Reproducing kernels for Dirichlet spaces associated to finitely supported measures

Constantin COSTARA

Ovidius University of Constanța, România

December 2017
IMAR

Definition

- Given b in the closed unit ball of H^∞, the associated de Branges–Rovnyak space $\mathcal{H}_b$ consists of the range of the operator $(I - T_b T_{\overline{b}})^{1/2}$ equipped with the range norm. (For $\varphi \in L^\infty(\mathbb{T})$, by $T_\varphi : H^2 \to H^2$ we have denoted the Toeplitz operator defined by $T_\varphi(f) = P_+(\varphi f)$, where $P_+ : L^2(\mathbb{T}) \to H^2$ is the standard Riesz projection onto the closed subspace H^2 of $L^2(\mathbb{T})$.) The range norm induces an inner product on $\mathcal{H}_b$ which makes $(I - T_b T_{\overline{b}})^{1/2}$ an isometry from H^2 onto $\mathcal{H}_b$.

Definition

- Given b in the closed unit ball of H^∞, the associated de Branges–Rovnyak space $\mathcal{H}_b$ consists of the range of the operator $(I - T_b T_{\overline{b}})^{1/2}$ equipped with the range norm. (For $\varphi \in L^\infty(\mathbb{T})$, by $T_\varphi : H^2 \to H^2$ we have denoted the Toeplitz operator defined by $T_\varphi(f) = P_+(\varphi f)$, where $P_+ : L^2(\mathbb{T}) \to H^2$ is the standard Riesz projection onto the closed subspace H^2 of $L^2(\mathbb{T})$.) The range norm induces an inner product on $\mathcal{H}_b$ which makes $(I - T_b T_{\overline{b}})^{1/2}$ an isometry from H^2 onto $\mathcal{H}_b$.
- $\mathcal{H}_b$ consists of those $f \in H^2$ satisfying

$$\|f\|_b^2 := \sup_{g \in H^2} \left(\|f + bg\|_{H^2}^2 - \|g\|_{H^2}^2 \right) < \infty.$$

Definition

- Given b in the closed unit ball of H^∞, the associated de Branges–Rovnyak space $\mathcal{H}_b$ consists of the range of the operator $(I - T_b T_{\overline{b}})^{1/2}$ equipped with the range norm. (For $\varphi \in L^\infty(\mathbb{T})$, by $T_\varphi : H^2 \to H^2$ we have denoted the Toeplitz operator defined by $T_\varphi(f) = P_+(\varphi f)$, where $P_+ : L^2(\mathbb{T}) \to H^2$ is the standard Riesz projection onto the closed subspace H^2 of $L^2(\mathbb{T})$.) The range norm induces an inner product on $\mathcal{H}_b$ which makes $(I - T_b T_{\overline{b}})^{1/2}$ an isometry from H^2 onto $\mathcal{H}_b$.
- $\mathcal{H}_b$ consists of those $f \in H^2$ satisfying

$$\|f\|_b^2 := \sup_{g \in H^2} \left(\|f + bg\|_{H^2}^2 - \|g\|_{H^2}^2\right) < \infty.$$

Reproducing kernels for the de Branges–Rovnyak spaces

- One can easily check that $\mathcal{H}_b$ is a reproducing kernel Hilbert space: the reproducing kernels can be computed explicitly and have a natural form,

$$K_b(z, \lambda) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z} \qquad (z; \lambda \in \mathbb{D}).$$

Reproducing kernels for the de Branges–Rovnyak spaces

- One can easily check that $\mathcal{H}_b$ is a reproducing kernel Hilbert space: the reproducing kernels can be computed explicitly and have a natural form,

$$K_b(z,\lambda) = \frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z} \qquad (z;\lambda \in \mathbb{D}).$$

- L. de Branges, J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York, 1966.

Reproducing kernels for the de Branges–Rovnyak spaces

- One can easily check that $\mathcal{H}_b$ is a reproducing kernel Hilbert space: the reproducing kernels can be computed explicitly and have a natural form,

$$K_b(z,\lambda) = \frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z} \qquad (z;\lambda \in \mathbb{D}).$$

- L. de Branges, J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York, 1966.
- D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons Inc., New York, 1994.

Reproducing kernels for the de Branges–Rovnyak spaces

- One can easily check that $\mathcal{H}_b$ is a reproducing kernel Hilbert space: the reproducing kernels can be computed explicitly and have a natural form,

$$K_b(z,\lambda) = \frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z} \qquad (z;\lambda \in \mathbb{D}).$$

- L. de Branges, J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York, 1966.
- D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons Inc., New York, 1994.
- E. Fricain, J. Mashreghi, *The theory of $\mathcal{H}(b)$-Spaces*, Cambridge University Press, 2016.

Reproducing kernels for the de Branges–Rovnyak spaces

- One can easily check that $\mathcal{H}_b$ is a reproducing kernel Hilbert space: the reproducing kernels can be computed explicitly and have a natural form,

$$K_b(z,\lambda) = \frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z} \qquad (z;\lambda \in \mathbb{D}).$$

- L. de Branges, J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York, 1966.
- D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons Inc., New York, 1994.
- E. Fricain, J. Mashreghi, *The theory of $\mathcal{H}(b)$-Spaces*, Cambridge University Press, 2016.

Dirichlet spaces

- For a finite, positive Borel measure μ on the unit circle, let P_μ denote its Poisson integral. The corresponding generalized Dirichlet space $\mathcal{D}_\mu$ is defined to be the set of all holomorphic functions f on $\mathbb{D}$ such that the Dirichlet integral with respect to μ satisfies

$$D_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z)\, dA(z) < \infty,$$

where dA is the normalized area measure on the unit disc.

Dirichlet spaces

- For a finite, positive Borel measure μ on the unit circle, let P_μ denote its Poisson integral. The corresponding generalized Dirichlet space $\mathcal{D}_\mu$ is defined to be the set of all holomorphic functions f on $\mathbb{D}$ such that the Dirichlet integral with respect to μ satisfies

$$D_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z)\, dA(z) < \infty,$$

where dA is the normalized area measure on the unit disc.
- If μ is the normalized Lebesgue measure on the unit circle, then $\mathcal{D}_\mu$ coincides with the classical Dirichlet space $\mathcal{D}$. In the general case, $\mathcal{D}_\mu$ is always a subspace of the classical Hardy space H^2, and we may define a norm on it by putting

$$\|f\|_\mu^2 := \|f\|_2^2 + D_\mu(f) \qquad (f \in \mathcal{D}_\mu),$$

Dirichlet spaces

- For a finite, positive Borel measure μ on the unit circle, let P_μ denote its Poisson integral. The corresponding generalized Dirichlet space $\mathcal{D}_\mu$ is defined to be the set of all holomorphic functions f on $\mathbb{D}$ such that the Dirichlet integral with respect to μ satisfies

$$D_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z)\, dA(z) < \infty,$$

  where dA is the normalized area measure on the unit disc.
- If μ is the normalized Lebesgue measure on the unit circle, then $\mathcal{D}_\mu$ coincides with the classical Dirichlet space $\mathcal{D}$. In the general case, $\mathcal{D}_\mu$ is always a subspace of the classical Hardy space H^2, and we may define a norm on it by putting

$$\|f\|_\mu^2 := \|f\|_2^2 + D_\mu(f) \qquad (f \in \mathcal{D}_\mu),$$

Dirichlet spaces

- S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Amer. Math. Soc.* 328 (1991), 325–349.

Dirichlet spaces

- S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Amer. Math. Soc.* 328 (1991), 325–349.
- S. Richter and C. Sundberg - series of papers;

Dirichlet spaces

- S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Amer. Math. Soc.* 328 (1991), 325–349.
- S. Richter and C. Sundberg - series of papers;
- O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A Primer on Dirichlet Spaces*, Cambridge University Press, 2014.

Dirichlet spaces

- S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Amer. Math. Soc.* 328 (1991), 325–349.
- S. Richter and C. Sundberg - series of papers;
- O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A Primer on Dirichlet Spaces*, Cambridge University Press, 2014.
- Not easy to work with the standard norm/inner product of $\mathcal{D}_\mu$, mainly because of the way the Dirichlet integral is defined.

Dirichlet spaces

- S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Amer. Math. Soc.* 328 (1991), 325–349.
- S. Richter and C. Sundberg - series of papers;
- O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A Primer on Dirichlet Spaces*, Cambridge University Press, 2014.
- Not easy to work with the standard norm/inner product of $\mathcal{D}_\mu$, mainly because of the way the Dirichlet integral is defined.

Evaluation functionals

- For $\lambda \in \mathbb{D}$, the evaluation functional at λ is continuous on $\mathcal{D}_\mu$.

Evaluation functionals

- For $\lambda \in \mathbb{D}$, the evaluation functional at λ is continuous on $\mathcal{D}_\mu$.
- How to compute the corresponding reproducing kernels?

Evaluation functionals

- For $\lambda \in \mathbb{D}$, the evaluation functional at λ is continuous on $\mathcal{D}_\mu$.
- How to compute the corresponding reproducing kernels?
- If $\xi \in \mathbb{T}$ and $\mu(\{\xi\}) > 0$, the evaluation functional at ξ is well-defined and continuous on $\mathcal{D}_\mu$.

Evaluation functionals

- For $\lambda \in \mathbb{D}$, the evaluation functional at λ is continuous on $\mathcal{D}_\mu$.
- How to compute the corresponding reproducing kernels?
- If $\xi \in \mathbb{T}$ and $\mu(\{\xi\}) > 0$, the evaluation functional at ξ is well-defined and continuous on $\mathcal{D}_\mu$.
- How to compute the corresponding reproducing kernels?

Evaluation functionals

- For $\lambda \in \mathbb{D}$, the evaluation functional at λ is continuous on $\mathcal{D}_\mu$.
- How to compute the corresponding reproducing kernels?
- If $\xi \in \mathbb{T}$ and $\mu(\{\xi\}) > 0$, the evaluation functional at ξ is well-defined and continuous on $\mathcal{D}_\mu$.
- How to compute the corresponding reproducing kernels?

Local Dirichlet spaces as de Branges–Rovnyak spaces

- The connection between the spaces $\mathcal{D}_\mu$ and $\mathcal{H}_b$ was first noticed by S. Richter and C. Sundberg.
  (A formula for the local Dirichlet integral, *Michigan Math. J.* 38 (1991), 355–379.)

Local Dirichlet spaces as de Branges–Rovnyak spaces

- The connection between the spaces $\mathcal{D}_\mu$ and $\mathcal{H}_b$ was first noticed by S. Richter and C. Sundberg.
  (A formula for the local Dirichlet integral, *Michigan Math. J.* 38 (1991), 355–379.)
- Improving their observation, D. Sarason proved that by taking $\mu = \delta_\xi$, the point mass measure at $\xi \in \mathbb{T}$, then $\mathcal{D}_{\delta_\xi} = \mathcal{H}_{b_\xi}$ with equality of norms, where

$$b_\xi(z) = \frac{((\sqrt{5}-1)/2)\overline{\xi}z}{1-((3-\sqrt{5})/2)\overline{\xi}z} = -\frac{1+\sqrt{5}}{2}\frac{z}{z-w_0} \qquad (z \in \mathbb{D}),$$

where $w_0 = (3+\sqrt{5})\xi/2$.
(Local Dirichlet spaces as de Branges-Rovnyak spaces, *Proc. Amer. Math. Soc.* 125 (1997), 2133–2139.)

Local Dirichlet spaces as de Branges–Rovnyak spaces

- The connection between the spaces $\mathcal{D}_\mu$ and $\mathcal{H}_b$ was first noticed by S. Richter and C. Sundberg.
  (A formula for the local Dirichlet integral, *Michigan Math. J.* 38 (1991), 355–379.)
- Improving their observation, D. Sarason proved that by taking $\mu = \delta_\xi$, the point mass measure at $\xi \in \mathbb{T}$, then $\mathcal{D}_{\delta_\xi} = \mathcal{H}_{b_\xi}$ with equality of norms, where

$$b_\xi(z) = \frac{((\sqrt{5}-1)/2)\bar{\xi}z}{1-((3-\sqrt{5})/2)\bar{\xi}z} = -\frac{1+\sqrt{5}}{2}\frac{z}{z-w_0} \qquad (z \in \mathbb{D}),$$

  where $w_0 = (3+\sqrt{5})\xi/2$.
  (Local Dirichlet spaces as de Branges-Rovnyak spaces, *Proc. Amer. Math. Soc.* 125 (1997), 2133–2139.)

Reproducing kernels for local Dirichlet spaces

- We have

$$\frac{1-\overline{b_\xi(\lambda)}b_\xi(z)}{1-\overline{\lambda}z} = \frac{3+\sqrt{5}}{2}\frac{-z(\overline{\lambda}(\sqrt{5}-1)/2+\overline{\xi})-(\overline{\lambda}-\overline{w}_0)\xi}{(\overline{\lambda}-\overline{w}_0)(z-w_0)(1-\overline{\lambda}z)}.$$

Reproducing kernels for local Dirichlet spaces

- We have
$$\frac{1-\overline{b_\xi(\lambda)}b_\xi(z)}{1-\bar\lambda z}=\frac{3+\sqrt5}{2}\frac{-z(\bar\lambda(\sqrt5-1)/2+\bar\xi)-(\bar\lambda-\overline{w_0})\xi}{(\bar\lambda-\overline{w_0})(z-w_0)(1-\bar\lambda z)}.$$

- For any fixed $\xi\in\mathbb{T}$, for $z,\lambda\in\mathbb{D}$ we have
$$K_{\mathcal{D}_{\delta_\xi}}(z,\lambda)=-\frac{1+\sqrt5}{2}\frac{z(\bar\lambda+(\sqrt5+1)\bar\xi/2)+(\bar\lambda-\overline{w_0})(\sqrt5+1)\xi/2}{(\bar\lambda-\overline{w_0})(z-w_0)(1-\bar\lambda z)}$$

Reproducing kernels for local Dirichlet spaces

- We have
$$\frac{1-\overline{b_\xi(\lambda)}b_\xi(z)}{1-\bar{\lambda}z} = \frac{3+\sqrt{5}}{2}\frac{-z(\bar{\lambda}(\sqrt{5}-1)/2+\bar{\xi})-(\bar{\lambda}-\overline{w}_0)\xi}{(\bar{\lambda}-\overline{w}_0)(z-w_0)(1-\bar{\lambda}z)}.$$
- For any fixed $\xi \in \mathbb{T}$, for $z, \lambda \in \mathbb{D}$ we have
$$K_{\mathcal{D}_{\delta_\xi}}(z,\lambda) = -\frac{1+\sqrt{5}}{2}\frac{z(\bar{\lambda}+(\sqrt{5}+1)\bar{\xi}/2)+(\bar{\lambda}-\overline{w_0})(\sqrt{5}+1)\xi/2}{(\bar{\lambda}-\overline{w}_0)(z-w_0)(1-\bar{\lambda}z)}$$
- The converse for Sarason's result also holds: the local Dirichlet space is the only case when $\mathcal{D}_\mu$ arises as a de Branges–Rovnyak space, with equality of norms.
(N. Chevrot, D. Guillot, T. Ransford, De Branges–Rovnyak spaces and Dirichlet spaces, *J. Funct. Anal.* 259 (2010), 2366–2383.)

Reproducing kernels for local Dirichlet spaces

- We have
$$\frac{1-\overline{b_\xi(\lambda)}b_\xi(z)}{1-\bar\lambda z}=\frac{3+\sqrt{5}}{2}\,\frac{-z(\bar\lambda(\sqrt{5}-1)/2+\bar\xi)-(\bar\lambda-\overline{w}_0)\xi}{(\bar\lambda-\overline{w}_0)(z-w_0)(1-\bar\lambda z)}.$$
- For any fixed $\xi\in\mathbb{T}$, for $z,\lambda\in\mathbb{D}$ we have
$$K_{\mathcal{D}_{\delta_\xi}}(z,\lambda)=-\frac{1+\sqrt{5}}{2}\,\frac{z(\bar\lambda+(\sqrt{5}+1)\bar\xi/2)+(\bar\lambda-\overline{w_0})(\sqrt{5}+1)\xi/2}{(\bar\lambda-\overline{w}_0)(z-w_0)(1-\bar\lambda z)}$$
- The converse for Sarason's result also holds: the local Dirichlet space is the only case when $\mathcal{D}_\mu$ arises as a de Branges–Rovnyak space, with equality of norms. (N. Chevrot, D. Guillot, T. Ransford, De Branges–Rovnyak spaces and Dirichlet spaces, *J. Funct. Anal.* 259 (2010), 2366–2383.)

Necessary conditions

- (C. Costara, T. Ransford, Which de Branges–Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* 265 (2013), 3204–3218.)

Necessary conditions

- (C. Costara, T. Ransford, Which de Branges–Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* 265 (2013), 3204–3218.)
- Equivalent norms!

Necessary conditions

- (C. Costara, T. Ransford, Which de Branges–Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* 265 (2013), 3204–3218.)
- Equivalent norms!
- b is not extremal in the unit ball of H^∞.

Necessary conditions

- (C. Costara, T. Ransford, Which de Branges–Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* 265 (2013), 3204–3218.)
- Equivalent norms!
- b is not extremal in the unit ball of H^∞.
- There is a unique outer function a satisfying

$$|a|^2 + |b|^2 = 1 \ m\text{-a.e. on } \mathbb{T} \qquad \text{and} \qquad a(0) > 0.$$

(We say that (b, a) is a *pair*.)

Necessary conditions

- (C. Costara, T. Ransford, Which de Branges–Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* 265 (2013), 3204–3218.)
- Equivalent norms!
- b is not extremal in the unit ball of H^∞.
- There is a unique outer function a satisfying

$$|a|^2 + |b|^2 = 1 \ m\text{-a.e. on } \mathbb{T} \qquad \text{and} \qquad a(0) > 0.$$

(We say that (b, a) is a *pair*.)

Necessary conditions

- Define $V_\mu : \mathbb{C} \to [0, \infty]$ by

$$V_\mu(z) := \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} \qquad (z \in \mathbb{C}).$$

Necessary conditions

- Define $V_\mu : \mathbb{C} \to [0, \infty]$ by

$$V_\mu(z) := \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} \qquad (z \in \mathbb{C}).$$

- V_μ is lower semicontinuous on $\mathbb{C}$ and continuous on $\mathbb{C} \setminus \text{supp}\mu$,
- $\mu(\mathbb{T})/(1 + |z|)^2 \leq V_\mu(z) \leq \mu(\mathbb{T})/\text{dist}(z, \text{supp}\mu)^2$,
- $V_\mu(\zeta) = \lim_{r \to 1^-} V_\mu(r\zeta)$ for all $\zeta \in \mathbb{T}$,
- $V_\mu = \infty$ μ-a.e. on $\mathbb{T}$.

Necessary conditions

- Define $V_\mu : \mathbb{C} \to [0, \infty]$ by

$$V_\mu(z) := \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} \qquad (z \in \mathbb{C}).$$

- V_μ is lower semicontinuous on $\mathbb{C}$ and continuous on $\mathbb{C} \setminus \mathrm{supp}\mu$,
- $\mu(\mathbb{T})/(1 + |z|)^2 \le V_\mu(z) \le \mu(\mathbb{T})/\mathrm{dist}(z, \mathrm{supp}\mu)^2$,
- $V_\mu(\zeta) = \lim_{r\to 1^-} V_\mu(r\zeta)$ for all $\zeta \in \mathbb{T}$,
- $V_\mu = \infty$ μ-a.e. on $\mathbb{T}$.

Necessary conditions

- We have $1 + V_\mu \asymp 1/|a|^2$ m-a.e. on $\mathbb{T}$.

Necessary conditions

- We have $1 + V_\mu \asymp 1/|a|^2$ m-a.e. on $\mathbb{T}$.
- μ and m are mutually singular.

Necessary conditions

- We have $1 + V_\mu \asymp 1/|a|^2$ m-a.e. on $\mathbb{T}$.
- μ and m are mutually singular.
- We have $\log^+ V_\mu \in L^1(\mathbb{T})$.

Necessary conditions

- We have $1 + V_\mu \asymp 1/|a|^2$ m-a.e. on $\mathbb{T}$.
- μ and m are mutually singular.
- We have $\log^+ V_\mu \in L^1(\mathbb{T})$.
- If b is a rational function, then the corresponding function a is also rational, all its zeros on $\mathbb{T}$ are simple, and the support of μ is exactly equal to this set of zeros.

Necessary conditions

- We have $1 + V_\mu \asymp 1/|a|^2$ m-a.e. on $\mathbb{T}$.
- μ and m are mutually singular.
- We have $\log^+ V_\mu \in L^1(\mathbb{T})$.
- If b is a rational function, then the corresponding function a is also rational, all its zeros on $\mathbb{T}$ are simple, and the support of μ is exactly equal to this set of zeros.
- We have

$$\{f \in \mathcal{D}_\mu : f = 0 \ \mu\text{-a.e.}\} = aH^2.$$

Necessary conditions

- We have $1 + V_\mu \asymp 1/|a|^2$ m-a.e. on $\mathbb{T}$.
- μ and m are mutually singular.
- We have $\log^+ V_\mu \in L^1(\mathbb{T})$.
- If b is a rational function, then the corresponding function a is also rational, all its zeros on $\mathbb{T}$ are simple, and the support of μ is exactly equal to this set of zeros.
- We have

$$\{f \in \mathcal{D}_\mu : f = 0 \ \mu\text{-a.e.}\} = aH^2.$$

The case of finitely supported measures

- If μ is finitely supported, then $\mathcal{D}_\mu$ is a de Branges–Rovnyak space, but this time we have only equality as sets.

The case of finitely supported measures

- If μ is finitely supported, then $\mathcal{D}_\mu$ is a de Branges–Rovnyak space, but this time we have only equality as sets.
- Of course, unlike the case $\mu = \delta_\xi$, the inner products from the Dirichlet and de Branges–Rovnyak spaces (which, in fact, define the reproducing kernels) do not coincide, so we cannot use the above result to compute directly the reproducing kernels of $\mathcal{D}_\mu$.

The case of finitely supported measures

- If μ is finitely supported, then $\mathcal{D}_\mu$ is a de Branges–Rovnyak space, but this time we have only equality as sets.
- Of course, unlike the case $\mu = \delta_\xi$, the inner products from the Dirichlet and de Branges–Rovnyak spaces (which, in fact, define the reproducing kernels) do not coincide, so we cannot use the above result to compute directly the reproducing kernels of $\mathcal{D}_\mu$.
- The identification was obtained by studying the subspace of $\mathcal{D}_\mu$ consisting of all functions which are zero on the support of μ.

The case of finitely supported measures

- If μ is finitely supported, then $\mathcal{D}_\mu$ is a de Branges–Rovnyak space, but this time we have only equality as sets.
- Of course, unlike the case $\mu = \delta_\xi$, the inner products from the Dirichlet and de Branges–Rovnyak spaces (which, in fact, define the reproducing kernels) do not coincide, so we cannot use the above result to compute directly the reproducing kernels of $\mathcal{D}_\mu$.
- The identification was obtained by studying the subspace of $\mathcal{D}_\mu$ consisting of all functions which are zero on the support of μ.

Zero on the support of μ

- Let $c_1, ..., c_n > 0$ and $\xi_1, ..., \xi_n$ pairwise distinct on the unit circle, and consider $\mu = \sum_{j=1}^{n} c_j \delta_{\xi_j}$.

Zero on the support of μ

- Let $c_1, ..., c_n > 0$ and $\xi_1, ..., \xi_n$ pairwise distinct on the unit circle, and consider $\mu = \sum_{j=1}^{n} c_j \delta_{\xi_j}$.
- Then $V_\mu : \mathbb{C} \to [0, +\infty]$ is given by

$$V_\mu(z) = \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} = \sum_{j=1}^{n} \frac{c_j}{|\xi_j - z|^2}.$$

Zero on the support of μ

- Let $c_1, \ldots, c_n > 0$ and $\xi_1, \ldots, \xi_n$ pairwise distinct on the unit circle, and consider $\mu = \sum_{j=1}^{n} c_j \delta_{\xi_j}$.
- Then $V_\mu : \mathbb{C} \to [0, +\infty]$ is given by

$$V_\mu(z) = \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} = \sum_{j=1}^{n} \frac{c_j}{|\xi_j - z|^2}.$$

- Construct the unique outer function $O_\mu \in H^2$ such that $O_\mu(0) > 0$ and $|O_\mu|^2 = 1/(1 + V_\mu)$ on $\mathbb{T}$.

Zero on the support of μ

- Let $c_1, \ldots, c_n > 0$ and $\xi_1, \ldots, \xi_n$ pairwise distinct on the unit circle, and consider $\mu = \sum_{j=1}^{n} c_j \delta_{\xi_j}$.
- Then $V_\mu : \mathbb{C} \to [0, +\infty]$ is given by

$$V_\mu(z) = \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} = \sum_{j=1}^{n} \frac{c_j}{|\xi_j - z|^2}.$$

- Construct the unique outer function $O_\mu \in H^2$ such that $O_\mu(0) > 0$ and $|O_\mu|^2 = 1/(1 + V_\mu)$ on $\mathbb{T}$.
- Apply Riesz–Fejér theorem to the trigonometric polynomial

$$\prod_{j=1}^{n} |z - \xi_j|^2 + \sum_{k=1}^{n} (c_k \prod_{j \neq k} |z - \xi_j|^2),$$

which is positive on $\mathbb{T}$.

Zero on the support of μ

- Let $c_1, \ldots, c_n > 0$ and $\xi_1, \ldots, \xi_n$ pairwise distinct on the unit circle, and consider $\mu = \sum_{j=1}^n c_j \delta_{\xi_j}$.
- Then $V_\mu : \mathbb{C} \to [0, +\infty]$ is given by

$$V_\mu(z) = \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|\lambda - z|^2} = \sum_{j=1}^n \frac{c_j}{|\xi_j - z|^2}.$$

- Construct the unique outer function $O_\mu \in H^2$ such that $O_\mu(0) > 0$ and $|O_\mu|^2 = 1/(1 + V_\mu)$ on $\mathbb{T}$.
- Apply Riesz–Fejér theorem to the trigonometric polynomial

$$\prod_{j=1}^n |z - \xi_j|^2 + \sum_{k=1}^n (c_k \prod_{j \neq k} |z - \xi_j|^2),$$

which is positive on $\mathbb{T}$.

Zero on the support of μ

- There exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, all of modulus strictly bigger than one, and $c > 0$ such that, on $\mathbb{T}$,

$$\prod_{j=1}^{n} |z - \xi_j|^2 + \sum_{k=1}^{n} \left(c_k \prod_{j \neq k} |z - \xi_j|^2 \right) = c \prod_{j=1}^{n} |z - \alpha_j|^2 .$$

Zero on the support of μ

- There exist $\alpha_1, ..., \alpha_n \in \mathbb{C}$, all of modulus strictly bigger than one, and $c > 0$ such that, on $\mathbb{T}$,

$$\prod_{j=1}^{n} |z-\xi_j|^2 + \sum_{k=1}^{n} \left(c_k \prod_{j\neq k} |z-\xi_j|^2\right) = c \prod_{j=1}^{n} |z-\alpha_j|^2 .$$

- Then

$$O_\mu(z) = \frac{\eta}{\sqrt{c}} \frac{\prod_{j=1}^{n}(z-\xi_j)}{\prod_{j=1}^{n}(z-\alpha_j)},$$

where η is a complex number of modulus one which gives $O_\mu(0) > 0$.

Zero on the support of μ

- There exist $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, all of modulus strictly bigger than one, and $c > 0$ such that, on $\mathbb{T}$,

$$\prod_{j=1}^{n} |z-\xi_j|^2 + \sum_{k=1}^{n} \left(c_k \prod_{j\neq k} |z-\xi_j|^2\right) = c \prod_{j=1}^{n} |z-\alpha_j|^2 .$$

- Then

$$O_\mu(z) = \frac{\eta}{\sqrt{c}} \frac{\prod_{j=1}^{n}(z-\xi_j)}{\prod_{j=1}^{n}(z-\alpha_j)},$$

where η is a complex number of modulus one which gives $O_\mu(0) > 0$.
- We have $\mathcal{D}_\mu = \mathcal{P}_{n-1} \oplus O_\mu H^2$, an algebraic direct sum!

Zero on the support of μ

- There exist $\alpha_1, ..., \alpha_n \in \mathbb{C}$, all of modulus strictly bigger than one, and $c > 0$ such that, on $\mathbb{T}$,

$$\prod_{j=1}^n |z-\xi_j|^2 + \sum_{k=1}^n \Big(c_k \prod_{j\neq k} |z-\xi_j|^2\Big) = c \prod_{j=1}^n |z-\alpha_j|^2 .$$

- Then

$$O_\mu(z) = \frac{\eta}{\sqrt{c}} \frac{\prod_{j=1}^n (z-\xi_j)}{\prod_{j=1}^n (z-\alpha_j)},$$

where η is a complex number of modulus one which gives $O_\mu(0) > 0$.

- We have $\mathcal{D}_\mu = \mathcal{P}_{n-1} \oplus O_\mu H^2$, an algebraic direct sum!

On $O_\mu H^2$

- We have

$$\langle O_\mu f, O_\mu g \rangle_\mu = \langle f, g \rangle_2 \qquad (f, g \in H^2).$$

On $O_\mu H^2$

- We have

$$\langle O_\mu f, O_\mu g\rangle_\mu = \langle f, g\rangle_2 \qquad (f, g \in H^2).$$

- For $\lambda \in \mathbb{D}$, let

$$\widetilde{K}_\mu(z,\lambda) = \frac{\overline{O_\mu(\lambda)}}{1-\overline{\lambda}z} O_\mu(z) \qquad (z \in \mathbb{D}).$$

Then $\widetilde{K}_\mu(\cdot,\lambda) \in \mathcal{D}_\mu$ and

$$\left\langle f, \widetilde{K}_\mu(\cdot,\lambda)\right\rangle_\mu = f(\lambda) \qquad (f \in O_\mu H^2).$$

On $O_\mu H^2$

- We have

$$\langle O_\mu f, O_\mu g\rangle_\mu = \langle f, g\rangle_2 \qquad (f, g \in H^2).$$

- For $\lambda \in \mathbb{D}$, let

$$\widetilde{K}_\mu(z,\lambda) = \frac{\overline{O_\mu(\lambda)}}{1-\overline{\lambda}z} O_\mu(z) \qquad (z \in \mathbb{D}).$$

Then $\widetilde{K}_\mu(\cdot,\lambda) \in \mathcal{D}_\mu$ and

$$\left\langle f, \widetilde{K}_\mu(\cdot,\lambda)\right\rangle_\mu = f(\lambda) \qquad (f \in O_\mu H^2).$$

On $(O_\mu H^2)^\perp$

- For $k = 1, ..., n$, put

$$f_k(z) = \frac{O_\mu(z)}{O'_\mu(\xi_k)(z-\xi_k)} = \frac{\eta}{\sqrt{c}\,O'_\mu(\xi_k)} \frac{\prod_{j\neq k}(z-\xi_j)}{\prod_{j=1}^{n}(z-\alpha_j)}.$$

On $(O_\mu H^2)^\perp$

- For $k = 1, ..., n$, put

$$f_k(z) = \frac{O_\mu(z)}{O'_\mu(\xi_k)(z-\xi_k)} = \frac{\eta}{\sqrt{c}O'_\mu(\xi_k)} \frac{\prod_{j\neq k}(z-\xi_j)}{\prod_{j=1}^{n}(z-\alpha_j)}.$$

- Each f_k is a rational function with poles outside the closed unit disk, and therefore analytic on a neighborhood of it. In particular, $f_k \in \mathcal{D}_\mu$ for each k. Also, $|f_k| \leq 1$ on the closed unit disk, and $f_k(\xi_l)$ equals 1 for $k = l$ and 0 for $k \neq l$.

On $(O_\mu H^2)^\perp$

- For $k = 1, ..., n$, put

$$f_k(z) = \frac{O_\mu(z)}{O'_\mu(\xi_k)(z - \xi_k)} = \frac{\eta}{\sqrt{c}\,O'_\mu(\xi_k)} \frac{\prod_{j \neq k}(z - \xi_j)}{\prod_{j=1}^{n}(z - \alpha_j)}.$$

- Each f_k is a rational function with poles outside the closed unit disk, and therefore analytic on a neighborhood of it. In particular, $f_k \in \mathcal{D}_\mu$ for each k. Also, $|f_k| \leq 1$ on the closed unit disk, and $f_k(\xi_l)$ equals 1 for $k = l$ and 0 for $k \neq l$.

On $(O_\mu H^2)^\perp$

- We have

$$f_k \in (O_\mu H^2)^\perp$$

  for each k.

On $(O_\mu H^2)^\perp$

- We have

$$f_k \in (O_\mu H^2)^\perp$$

  for each k.

- For $k = 1, \dots, n$,

$$\|f_k\|_\mu^2 = c_k \xi_k f_k'(\xi_k) = c_k \frac{\xi_k O_\mu''(\xi_k)}{2O_\mu'(\xi_k)}.$$

On $(O_\mu H^2)^\perp$

- We have

$$f_k \in (O_\mu H^2)^\perp$$

for each k.

- For $k = 1, ..., n$,

$$\|f_k\|_\mu^2 = c_k \xi_k f_k'(\xi_k) = c_k \frac{\xi_k O_\mu''(\xi_k)}{2O_\mu'(\xi_k)}.$$

- For $k \neq j$, we have

$$\langle f_k, f_j \rangle_\mu = \frac{1}{O_\mu'(\xi_k)\,\overline{O_\mu'(\xi_j)}(1 - \overline{\xi}_j \xi_k)}.$$

On $(O_\mu H^2)^\perp$

- We have
$$f_k \in (O_\mu H^2)^\perp$$
for each k.
- For $k = 1, ..., n$,
$$\|f_k\|_\mu^2 = c_k \xi_k f_k'(\xi_k) = c_k \frac{\xi_k O_\mu''(\xi_k)}{2O_\mu'(\xi_k)}.$$
- For $k \neq j$, we have
$$\langle f_k, f_j \rangle_\mu = \frac{1}{O_\mu'(\xi_k)\,\overline{O_\mu'(\xi_j)}(1 - \overline{\xi}_j \xi_k)}.$$

On $(O_\mu H^2)^\perp$

- Let $\lambda \in \mathbb{D}$. Define

$$(\alpha_1(\lambda), \ldots, \alpha_n(\lambda))^t = \overline{B(f_1(\lambda), \ldots, f_n(\lambda))^t},$$

where $B = \left[(\langle f_k, f_j \rangle_\mu)_{k,j=1}^n \right]^{-1} \in \mathcal{M}_n(\mathbb{C})$.
Put

$$\widehat{K}_\mu(z, \lambda) = \sum_{j=1}^n \alpha_j(\lambda) f_j(z) \qquad (z \in \mathbb{D}).$$

Then $\widehat{K}_\mu(\cdot, \lambda) \in (O_\mu H^2)^\perp \subseteq \mathcal{D}_\mu$ and

$$\left\langle f, \widehat{K}_\mu(\cdot, \lambda) \right\rangle_\mu = f(\lambda) \qquad (f \in (O_\mu H^2)^\perp).$$

On $(O_\mu H^2)^\perp$

- Let $\lambda \in \mathbb{D}$. Define

$$(\alpha_1(\lambda), \ldots, \alpha_n(\lambda))^t = \overline{B(f_1(\lambda), \ldots, f_n(\lambda))^t},$$

where $B = \left[(\langle f_k, f_j \rangle_\mu)_{k,j=1}^n \right]^{-1} \in \mathcal{M}_n(\mathbb{C})$.
Put

$$\widehat{K}_\mu(z, \lambda) = \sum_{j=1}^n \alpha_j(\lambda) f_j(z) \qquad (z \in \mathbb{D}).$$

Then $\widehat{K}_\mu(\cdot, \lambda) \in (O_\mu H^2)^\perp \subseteq \mathcal{D}_\mu$ and

$$\left\langle f, \widehat{K}_\mu(\cdot, \lambda) \right\rangle_\mu = f(\lambda) \qquad (f \in (O_\mu H^2)^\perp).$$

Evaluations at points from the unit circle

- Let $j \in \{1, ..., n\}$, and define $(\alpha_1, ..., \alpha_n)^t := \overline{Be_j}$, where e_j is the column vector in $\mathbb{C}^n$ having its jth entry equal to 1 and zeros everywhere else. Then

$$\left\langle f, \sum_{l=1}^{n} \alpha_l f_l \right\rangle_\mu = f(\xi_j) \qquad (f \in \mathcal{D}_\mu).$$

Evaluations at points from the unit circle

- Let $j \in \{1, ..., n\}$, and define $(\alpha_1, ..., \alpha_n)^t := \overline{Be_j}$, where e_j is the column vector in $\mathbb{C}^n$ having its jth entry equal to 1 and zeros everywhere else. Then

$$\left\langle f, \sum_{l=1}^{n} \alpha_l f_l \right\rangle_\mu = f(\xi_j) \qquad (f \in \mathcal{D}_\mu).$$

- We have

$$K_\mu(z,\lambda)=\sum_{j=1}^{n}\alpha_j(\lambda)\frac{O_\mu(z)}{O'_\mu(\xi_j)(z-\xi_j)}+\frac{\overline{O_\mu(\lambda)}}{1-\overline{\lambda}z}O_\mu(z)\qquad (z,\lambda\in\mathbb{D}$$

where $(\alpha_1(\lambda),...,\alpha_n(\lambda))$ is the solution for the system of equations

$$\sum_{k\neq j}\frac{\alpha_k(\lambda)}{O'_\mu(\xi_k)\overline{O'_\mu(\xi_j)}(1-\overline{\xi}_j\xi_k)}+c_j\frac{\xi_j O''_\mu(\xi_j)}{2O'_\mu(\xi_j)}\alpha_j(\lambda)=\frac{\overline{O_\mu(\lambda)}}{\overline{O'_\mu(\xi_j)}(\overline{\lambda}-\overline{\xi}_j)},$$

for $j=1,...,n$.

- We have

$$K_\mu(z,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda) \frac{O_\mu(z)}{O'_\mu(\xi_j)(z-\xi_j)} + \frac{\overline{O_\mu(\lambda)}}{1-\overline{\lambda} z} O_\mu(z) \qquad (z,\lambda \in \mathbb{D}$$

where $(\alpha_1(\lambda), ..., \alpha_n(\lambda))$ is the solution for the system of equations

$$\sum_{k\neq j} \frac{\alpha_k(\lambda)}{O'_\mu(\xi_k)\overline{O'_\mu(\xi_j)}(1-\overline{\xi}_j \xi_k)} + c_j \frac{\xi_j O''_\mu(\xi_j)}{2O'_\mu(\xi_j)} \alpha_j(\lambda) = \frac{\overline{O_\mu(\lambda)}}{\overline{O'_\mu(\xi_j)}(\overline{\lambda}-\overline{\xi}_j)},$$

for $j = 1, ..., n$.

$\mu = \delta_\xi$

- Put $w_0 = (3+\sqrt{5})\xi/2$, and then we have

$$O_\mu(z) = \frac{1+\sqrt{5}}{2}\frac{z-\xi}{z-w_0}.$$

$\mu = \delta_\xi$

- Put $w_0 = (3+\sqrt{5})\xi/2$, and then we have

$$O_\mu(z) = \frac{1+\sqrt{5}}{2}\frac{z-\xi}{z-w_0}.$$

- We have

$$f_1(z) = -\xi\frac{1+\sqrt{5}}{2}\frac{1}{z-w_0}.$$

$\mu = \delta_\xi$

- Put $w_0 = (3+\sqrt{5})\xi/2$, and then we have

$$O_\mu(z) = \frac{1+\sqrt{5}}{2}\frac{z-\xi}{z-w_0}.$$

- We have

$$f_1(z) = -\xi\frac{1+\sqrt{5}}{2}\frac{1}{z-w_0}.$$

- Then

$$K_\mu(z,\lambda) = -\frac{1+\sqrt{5}}{2}\frac{z(\overline{\lambda}+(\sqrt{5}+1)\overline{\xi}/2)-(\overline{w_0}-\overline{\lambda})(\sqrt{5}+1)\xi/2}{\left(\overline{\lambda}-\overline{w}_0\right)(z-w_0)(1-\overline{\lambda}z)}.$$

$\mu = \delta_\xi$

- Put $w_0 = (3+\sqrt{5})\xi/2$, and then we have

$$O_\mu(z) = \frac{1+\sqrt{5}}{2}\frac{z-\xi}{z-w_0}.$$

- We have

$$f_1(z) = -\xi\frac{1+\sqrt{5}}{2}\frac{1}{z-w_0}.$$

- Then

$$K_\mu(z,\lambda) = -\frac{1+\sqrt{5}}{2}\frac{z(\overline{\lambda}+(\sqrt{5}+1)\overline{\xi}/2)-(\overline{w_0}-\overline{\lambda})(\sqrt{5}+1)\xi/2}{(\overline{\lambda}-\overline{w}_0)(z-w_0)(1-\overline{\lambda}z)}.$$

$\mu = \delta_{-1} + \delta_1$

- For $\mu = \delta_{-1} + \delta_1$, we have

$$1 + V_\mu(z) = \frac{\left|(\sqrt{2}-1)(z^2-(3+2\sqrt{2}))\right|^2}{|z+1|^2\,|z-1|^2} \qquad (z \in \mathbb{T}).$$

$\mu = \delta_{-1} + \delta_1$

- For $\mu = \delta_{-1} + \delta_1$, we have

$$1 + V_\mu(z) = \frac{\left|(\sqrt{2}-1)(z^2 - (3+2\sqrt{2}))\right|^2}{|z+1|^2\,|z-1|^2} \qquad (z \in \mathbb{T}).$$

- Put $w_0 = 3 + 2\sqrt{2}$, and then

$$O_\mu(z) = (\sqrt{2}+1)\frac{z^2 - 1}{z^2 - w_0}.$$

$\mu = \delta_{-1} + \delta_1$

- For $\mu = \delta_{-1} + \delta_1$, we have

$$1 + V_\mu(z) = \frac{\left|(\sqrt{2}-1)(z^2-(3+2\sqrt{2}))\right|^2}{|z+1|^2\,|z-1|^2} \qquad (z \in \mathbb{T}).$$

- Put $w_0 = 3 + 2\sqrt{2}$, and then

$$O_\mu(z) = (\sqrt{2}+1)\frac{z^2-1}{z^2-w_0}.$$

$\mu = \delta_{-1} + \delta_1$

- Then

$$f_1(z) = (\sqrt{2}+1)\frac{z-1}{z^2-w_0}$$

and

$$f_2(z) = -(\sqrt{2}+1)\frac{z+1}{z^2-w_0}.$$

$\mu = \delta_{-1} + \delta_1$

- Then

$$f_1(z) = (\sqrt{2}+1)\frac{z-1}{z^2-w_0}$$

and

$$f_2(z) = -(\sqrt{2}+1)\frac{z+1}{z^2-w_0}.$$

$\mu = \delta_{-1} + \delta_1$

- For $z, \lambda \in \mathbb{D}$ we have

$$K_\mu(z,\lambda) = C\,\frac{-(1+(\sqrt{2}-1)\overline{\lambda}^2)z^2 - \overline{\lambda}(2+\sqrt{2})z + (3+2\sqrt{2}-\overline{\lambda}^2)}{(1-\overline{\lambda}z)(z^2-(3+2\sqrt{2}))}$$

$$C = \frac{(1+\sqrt{2})^2}{\overline{\lambda}^2 - (3+2\sqrt{2})}.$$

$\mu = \delta_{-1} + \delta_1$

- For $z, \lambda \in \mathbb{D}$ we have

$$K_\mu(z,\lambda) = C\,\frac{-(1+(\sqrt{2}-1)\overline{\lambda}^{2})z^2 - \overline{\lambda}(2+\sqrt{2})z + (3+2\sqrt{2}-\overline{\lambda}^{2}}{(1-\overline{\lambda}z)(z^2-(3+2\sqrt{2}))}$$

$$C = \frac{(1+\sqrt{2})^2}{\overline{\lambda}^{2} - (3+2\sqrt{2})}.$$

- The evaluation functional at -1 is given by

$$h_1(z) = \frac{1+\sqrt{2}}{2}\,\frac{\sqrt{2}z - 2(1+\sqrt{2})}{z^2-(3+2\sqrt{2})} \qquad (z \in \mathbb{D}).$$

$\mu = \delta_{-1} + \delta_1$

- For $z, \lambda \in \mathbb{D}$ we have

$$K_\mu(z,\lambda) = C\,\frac{-(1+(\sqrt{2}-1)\overline{\lambda}^2)z^2 - \overline{\lambda}(2+\sqrt{2})z + (3+2\sqrt{2}-\overline{\lambda}^2}{(1-\overline{\lambda}z)(z^2-(3+2\sqrt{2}))}$$

$$C = \frac{(1+\sqrt{2})^2}{\overline{\lambda}^2 - (3+2\sqrt{2})}.$$

- The evaluation functional at -1 is given by

$$h_1(z) = \frac{1+\sqrt{2}}{2}\,\frac{\sqrt{2}z - 2(1+\sqrt{2})}{z^2-(3+2\sqrt{2})} \qquad (z\in\mathbb{D}).$$

- (C.Costara, Reproducing kernels for Dirichlet spaces associated to finitely supported measures, Complex Anal. Oper. Theory 10 (2016), 1277–1293.)

$\mu = \delta_{-1} + \delta_1$

- For $z, \lambda \in \mathbb{D}$ we have

$$K_\mu(z, \lambda) = C \frac{-(1 + (\sqrt{2} - 1)\overline{\lambda}^2)z^2 - \overline{\lambda}(2 + \sqrt{2})z + (3 + 2\sqrt{2} - \overline{\lambda}^2)}{(1 - \overline{\lambda} z)(z^2 - (3 + 2\sqrt{2}))}$$

$$C = \frac{(1 + \sqrt{2})^2}{\overline{\lambda}^2 - (3 + 2\sqrt{2})}.$$

- The evaluation functional at -1 is given by

$$h_1(z) = \frac{1 + \sqrt{2}}{2} \frac{\sqrt{2}z - 2(1 + \sqrt{2})}{z^2 - (3 + 2\sqrt{2})} \qquad (z \in \mathbb{D}).$$

- (C.Costara, Reproducing kernels for Dirichlet spaces associated to finitely supported measures, Complex Anal. Oper. Theory 10 (2016), 1277–1293.)

Thank You!