mathbf{w})} \mathbf{d}\mathbf{w}, \quad \alpha > 0,$
where $\gamma'_\alpha = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$and $\mu(\mathbf{d}\xi) = (1+|\xi|^2)^{-\alpha/2}\mathbf{d}\xi$.

# Examples

## Example

The heat kernel

$$\mathbf{f}(\mathbf{x}) = \mathbf{G}_\alpha(\mathbf{x}) := \gamma''_{\alpha,\mathbf{d}} \mathbf{e}^{-|\mathbf{x}|^2/(4\alpha)}, \quad \alpha > 0,$$

where $\gamma''_{\alpha,\mathbf{d}} = (4\pi\alpha)^{-\mathbf{d}/2}$ and $\mu(\mathbf{d}\xi) = \mathbf{e}^{-\pi^2\alpha|\xi|^2}\mathbf{d}\xi$.

## Example

The Poisson kernel

$$\mathbf{f}(\mathbf{x}) = \mathbf{P}_\alpha(\mathbf{x}) := \gamma'''_{\alpha,\mathbf{d}}(|\mathbf{x}|^2 + \alpha^2)^{-(\mathbf{d}+1)/2}, \quad \alpha > 0,$$

where $\gamma'''_{\alpha,\mathbf{d}}$ is aconstant. In this case, $\mu(\mathbf{d}\xi) = \mathbf{e}^{-4\pi^2\alpha|\xi|}\mathbf{d}\xi$.

Consider the so -called *white-colored noise*, meaning a Gaussian process $\mathbf{W} = \{\mathbf{W}(\mathbf{t}, \mathbf{A}); \mathbf{t} \geq 0, \mathbf{A} \in \mathcal{B}_{\mathbf{b}}(\mathbb{R}^{\mathbf{d}})\}$ with zero mean and covariance

$$\mathbf{E}\mathbf{W}(\mathbf{t}, \mathbf{A})\mathbf{W}(\mathbf{t}, \mathbf{B}) = (\mathbf{t} \wedge \mathbf{s}) \int_{\mathbf{A}} \int_{\mathbf{B}} \mathbf{f}(\mathbf{x} - \mathbf{y}) \mathbf{dxdy}. \tag{10}$$

The noise **W** behaves as a Brownian motion with respect to the time variable and it has a correlated spatial covariance.

Here the kernel **f** should be the the Fourier of a tempered non-negative measure $\mu$ on $\mathbb{R}^{\mathbf{d}}$ (as previously described)

Under this assumption the right hand side of (**??**) is a covariance function. There are several examples of such kernels **f** (Riesz, Besel,...)

With the Gaussian process $\mathbf{W}$ we can associated a canonical Hilbert space $\mathcal{P}$. The space $\mathcal{P}$ defined as the completion of $\mathcal{E}$, the linear space generated by the indicator functions $1_{[0,\mathbf{t}]\times\mathbf{A}}$, $\mathbf{t}\in[0,\mathbf{T}], \mathbf{A}\subset\mathcal{B}(\mathbb{R}^{\mathbf{d}})$ with respect to the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{P}}=\int_0^{\mathbf{T}}\int_{\mathbb{R}^{\mathbf{d}}}\int_{\mathbb{R}^{\mathbf{d}}}\varphi(\mathbf{t},\mathbf{x})\mathbf{f}(\mathbf{x}-\mathbf{y})\psi(\mathbf{t},\mathbf{y})\mathbf{dydxdt} \tag{11}$$

One can prove that $\mathcal{P}\subset\mathbf{L}^2((0,\mathbf{T})$; (this space may contain distributions)

### Proposition

*The stochastic heat equation with white -colored noise given by (**??**) admits a unique solution if and only if*

$$\int_{\mathbb{R}^{\mathbf{d}}} \frac{1}{1+|\xi|^2}\mu(\mathbf{d}\xi) < \infty. \tag{12}$$

-this is true also for nonlinear heat equation

$$
\begin{aligned}
&\mathbf{E}\mathbf{u}(\mathbf{t},\mathbf{x})^2 \\
&= \int_0^{\mathbf{t}} \mathbf{du} \int_{\mathbb{R}^{\mathbf{d}}} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{G}(\mathbf{t}-\mathbf{u},\mathbf{x}-\mathbf{y})\mathbf{G}(\mathbf{t}-\mathbf{y},\mathbf{x}-\mathbf{y}')\mathbf{f}(\mathbf{y}-\mathbf{y}')\mathbf{dydy}' \\
&= (2\pi)^{-\mathbf{d}} \int_0^{\mathbf{t}} \mathbf{du} \int_{\mathbb{R}^{\mathbf{d}}} \mu(\mathbf{d}\xi)\mathcal{F}\mathbf{G}(\mathbf{t}-\mathbf{u},\mathbf{x}-\cdot)(\xi)\overline{\mathcal{F}\mathbf{G}(\mathbf{t}-\mathbf{u},\mathbf{x}-\cdot)}(\xi) \\
&= (2\pi)^{-\mathbf{d}} \int_0^{\mathbf{t}} \mathbf{du} \int_{\mathbb{R}^{\mathbf{d}}} \mu(\mathbf{d}\xi)\mathbf{e}^{-\frac{1}{2}(\mathbf{t}-\mathbf{u})|\xi|^2}\mathbf{e}^{-\frac{1}{2}(\mathbf{t}-\mathbf{u})|\xi|^2} \\
&= (2\pi)^{-\mathbf{d}} \int_{\mathbb{R}^{\mathbf{d}}} \mu(\mathbf{d}\xi)\frac{1}{|\xi|^2}\left(1-\mathbf{e}^{-\mathbf{t}|\xi|^2}\right)
\end{aligned}
$$

and use

$$
\mathbf{c}_{1,\mathbf{t}}\frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}\left(1-\mathbf{e}^{-\mathbf{t}|\xi|^2}\right) \leq \mathbf{c}_{2,\mathbf{t}}\frac{1}{1+|\xi|^2}
$$

The condition (**??**) allows to consider higher dimension. It is also related to the expression of the measure $\mu$ and hence to the spatial covariance kernel **f**. For example in the case of the Riesz or Bessel kernels, we have the following.

### Corollary

*Suppose that the spatial covariance is given by the Riesz kernel or by the Bessel kernel. Then the stochastic heat equation with white-colore noise admits an unique solution if and only if*

$$\mathbf{d} < 2 + \alpha$$

This implies that one can consider every dimension $\mathbf{d} \geq 1$ (recall $0 < \alpha < \mathbf{d}$).

For the Riesz kernel of order $\alpha$ one can compute explicitly the covariance of the solution : for every $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ and for every $\mathbf{s}, \mathbf{t} \in [0, \mathbf{T}]$

$$\mathbf{E}\mathbf{u}(\mathbf{t}, \mathbf{x})\mathbf{u}(\mathbf{s}, \mathbf{x}) = \mathbf{C}_0^2 \left( (\mathbf{t} + \mathbf{s})^{-\frac{\mathbf{d}-\alpha}{2}+1} - (\mathbf{t} - \mathbf{s})^{-\frac{\mathbf{d}-\alpha}{2}+1} \right).$$

where $\mathbf{C}_0$ is a positive constant.

That means that the solution is again (in law) a bifractional Brownian motion (with $\mathbf{H} = \frac{1}{2}$) and $\mathbf{K} = -\frac{\mathbf{d}-\alpha}{2} + 1$.
White noise case : $\mathbf{d} = 1, \alpha = 0$ so $\mathbf{K} = \frac{1}{2}$.

Next step : put "a color" in time for the noise

Let $(\mathbf{B}^{\mathbf{H}}_{\mathbf{t}})_{\mathbf{t}\in[0,\mathbf{T}]}$ be a fractional Brownian motion with $\mathbf{H} > \frac{1}{2}$, i.e. $\mathbf{B}^{\mathbf{H}}$ is a centered Gaussian process with covariance

$$\mathbf{R}_{\mathbf{H}}(\mathbf{t}, \mathbf{s}) = \frac{1}{2}(\mathbf{t}^{2\mathbf{H}} + \mathbf{s}^{2\mathbf{H}} - |\mathbf{t} - \mathbf{s}|^{2\mathbf{H}}).$$

Let $\mathcal{H}(0, \mathbf{T})$ be the canonical Hilbert space of $\mathbf{B}^{\mathbf{H}}$.
The inner product in $\mathcal{H}(0, \mathbf{T})$ is

$$\langle \varphi, \psi \rangle_{\mathcal{H}(0,\mathbf{T})} = \alpha_{\mathbf{H}} \int_0^{\mathbf{T}} \int_0^{\mathbf{T}} \varphi(\mathbf{u}) |\mathbf{u} - \mathbf{v}|^{2\mathbf{H}-2} \psi(\mathbf{v}) \mathbf{dvdu}$$

Now, our aim is to study the linear heat equation driven by a fractional-colored Gaussian noise

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \frac{1}{2}\Delta \mathbf{u} + \dot{\mathbf{W}}^{\mathbf{H}}, \quad \mathbf{t} \in [0, \mathbf{T}], \mathbf{x} \in \mathbb{R}^{\mathbf{d}} \tag{13}$$

with vanishing initial condition, where $\{\mathbf{W}^{\mathbf{H}}(\mathbf{t}, \mathbf{x}), \mathbf{t} \in [0, \mathbf{T}], \mathbf{x} \in \mathbb{R}^{\mathbf{d}}\}$ is a centered Gaussian noise with covariance

$$E\left(\mathbf{W}^{\mathbf{H}}(\mathbf{t}, \mathbf{A})\mathbf{W}^{\mathbf{H}}(\mathbf{s}, \mathbf{B})\right) = \mathbf{R}_{\mathbf{H}}(\mathbf{t}, \mathbf{s}) \int_{\mathbf{A}} \int_{\mathbf{B}} \mathbf{f}(\mathbf{z} - \mathbf{z}')\mathbf{dz}\mathbf{dz}', \tag{14}$$

where

- $\mathbf{R_H}(\mathbf{t},\mathbf{s}) := \frac{1}{2}(\mathbf{t}^{\mathbf{2H}} + \mathbf{s}^{\mathbf{2H}} - |\mathbf{t}-\mathbf{s}|^{\mathbf{2H}})$ is the covariance of the fractional Brownian motion with index $\mathbf{H}$
- $\mathbf{f}$ is the Fourier transform of a tempered measure $\mu$. This noise is usually called *fractional-colored* noise.

- When the structure of the noise with respect to the time variable changes and the white noise is replaced by a fractional noise, the solution does not coincide with a bifractional Brownian motion anymore .
- Some new methods are needed for analyzing the path properties of the solution with respect to the time and to the space variables.
- for the non-liunear case, the noise is not a (semi)-masrtingale anymore, classical methods does not work

The necessary and sufficient condition for the existence of the mild solution to the fractional-colored heat equation (**??**) is Namely, (**??**) has a solution $\{\mathbf{u}(\mathbf{t},\mathbf{x}), \mathbf{t} \geq 0, \mathbf{x} \in \mathbb{R}^{\mathbf{d}}\}$ that satisfies

$$\sup_{\mathbf{t}\in[0,\mathbf{T}],\mathbf{x}\in\mathbb{R}^{\mathbf{d}}} \mathrm{E}\left(\mathbf{u}(\mathbf{t},\mathbf{x})^2\right) < +\infty$$

if and only if

$$\int_{\mathbb{R}^{\mathbf{d}}} \left(\frac{1}{1+|\xi|^2}\right)^{2\mathbf{H}} \mu(\mathbf{d}\xi) < \infty. \qquad (15)$$

For the Riesz kernel $\mu(\mathbf{d}\xi) = |\xi|^{-\alpha}\mathbf{d}\xi$, the "iff" condition for the existence of the solution is

$$\mathbf{d} < 4\mathbf{H} + \alpha.$$

- If $\alpha = 0$ ("white noise in space"), then $\mathbf{d} < \mathbf{H}$, i.e. we may consider spatial dimension $\mathbf{d} = 1, 2, 3$.
- If $\alpha = 0$ and $\mathbf{H} = \frac{1}{2}$ ("space -time white noise"), then $\mathbf{d} < 2$, i.e. $\mathbf{d} = 1$.

To see the law of the process in time, we introduce the pinned string process in time $\{\mathbf{U}(\mathbf{t}), \mathbf{t} \geq 0\}$ defined by

$$\mathbf{U}(\mathbf{t}) = \int_{-\infty}^{0} \int_{\mathbb{R}^{\mathbf{d}}} \left(\mathbf{G}(\mathbf{t}-\mathbf{u}, \mathbf{x}-\mathbf{y}) - \mathbf{G}(-\mathbf{u}, \mathbf{x}-\mathbf{y})\right) \mathbf{W}^{\mathbf{H}}(\mathbf{du}, \mathbf{dy}) + \int_{0}^{\mathbf{t}} \int_{\mathbb{R}^{\mathbf{d}}} \mathbf{G}(\mathbf{t}-\mathbf{u}, \mathbf{x}-\mathbf{u}) \mathbf{W}^{\mathbf{H}}(\mathbf{du}, \mathbf{du}).$$

Note that $\mathbf{U}(0) = 0$ and $\mathbf{U}(\mathbf{t})$ can be expressed as

$$\mathbf{U}(\mathbf{t}) = \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbf{d}}} \left(\mathbf{G}((\mathbf{t}-\mathbf{u})_+, \mathbf{x}-\mathbf{y}) - \mathbf{G}((-\mathbf{u})_+, \mathbf{x}-\mathbf{y})\right) \mathbf{W}^{\mathbf{H}}(\mathbf{du}, \mathbf{dy}). \tag{16}$$

for every $\mathbf{t} \geq 0$ we have the following decomposition

$$\mathbf{u}(\mathbf{t}, \mathbf{x}) = \mathbf{U}(\mathbf{t}) - \mathbf{Y}(\mathbf{t}),$$

where

$$\mathbf{Y}(\mathbf{t}) = \int_{-\infty}^{0} \int_{\mathbb{R}^{\mathbf{d}}} (\mathbf{G}(\mathbf{t} - \mathbf{u}, \mathbf{x} - \mathbf{y}) - \mathbf{G}(-\mathbf{u}, \mathbf{x} - \mathbf{y})) \, \mathbf{W}^{\mathbf{H}}(\mathbf{du}, \mathbf{dy}).$$

The Gaussian process $\{\mathbf{U(t)}, \mathbf{t} \geq 0\}$ given by (**??**) has stationary increments and its spectral density is given by

$$\mathbf{f_U}(\tau) = \frac{2(2\pi)^{-\mathbf{d}}\alpha_\mathbf{H}}{|\tau|^{2\mathbf{H}-1}} \int_{\mathbb{R}^\mathbf{d}} \frac{\mu(\mathbf{d}\xi)}{\tau^2 + \frac{|\xi|^4}{4}}. \tag{17}$$

For he Riesz kernel : the Gaussian process $\mathbf{U}$ coincides in distribution with $\mathbf{C}_0\mathbf{B}^\gamma$ with $\gamma = \mathbf{H} - \frac{\mathbf{d}-\alpha}{4}$ and

$$\mathbf{C}_0^2 = \frac{(2\pi)^{-\mathbf{d}+\frac{1}{2}}\alpha_{\mathbf{H}} 2^{2\mathbf{H}-1}\Gamma(\mathbf{H}-\frac{1}{2})}{\sin\left(\pi(\mathbf{d}-\frac{\mathbf{H}-\alpha}{4})\right)\Gamma(1+2\mathbf{H}-\frac{\mathbf{d}-\alpha}{2})\Gamma(1-\mathbf{H})}\int_{\mathbb{R}^\mathbf{d}}\frac{\mathbf{d}\eta}{|\eta|^\alpha\left(1+\frac{|\eta|^4}{4}\right)}.$$

For the process $\mathbf{Y}$ : Let $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ be fixed and let $[\mathbf{a}, \mathbf{b}] \subset (0, \infty)$. Then for any $\mathbf{k} \geq 1$ there is a modification of $\{\mathbf{Y}(\mathbf{t}), \mathbf{t} \geq 0\}$ such that its sample function is almost surely continuously differentiable on $[\mathbf{a}, \mathbf{b}]$ of order $\mathbf{k}$.

The mean square derivative of $\mathbf{Y}$ at $\mathbf{t} \in (0, \infty)$ can be expressed as

$$\mathbf{Y}'(\mathbf{t}) = \int_{-\infty}^{0} \int_{\mathbb{R}^\mathbf{d}} \mathbf{G}'(\mathbf{t} - \mathbf{u}, \mathbf{x} - \mathbf{y}) \, \mathbf{W}^\mathbf{H}(\mathbf{du}, \mathbf{dy}),$$

where $\mathbf{G}' := \partial \mathbf{G} / \partial \mathbf{t}$.
We can show

$$\mathbb{E}\left( |\mathbf{Y}'(\mathbf{t}) - \mathbf{Y}'(\mathbf{s})|^2 \right) \leq \mathbf{C} |\mathbf{t} - \mathbf{s}|^2$$

by using Kolmogorov's continuity theorem, we can find a modification of $\mathbf{Y}$ such that $\mathbf{Y}(\mathbf{t})$ is continuously differentiable on $[\mathbf{a}, \mathbf{b}]$. Iterating this argument yields the conclusion

- Our theorem shows that the time behavior of the process **u** is very similar to the behavior of the bifractional Brownian motion
- Precisely, the solution is a fbm plus a "smooth process"
- many properties of the solution from be deduced from the analysis of the fBm

Moduli of continuity Let $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ be fixed. Then for any $0 < \mathbf{a} < \mathbf{b} < \infty$, we have

$$\lim_{\varepsilon \to 0} \sup_{\mathbf{s},\mathbf{t} \in [\mathbf{a},\mathbf{b}], |\mathbf{s}-\mathbf{t}| \leq \varepsilon} \frac{|\mathbf{u}(\mathbf{t},\mathbf{x}) - \mathbf{u}(\mathbf{s},\mathbf{x})|}{|\mathbf{s}-\mathbf{t}|^{\gamma} \sqrt{\log(1/(\mathbf{t}-\mathbf{s})}} = \mathbf{c}_3 \quad \text{a.s.},$$

where $0 < \mathbf{c}_3 < \infty$ is a constant that may depend on $\mathbf{a}, \mathbf{b}, \gamma$ and $\mathbf{x}$. Or

$$\limsup_{\varepsilon \to 0} \sup_{\mathbf{s},\mathbf{t} \in [\mathbf{a},\mathbf{b}], |\mathbf{s}-\mathbf{t}| \leq \varepsilon} \frac{|\mathbf{u}(\mathbf{t},\mathbf{x}) - \mathbf{u}(\mathbf{s},\mathbf{x})|}{\varepsilon^{\gamma} \sqrt{\log(1/\varepsilon)}} = \mathbf{c}_3 \quad \text{a.s.}$$

Here and in Propositions **??** and **??**, $\gamma = \mathbf{H} - \frac{\mathbf{d}-\alpha}{4}$.

Let $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ be fixed. Then for any $\mathbf{t}_0 > 0$ we have

$$\liminf_{\varepsilon \to 0} \frac{\sup_{|\mathbf{t}-\mathbf{t}_0| \leq \varepsilon} |\mathbf{u}(\mathbf{t}, \mathbf{x}) - \mathbf{u}(\mathbf{t}_0, \mathbf{x})|}{(\varepsilon / \log\log(1/\varepsilon))^{\gamma}} = \mathbf{c}_5 \quad \text{a.s.},$$

where $0 < \mathbf{c}_5 < \infty$ is a constant depending on the small ball probability of the Gaussian process $\mathbf{U}$ in Theorem **??**.

**Space regularity :** we fix $\mathbf{t} > 0$ and analyze the space regularity of the solution $\{\mathbf{u}(\mathbf{t},\mathbf{x}), \mathbf{x} \in \mathbb{R}^{\mathbf{d}}\}$.

We have the following result.
For each $\mathbf{t} > 0$, the Gaussian random field $\{\mathbf{u}(\mathbf{t},\mathbf{x}), \mathbf{x} \in \mathbb{R}^{\mathbf{d}}\}$ is stationary with spectral measure

$$\Delta(\mathbf{d}\xi) = \alpha_{\mathbf{H}}(2\pi)^{-\mathbf{d}} \int_0^{\mathbf{t}} \int_0^{\mathbf{t}} \frac{\mathbf{dudv}}{|\mathbf{u}-\mathbf{v}|^{2-2\mathbf{H}}} \mathbf{e}^{-\frac{(\mathbf{u}+\mathbf{v})|\xi|^2}{2}} \mu(\mathbf{d}\xi).$$

We can show :

$$c_{1,H}(t^{2H} \wedge 1)\left(\frac{1}{1+|\xi|^2}\right)^{2H} \leq \int_0^t\int_0^t \frac{dudv}{|u-v|^{2-2H}} e^{-\frac{(u+v)|\xi|^2}{2}} \leq c_{2,H}(t^{2H}+1)\left(\frac{1}{1+|\xi|^2}\right)^{2H} \quad (18)$$

- This implies that the spectral measure $\Delta(d\xi)$ is comparable with an absolutely continuous measure with a density function close to $|\xi|^{-(\alpha+4H)}$ for all $\xi \in \mathbb{R}^d$ with $|\xi| \geq 1$.
- this is very useful for studying regularity and other sample path properties of the Gaussian random field $\{u(t,x), x \in \mathbb{R}^d\}$.

For example : For any $\mathbf{M} > 0$ and $\mathbf{t} > 0$, there exist positive and finite constants $\mathbf{c_6}$, $\mathbf{c_7}$ such that for any $\mathbf{x}, \mathbf{y} \in [-\mathbf{M}, \mathbf{M}]^{\mathbf{d}}$,

$$\mathbf{c}|\mathbf{x}-\mathbf{y}|^{2\beta}\left(\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)^{\rho} \leq E\left(|\mathbf{u}(\mathbf{t},\mathbf{x})-\mathbf{u}(\mathbf{t},\mathbf{y})|^2\right)$$
$$\leq \mathbf{C}|\mathbf{x}-\mathbf{y}|^{2\beta}\left(\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)^{\rho}.$$

Here $\beta = \min\{1, 2\mathbf{H} - \frac{\mathbf{d}-\alpha}{2}\}$, and

$$\rho = \begin{cases} 1 & \text{if } \beta = 1, \\ 0 & \text{otherwise.} \end{cases}$$

# Quadratic variations of the solution

Suppose $\mathbf{U}(\mathbf{t}, \mathbf{x})$ is the solution of the heat equation with fractional noise in time.
Then for every $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ and for every $\mathbf{s}, \mathbf{t}$

$$\mathbf{E}\left(\mathbf{u}(\mathbf{t}, \mathbf{x})\mathbf{u}(\mathbf{s}, \mathbf{x})\right) = \mathbf{D} \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} |\mathbf{u} - \mathbf{v}|^{2\mathbf{H}-2} \left((\mathbf{t} + \mathbf{s}) - (\mathbf{u} + \mathbf{v})\right)^{-\beta} \mathbf{dvdu} \tag{19}$$

with $0 < 2\beta \leq \mathbf{d}$ and $2\beta < 4\mathbf{H}$, and with $\mathbf{D}$ a fixed positive scaling constant. if the noise is white in space (and $\mu$ is the Lebesgue measure).
Here

$$\beta = \frac{\mathbf{d} - \alpha}{2}.$$

Since we study the behavior with repect to the time variable, consider a centered Gaussian process $(\mathbf{u_t})_{\mathbf{t}\in[0,1]}$ with covariance and

$$\mathbf{R}(\mathbf{t},\mathbf{s}) = \mathbf{D}\int_0^{\mathbf{t}}\int_0^{\mathbf{s}} |\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\left((\mathbf{t}+\mathbf{s})-(\mathbf{u}+\mathbf{v})\right)^{-\beta}\mathbf{dvdu} \quad (20)$$

Recall that $\mathbf{u}$ is very close to a fBm with Hurst index $\mathbf{H}-\frac{\mathbf{d}-\alpha}{4} = \mathbf{H}-\frac{\beta}{2}$

important difference : the process $\mathbf{u}$ does not have stationary increments.

Define the centered quadratic variation of the process $(\mathbf{u_t})$

$$\mathbf{V_N} := \sum_{\mathbf{i}=0}^{\mathbf{N}-1} \left[ \left( \mathbf{u}_{\mathbf{t}_{\mathbf{i}+1}} - \mathbf{u}_{\mathbf{t}_{\mathbf{i}}} \right)^2 - \mathbf{E} \left( \mathbf{u}_{\mathbf{t}_{\mathbf{i}+1}} - \mathbf{u}_{\mathbf{t}_{\mathbf{i}}} \right)^2 \right].$$

Purpose : find the limit in distribution, as $\mathbf{N} \to \infty$, of the sequence $\mathbf{V_N}$

Recall what happens in the fBm case. Let $\mathbf{B}$ be a fBm with $\mathbf{H} \in (0,1)$. Define

$$\mathbf{V_N} := \sum_{\mathbf{i}=0}^{\mathbf{N}-1} \left[ \left( \mathbf{B_{t_{i+1}}} - \mathbf{B_{t_i}} \right)^2 - \mathbf{E} \left( \mathbf{B_{t_{i+1}}} - \mathbf{B_{t_i}} \right)^2 \right]. \tag{21}$$

Then, if $\mathbf{H} < \frac{3}{4}$,

$$\mathbf{c}\mathbf{N}^{2\mathbf{H}-\frac{1}{2}} \mathbf{V_N} \to^{\mathbf{d}}_{\mathbf{N}} \mathbf{N}(0,1)$$

and if $\mathbf{H} > \frac{3}{4}$ then

$$\mathbf{N}\mathbf{V_N} \to_{\mathbf{N}} \textbf{Rosenblatt}.$$

The last convergence holds also in $\mathbf{L}^2$.

When $\mathbf{H} = \frac{1}{2}$ then the solution $\mathbf{U}$ is a bi-fBm with

$$\mathbf{H} = \frac{1}{2}, \mathbf{K} = 1 - \frac{\mathbf{d} - \alpha}{2} = 1 - \beta$$

Then the asymptotic behavior of the solution can be deduced from the results for bi-fBm (which is similar to the case of the fBm ; $\mathbf{H}$ replaced by $\mathbf{HK}$) : Since $\mathbf{HK} = \frac{1}{2} - \frac{\mathbf{d}-\alpha}{4} < \frac{3}{4}$,

$$\mathbf{c}\mathbf{N}^{2\mathbf{HK}-\frac{1}{2}}\mathbf{V_N} \to^{\mathbf{d}}_{\mathbf{N}} \mathbf{N}(0,1)$$

In the fractional case : Let $\mathbf{I_n}$ denote the multiple integral with respect to the Gaussian process $(\mathbf{u_t})$.

-multiple integrals can be defined with respect to any Gaussian process ; a multiple integral of order $\mathbf{n}$ is an isometry between $\mathcal{H}^{\odot \mathbf{n}}$ and $\mathbf{L}^2(\Omega)$ ($\mathcal{H}$ is the canonical Hilbert space associated to the Gaussian process)

Back to the process $\mathbf{u}$ : we write $\mathbf{V_N}$ as a multiple integral. Since

$$\mathbf{u}_{\mathbf{t_{i+1}}} - \mathbf{u}_{\mathbf{t_i}} = \mathbf{I}_1\left(1_{(\mathbf{t_i},\mathbf{t_{i+1}})}\right)$$

and thanks to the product formula we can express the sequence $\mathbf{V_N}$ as a multiple integral of order 2 :

$$\mathbf{V_N} = \mathbf{I}_2\left(\sum_{\mathbf{i}=0}^{\mathbf{N}-1} 1_{(\mathbf{t_i},\mathbf{t_{i+1}})}^{\otimes 2}\right).$$

We use the so-called Fourth Moment Theorem.
If $\mathbf{F_N} = \mathbf{I_q}(\mathbf{f_N})$ is a sequence of multiple integrals (in the "$\mathbf{q}$th Wiener chaos") such that $\mathbf{E}\mathbf{F}_\mathbf{N}^2 \to_\mathbf{N} 1$ then

$$\mathbf{F_N} \to_\mathbf{N} \mathbf{N}(0,1)$$

if and only if

$$\mathbf{E}\mathbf{F}_\mathbf{N}^4 \to 3$$

if and only if

$$\|\mathbf{D}\mathbf{F_N}\|^2 \to_\mathbf{N} \mathbf{q}$$

in $\mathbf{L}^2$. ($\mathbf{D}$ is the Malliavin derivative)

consider only the even partition of the unit interval $[0,1]$ : $\mathbf{t_i} := \frac{\mathbf{i}}{\mathbf{N}}$ for $\mathbf{i} = 0, .., \mathbf{N}$.
Fist step : analyze $\mathbf{E}\mathbf{V}_\mathbf{N}^2$.
We have, using the isometry of multiple stochastic integrals

$$\mathbf{E}\mathbf{V}_\mathbf{N}^2 = 2! \sum_{\mathbf{i},\mathbf{j}=0}^{\mathbf{N}-1} \langle 1_{(\mathbf{t_i},\mathbf{t_{i+1}})}^{\otimes 2}, 1_{(\mathbf{t_j},\mathbf{t_{j+1}})}^{\otimes 2} \rangle = 2 \sum_{\mathbf{i},\mathbf{j}=0}^{\mathbf{N}-1} \langle 1_{(\mathbf{t_i},\mathbf{t_{i+1}})}, 1_{(\mathbf{t_j},\mathbf{t_{j+1}})} \rangle^2.$$

Here $\langle \cdot, \cdot \rangle_{\mathcal{U}} := \langle \cdot, \cdot \rangle$ denotes the scalar product in the canonical Hilbert space $\mathcal{U}$ associated with the process $\mathbf{U}$ which is defined as the closure of the set of indicator functions $\left(1_{[0,\mathbf{t}]}, \mathbf{t} \in [0, \mathbf{T}]\right)$ with respect to the scalar product

$$\langle 1_{[0,\mathbf{t}]}, 1_{[0,\mathbf{s}]} \rangle = \mathbf{R}(\mathbf{t}, \mathbf{s})$$

Recall : the covariance of $(\mathbf{u_t}, \mathbf{t} \in [0,1])$ is

$$\mathbf{R}(\mathbf{t},\mathbf{s}) = \mathbf{D}\int_0^{\mathbf{t}}\int_0^{\mathbf{s}} |\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\left((\mathbf{t}+\mathbf{s})-(\mathbf{u}+\mathbf{v})\right)^{-\beta}\mathbf{dvdu}$$

Continuation : : compute $\mathbf{E}\mathbf{V}_\mathbf{N}^2$ in order to renormalize it.
We have, using the isometry of multiple stochastic integrals

$$\mathbf{E}\mathbf{V}_\mathbf{N}^2 = 2! \sum_{\mathbf{i},\mathbf{j}=0}^{\mathbf{N}-1} \langle 1_{(\mathbf{t_i},\mathbf{t_{i+1}})}^{\otimes 2}, 1_{(\mathbf{t_j},\mathbf{t_{j+1}})}^{\otimes 2} \rangle = 2 \sum_{\mathbf{i},\mathbf{j}=0}^{\mathbf{N}-1} \langle 1_{(\mathbf{t_i},\mathbf{t_{i+1}})}, 1_{(\mathbf{t_j},\mathbf{t_{j+1}})} \rangle^2.$$

Recall : $\langle \cdot, \cdot \rangle_{\mathcal{U}} := \langle \cdot, \cdot \rangle$ denotes the scalar product in the canonical Hilbert space $\mathcal{U}$ associated with the process $\mathbf{u}$

the behavior of the covariance of the increments is crucial here and

$$
\begin{aligned}
& \mathbf{D}^{-1}\left\langle 1_{(\mathbf{t_i},\mathbf{t_{i+1}})}, 1_{(\mathbf{t_j},\mathbf{t_{j+1}})}\right\rangle = \\
& \mathbf{N}^{-2\mathbf{H}+\beta}\left[\int_0^{\mathbf{i}+1}\mathbf{du}\int_0^{\mathbf{j}+1}\mathbf{dv}|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\left(\mathbf{i}+1+\mathbf{j}+1-(\mathbf{u}+\mathbf{v})\right)^{-\beta}\right. \\
& -\int_0^{\mathbf{i}+1}\mathbf{du}\int_0^{\mathbf{j}}\mathbf{dv}|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\left(\mathbf{i}+1+\mathbf{j}-(\mathbf{u}+\mathbf{v})\right)^{-\beta} \\
& -\int_0^{\mathbf{i}}\mathbf{du}\int_0^{\mathbf{j}+1}\mathbf{dv}|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\left(\mathbf{i}+\mathbf{j}+1-(\mathbf{u}+\mathbf{v})\right)^{-\beta} \\
& \left.+\int_0^{\mathbf{i}}\mathbf{du}\int_0^{\mathbf{j}}\mathbf{dv}|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2}\left(\mathbf{i}+\mathbf{j}-(\mathbf{u}+\mathbf{v})\right)^{-\beta}\right] \\
:= \quad & \mathbf{N}^{-2\mathbf{H}+\beta}\left[\mathbf{A}(\mathbf{i},\mathbf{j})+\mathbf{B}(\mathbf{i},\mathbf{j})+\mathbf{C}(\mathbf{i},\mathbf{j})\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbf{EV_N^2} &= 2\mathbf{D}^2\mathbf{N}^{-4\mathbf{H}+2\beta}\sum_{\mathbf{i},\mathbf{j}=0}^{\mathbf{N}-1}[\mathbf{A}(\mathbf{i},\mathbf{j})+\mathbf{B}(\mathbf{i},\mathbf{j})+\mathbf{C}(\mathbf{i},\mathbf{j})]^2\\
&= 2\mathbf{D}^2\mathbf{N}^{-4\mathbf{H}+2\beta}\sum_{\mathbf{i},\mathbf{j}=0}^{\mathbf{N}-1}\Big[\mathbf{A}(\mathbf{i},\mathbf{j})^2+\mathbf{B}(\mathbf{i},\mathbf{j})^2+\mathbf{C}(\mathbf{i},\mathbf{j})^2\\
&\quad +2\mathbf{A}(\mathbf{i},\mathbf{j})\mathbf{B}(\mathbf{i},\mathbf{j})+2\mathbf{A}(\mathbf{i},\mathbf{j})\mathbf{C}(\mathbf{i},\mathbf{j})+2\mathbf{B}(\mathbf{i},\mathbf{j})\mathbf{C}(\mathbf{i},\mathbf{j})\Big]\\
&: = 2\mathbf{D}^2(\mathbf{T}_{1,\mathbf{N}}+\mathbf{T}_{2,\mathbf{N}}+\mathbf{T}_{3,\mathbf{N}}+\mathbf{T}_{4,\mathbf{N}}+\mathbf{T}_{5,\mathbf{N}}+\mathbf{T}_{6,\mathbf{N}}).'
\end{aligned}
$$

We will evaluate the asymptotic behavior, as $\mathbf{N} \to \infty$ of the six terms from above.

Actually, it happens that the six summands that appear in the decomposition of $\mathbf{EV_N^2}$ are all of them of the same magnitude. This makes our computations delicate and lengthy.

There is no negligible part that can be ignored in the estimation of $\mathbf{EV_N^2}$.

Result :

- If $\mathbf{H} < \frac{3}{4}$, then $\lim_{\mathbf{N}\to\infty} \mathbf{N}^{4\mathbf{H}-2\beta-1}\mathbf{T}_{\mathbf{i},\mathbf{N}} = \mathbf{K}_{1,\mathbf{i}}$ .
- If $\mathbf{H} > \frac{3}{4}$ then $\lim_{\mathbf{N}\to\infty} \mathbf{N}^{2-2\beta}\mathbf{T}_{\mathbf{i},\mathbf{N}} = \mathbf{K}_{2,\mathbf{i}}$

for $\mathbf{i} = 1, .., 6$.

The behavior of the variance depends on $\mathbf{H}$ (and not on $\mathbf{H} - \beta/2$).

Let

$$\mathbf{F_N} := \mathbf{K}_1^{-\frac{1}{2}} \mathbf{N}^{2\mathbf{H}-\beta-\frac{1}{2}} \mathbf{V_N}. \tag{22}$$

This is the renormalized quadratic variation since

$$\mathbf{E}\mathbf{F}_\mathbf{N}^2 \to_\mathbf{N} 1.$$

Then if $\mathbf{H} < \frac{3}{4}$,

$$\mathbf{F_N} \to_\mathbf{N}^\mathbf{d} \mathbf{N}(0,1).$$

The proof uses the Fourth Moment Theorem.

The case $\mathbf{H} > \frac{3}{4}$.

-a Rosenblatt-distributed limit does occur for the renormalized $\mathbf{V_N}$, for the threshold $\mathbf{H} > 3/4$, not the naive threshold $\mathbf{H} > 3/4 + \beta/2$.

the proof is based on the concept of *cumulant*.