Some new thoughts on systems of exponentials.

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Espaces de fonctions et théories des opérateurs

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Introduction

The trigonometric system $(e^{int})_{n\in\mathbb{Z}}$ is an ONB for $L^2(0,2\pi)$.

Every $f \in L^2(0,2\pi)$ can be written as

$$f(t) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{int},$$

where

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt.$$

Note:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2.$$

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What about changing

$$(e^{int})_{n\in\mathbb{Z}} \longrightarrow (e^{i\lambda_n t})_{n\in\mathbb{Z}}?$$

Question:

Does the system $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ form a basis for $L^2(0, 2\pi)$?

If λ_n is close to n, is $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ a basis?

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Theorem (Paley–Wiener, 1934)

Let $\lambda_n \in \mathbb{R}$, $n \in \mathbb{Z}$ and assume that

$$\sup_{n\in\mathbb{Z}} |\lambda_n - n| < \frac{1}{\pi^2}.$$

Then $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,2\pi)$.

Duffin–Eachus (1942): $\frac{1}{\pi^2} \approx 0.101 \rightsquigarrow \ln(2)/\pi \approx 0.22.$

Kadets (1964) : $\ln(2)/\pi \rightsquigarrow 1/4 = 0.25.$

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$(e_n)_{n\geq 1}$ is a Riesz basis for a Hilbert space $\mathcal{H}$ if there is an isomorphism $U : \mathcal{H} \longrightarrow \mathcal{H}$ such that $(Ue_n)_{n\geq 1}$ is an ONB for $\mathcal{H}$.

Equivalent to: $(e_n)_{n\geq 1}$ is complete and

$$c_1 \sum_{n\geq 1} |a_n|^2 \leq \left\| \sum_{n\geq 1} a_n e_n \right\|_{\mathcal{H}}^2 \leq c_2 \sum_{n\geq 1} |a_n|^2.$$

An orthonormal basis is a Riesz basis with $c_1 = c_2 = 1$.

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Another type of basis

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We say that $(e_n)_{n\geq 1}$ is an asymptotically orthonormal basis (AOB) for $\mathcal{H}$ if

- $(e_n)_{n\geq 1}$ is complete;
- For all $N \geq 1$, $\exists\, c_N, C_N > 0$ s.t.

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Note: $ONB \implies AOB \implies$ Riesz basis.

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Thin sequences

Hardy space of the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$:

$$H^2 = \left\{ f \in \mathrm{Hol}(\mathbb{C}_+) : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty \right\}.$$

Reproducing kernel: $k_\lambda(z) = \frac{i}{2\pi} \frac{1}{z - \overline{\lambda}}$, $z, \lambda \in \mathbb{C}_+$.

Theorem (Volberg (1982), Gorkin–McCarthy–Pott–Wick (2014))

Let $(\lambda_n)_n \subset \mathbb{C}_+$. The sequence $(k_{\lambda_n}/\|k_{\lambda_n}\|)_n$ is an AOB for its closed linear span if and only if $(\lambda_n)_n$ is thin, i.e.

$$\lim_{n\to\infty} \prod_{k\neq n} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \overline{\lambda_k}} \right| = 1.$$

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An approach through complex analysis.

Paley–Wiener: the Fourier transform $\mathcal{F}$ maps unitarily H^2 onto $L^2([0,\infty))$. Moreover, if $a > 0$, then

$$\begin{aligned}\mathcal{F}^{-1}(L^2(0,a)) =& (\Theta_a H^2)^\perp \\ =& \left\{ f \in H^2 : \int_{\mathbb{R}} f(t)\overline{\Theta_a(t) g(t)}\, dt = 0,\ \forall g \in H^2 \right\},\end{aligned}$$

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A simple computation shows that

$$\mathcal{F}^{-1}\left(e^{-i\overline{\lambda}t}\chi_{(0,a)}\right)(z) = \frac{i}{2\pi}\frac{1-\overline{\Theta_a(\lambda)}\Theta_a(z)}{z-\overline{\lambda}}.$$

Let

$$K_{\Theta_a} = (\Theta_a H^2)^{\perp}, \qquad \text{and} \qquad k_{\lambda}^{\Theta_a}(z) = \frac{i}{2\pi}\frac{1-\overline{\Theta_a(\lambda)}\Theta_a(z)}{z-\overline{\lambda}}.$$

$(e^{i\lambda_n t})_{n\geq 1}$ is an AOB for $L^2(0,a)$ if and only if $(k_{\lambda_n}^{\Theta_a})_{n\geq 1}$ is an AOB for K_{Θ_a}.

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Model spaces

Θ : inner function: that is Θ is a bounded and analytic function in $\mathbb{C}_+$ s.t.

$$\lim_{y\to 0} |\Theta(x+iy)| = 1 \qquad \text{for almost all } x \in \mathbb{R}.$$

Model space:

$$K_\Theta = (\Theta H^2)^\perp.$$

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Theorem (Chalendar-F.-Timotin, 2003)

Let $(\lambda_n)_n$ be a sequence in $\mathbb{C}_+$ and Θ be an inner function such that

$$\sup_n |\Theta(\lambda_n)| < 1. \tag{1}$$

The following are equivalent :

1. $(k^\Theta_{\lambda_n}/\|k^\Theta_{\lambda_n}\|)_n$ *is an AOB for* K_Θ.
2. $(\lambda_n)_n$ *is thin and* $(k^\Theta_{\lambda_n})_n$ *is minimal.*

Remark: in the case when $\Theta = \Theta_a$,

$$(1) \iff \inf_n \Im(\lambda_n) > 0.$$

Theorem (F.–Rupam, 2016)

Let $(\lambda_n)_n$ be a sequence in $\mathbb{C}_+$ such that $(k^\Theta_{\lambda_n}/\|k^\Theta_{\lambda_n}\|)_n$ is an AOB for K_Θ. Let $(\lambda'_n)_n \subset \mathbb{C}_+$ such that

$$\lim_{n\to\infty} \frac{|\lambda'_n - \lambda_n|}{\Im\mathrm{m}(\lambda_n)} = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{\|k^\Theta_{\lambda_n}\|^2} \int_{[\lambda_n,\lambda'_n]} \omega^{-2}(z)|dz| = 0.$$

Then $(k^\Theta_{\lambda'_n}/\|k^\Theta_{\lambda'_n}\|)_n$ is also an AOB for K_Θ.

Here ω is some weight involved in the Bernstein type inequality proved by Baranov for model spaces.

Corollary (F.–Rupam, 2016)

Let $(\lambda_n)_n$ *be a sequence in* $\mathbb{C}_+$ *such that* $(k^{\Theta}_{\lambda_n}/\|k^{\Theta}_{\lambda_n}\|)_n$ *is an AOB for* K_Θ*. Let* $(\lambda'_n)_n \subset \mathbb{C}_+$ *such that, for some* $\gamma > 1/3$*, we have*

$$\lim_{n\to\infty} \left| \frac{\lambda_n - \lambda'_n}{\lambda_n - \overline{\lambda'_n}} \right| (1 - |\Theta(\lambda_n)|)^{-\gamma} = 0.$$

Then $(k^{\Theta}_{\lambda'_n}/\|k^{\Theta}_{\lambda'_n}\|)_n$ *is also an AOB for* K_Θ*.*

Theorem (F.–Rupam, 2016)

Let Θ be a meromorphic inner function satisfying the connected level condition. Let $(\lambda_n)_n$ and $(\lambda'_n)_n$ be two real sequences and assume that $(k^\Theta_{\lambda_n}/\|k^\Theta_{\lambda_n}\|)_n$ is an AOB for K_Θ. If

$$\lim_{n\to\infty} |\arg\Theta(\lambda_n) - \arg\Theta(\lambda'_n)| = 0,$$

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Corollary (Chalendar-F.-Timotin,2003)

If $(\lambda_n)_n \subset \mathbb{R}$ and assume that

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THANK YOU FOR YOUR ATTENTION!