Espaces de type Dirichlet sur le Polydisque

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Espaces de fonctions et théorie des opérateurs

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Holomorphic Functions on the Polydisc

Let N be a fixed natural number.

The unit polydisc $\mathbb{D}^N = \mathbb{D} \times \cdots \times \mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

For any multi-index $k = (k_1, \ldots, k_N) \in \mathbb{Z}_+^N$ and any $z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ let $z^k = z_1^{k_1} \cdots z_N^{k_N}$.

Consider $H(\mathbb{D}^N)$ the algebra of all functions $f \colon \mathbb{D}^N \to \mathbb{C}$ that are holomorphic in each variable, equivalently, there exists $(a_k)_{k \in \mathbb{Z}_+^N}$ with the property that

$$f(z) = \sum_{k \in \mathbb{Z}_+^N} a_k z^k, \quad z \in \mathbb{D}^N, \tag{2.1}$$

where the series converges absolutely and uniformly on any compact subset in $\mathbb{D}^N$.
W. Rudin et al.

Dirichlet Type Spaces on the Polydisc

Let $\alpha \in \mathbb{R}^N$ be fixed. The *Dirichlet type space* $\mathcal{D}_\alpha$ consists on all functions $f \in H(\mathbb{D}^N)$ subject to the condition

$$f(z) = \sum_{k \in \mathbb{Z}_+^N} a_k z^k, \quad z \in \mathbb{D}^N, \quad \sum_{k \in \mathbb{Z}_+^N} (k+1)^\alpha |a_k|^2 < \infty, \tag{2.2}$$

where, with an abuse of notation, $(k+1)^\alpha = (k_1+1)^{\alpha_1} \cdots (k_N+1)^{\alpha_N}$. $\mathcal{D}_\alpha$ is naturally organized as a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\alpha$

$$\langle f, g \rangle_\alpha = \sum_{k \in \mathbb{Z}_+^N} (k+1)^\alpha a_k \overline{b_k}, \tag{2.3}$$

where f has representation (2.2) and similarly $g(z) = \sum_{k \in \mathbb{Z}_+^N} b_k z^k$, and norm $\| \cdot \|_\alpha$

$$\|f\|_\alpha^2 = \sum_{k \in \mathbb{Z}_+^N} (k+1)^\alpha |a_k|^2. \tag{2.4}$$

G.D. Taylor 1966 (for $N = 1$) and D. Jupiter, D. Redett 2006 (for $N \geq 2$)

Reproducing Kernel

For any $\alpha \in \mathbb{R}^N$, on the polydisc $\mathbb{D}^N$ one can define the following kernel

$$K^\alpha(w,z) = \sum_{k\in\mathbb{Z}_+^N} (k+1)^{-\alpha}\overline{w}^k z^k, \quad z,w \in \mathbb{D}^N, \tag{2.5}$$

where, for $w = (w_1, \ldots, w_N) \in \mathbb{D}^N$ one denotes $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_N)$.

As usually, we let $K_w^\alpha = K^\alpha(w, \cdot)$.

K^α is the reproducing kernel for the space $\mathcal{D}_\alpha$ in the sense that the following two properties hold:

(rk1) $K_w^\alpha \in \mathcal{D}_\alpha$ for all $w \in \mathbb{D}^N$.

(rk2) $f(w) = \langle f, K_w^\alpha\rangle_\alpha$ for all $f \in \mathcal{D}_\alpha$ and all $w \in \mathbb{D}^N$.

It follows (this is actually a more general statement) that the set $\{K_w^\alpha \mid w \in \mathbb{D}^N\}$ is total in $\mathcal{D}_\alpha$ and that the kernel K^α is positive semidefinite.

$\mathcal{D}_0$ is the Hardy Space $H^2(\mathbb{D}^N)$

Let $\mathbb{T} = \partial\mathbb{D}$ denote the one-dimensional torus and let $\mathbb{T}^N = \mathbb{T} \times \cdots \times \mathbb{T}$ be the N-dimensional torus, also called the distinguished boundary of the unit polydisc $\mathbb{D}^N$ (which is only a subset of $\partial\mathbb{D}^N$).

We consider the product measure $\mathrm{d}\, m_N = \mathrm{d}\, m_1 \times \cdots \times \mathrm{d}\, m_1$ on $\mathbb{D}^N$, where $\mathrm{d}\, m_1$ denotes the normalized Lebesgue measure on $\mathbb{T}$, and for any function $f \in H(\mathbb{D}^N)$ and $0 \leq r < 1$ let $f_r(z) = f(rz)$ for $z \in \mathbb{D}^N$.

Then $f \in H(\mathbb{D}^N)$ belongs to $H^2(\mathbb{D}^N)$ if and only if

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}^N} |f_r|^2 \,\mathrm{d}\, m_N < \infty.$$

W. Rudin at al.

Norm, Inner product, and Reproducing Kernel on $H^2(\mathbb{D}^N)$

The norm $\|\cdot\|_0$ and inner product $\langle\cdot,\cdot\rangle_0$ on the Hardy space $H^2(\mathbb{D}^N)$ are defined by

$$\|f\|_0^2 = \sup_{0\leq r<1} \int_{\mathbb{T}^N} |f_r|^2 \,\mathrm{d}\, m_N = \lim_{r\to 1-} \int_{\mathbb{T}^N} |f_r|^2 \,\mathrm{d}\, m_N, \quad f \in H^2(\mathbb{D}^N),$$

$$\langle f, g\rangle_0 = \lim_{r\to 1-} \int_{\mathbb{T}^N} f_r \overline{g_r} \,\mathrm{d}\, m_N, \quad f, g \in H^2(\mathbb{D}^N).$$

$\mathcal{D}_0$ coincides as a Hilbert space with $H^2(\mathbb{D}^N)$.

The reproducing kernel K^0 has a simple representation in this case

$$K^0(w,z) = \frac{1}{1-\overline{w}_1 z_1} \cdots \frac{1}{1-\overline{w}_N z_N}.$$

The Operator T_α: Radial Derivative

Let $\mathcal{P}_N$ denote the complex vector space of polynomial functions in N complex variables, that is, those functions f that admit a representation

$$f(z) = \sum_{k \in \mathbb{Z}_+^N} a_k z^k, \quad \text{supp}\{a_k\}_{k \in \mathbb{Z}_+^N} \text{ finite.}$$

On the additive group $\mathbb{R}^N$ define a representation $T_\cdot : \mathbb{R}^N \to \mathcal{L}(\mathcal{P}_N)$ by

$$(T_\alpha f)(z) = \sum_{k \in \mathbb{Z}_+^N} (k+1)^\alpha a_k z^k, \quad \alpha \in \mathbb{R}^N,\ f \in \mathcal{P}_N,\ z \in \mathbb{D}^N. \tag{2.6}$$

Radial Derivative, cf. F. Beatrous, J. Burbea 1989

Triplets of Dirichlet Type Spaces: The Embeddings

Theorem

Let $\alpha \in \mathbb{R}^N$ be positive, in the sense that $\alpha_k \geq 0$ for all $k = 1, \ldots, N$ and $\alpha_k > 0$ for at least one of them. Then, $(\mathcal{D}_\alpha; \mathcal{D}_0; \mathcal{D}_{-\alpha})$ is a triplet of Hilbert spaces with the following properties:

(a) *The embeddings $j_+ : \mathcal{D}_\alpha \hookrightarrow \mathcal{D}_0$ and $j_- : \mathcal{D}_0 \hookrightarrow \mathcal{D}_{-\alpha}$ are compact.*

(b) *The adjoint j_+^* is defined by $j_+^* f = T_{-\alpha} f$ for all $f \in \mathcal{D}_0$.*

Triplets of Dirichlet Type Spaces: The Kernel Operator

Theorem (continued)

(c) *The kernel operator $A_\alpha = j_+ j_+^*$ is a bounded positive operator on $\mathcal{D}_0 = H^2(\mathbb{D}^N)$ and is determined, for all $f \in \mathrm{Dom}(A_\alpha)$, by*

$$\begin{aligned}(A_\alpha f)(z) &= (T_{-\alpha} f)(z) = \langle f, \overline{K_\cdot^\alpha(z)} \rangle_0 \\ &= \lim_{r \to 1-} \int_{\mathbb{T}^N} f_r(w) K^\alpha(rw, z) \, \mathrm{d}\, m_N(w), \quad z \in \mathbb{D}^N.\end{aligned}$$

in particular, it is an integral operator with kernel K^α,

Triplets of Dirichlet Type Spaces: Hamiltonian and Spectra

Theorem (continued)

(d) *The Hamiltonian operator $H_\alpha = A_\alpha^{-1}$ is a positive selfadjoint operator*

$$H_\alpha f = T_\alpha f,\ \text{for all } f \in \mathrm{Dom}(H_\alpha) = \{f \in \mathcal{D}_0 \mid T_\alpha f \in \mathcal{D}_0\}.$$

(e) *The operator $T_\alpha \colon \mathcal{D}_\alpha \to \mathcal{D}_{-\alpha}$ is the canonical unitary identification of $\mathcal{D}_\alpha$ with $\mathcal{D}_{-\alpha}$.*

In addition,

$$\sigma(A_\alpha) \setminus \{0\} = \{(k+1)^{-\alpha} \mid k \in \mathbb{Z}_+^N\}$$

and

$$\sigma(H_\alpha) = \{(k+1)^{\alpha} \mid k \in \mathbb{Z}_+^N\},$$

are all eigenvalues.

A Projective Limit of Dirichlet Type Spaces

For any $\alpha \geq 0$, $\mathcal{D}_\alpha(\mathbb{D}^N)$ is continuously embedded in $H^2(\mathbb{D}^N) = \mathcal{D}_0(\mathbb{D}^N)$ and, if $\alpha \neq 0$ then the embedding $\mathcal{D}_\alpha(\mathbb{D}^N) \hookrightarrow H^2(\mathbb{D}^N)$ is contractive and compact.

The same is true for the embedding $\mathcal{D}_\beta(\mathbb{D}^N) \hookrightarrow \mathcal{D}_\alpha(\mathbb{D}^N)$ whenver $\beta \geq \alpha$ and $\beta \neq \alpha$.

With respect to continuous embeddings, the family $\{\mathcal{D}_\alpha(\mathbb{D}^N)\}_{\alpha \geq 0}$ is a projective system of Hilbert spaces and let

$$\mathcal{S}(\mathbb{D}^N) = \lim_{\substack{\alpha \geq 0}} \operatorname{proj} \mathcal{D}_\alpha(\mathbb{D}^N),$$

that is,

$$\mathcal{S}(\mathbb{D}^N) = \bigcap_{\alpha \geq 0} \mathcal{D}_\alpha(\mathbb{D}^N),$$

with the locally convex topology defined by the family of quadratic norms $\{\|\cdot\|_\alpha \mid \alpha \in \mathbb{R}^N\}$, with norms $\|\cdot\|_\alpha$ defined at (2.4) and restricted to $\mathcal{S}(\mathbb{D}^N)$.

$\mathcal{S}(\mathbb{D}^N) = \operatorname{lim\,proj}_{\alpha \geq 0} \mathcal{D}_\alpha(\mathbb{D}^N)$ is a Nuclear Fréchet Space

To see this we construct a scale of continuously embedded Dirichlet type spaces that has the same projective limit and with corresponding nuclear embeddings.

For each $n \in \mathbb{N}$ let, with an abuse of notation, $\mathcal{D}_n$ denote the Dirichlet type space corresponding to the multi-index $(n, \ldots, n) \in \mathbb{N}^N \subset \mathbb{R}^N$. Because, for each $\alpha \in \mathbb{R}^N$ there exists $n \in \mathbb{N}$ with $\alpha \leq (n, \ldots, n)$, it follows that

$$\mathcal{S}(\mathbb{D}^N) = \operatorname{lim\,proj}_{n \in \mathbb{N}} \mathcal{D}_n,$$

hence $\mathcal{S}(\mathbb{D}^N)$ is metrizable and complete (standard argument).

$\mathcal{S}(\mathbb{D}^N) = \lim\limits_{\alpha \geq 0} \operatorname{proj} \mathcal{D}_\alpha(\mathbb{D}^N)$ is a Nuclear Fréchet Space

We actually have a (left) scale of Hilbert spaces

$$\cdots \hookrightarrow \mathcal{D}_{n+1} \hookrightarrow \mathcal{D}_n \hookrightarrow \cdots \mathcal{D}_1 \hookrightarrow \mathcal{D}_0 = H^2(\mathbb{D}^N).$$

All the embeddings are continuous, actually compact, and each space in the scale is dense in its successor (and hence in all of its successors).

Moreover, for any $n \geq 1$, letting A denote the kernel operator of the embedding $\mathcal{D}_{n+2} \hookrightarrow \mathcal{D}_n$, we have

$$\sigma(A) = \{(1+k_1)^{-2} \cdots (1+k_N)^{-2} \mid k \in \mathbb{Z}_+^N\} \cup \{0\},$$

hence A is a trace-class (nuclear) operator, which makes the corresponding embedding $\mathcal{D}_{n+2} \hookrightarrow \mathcal{D}_n$ to be a Hilbert-Schmidt (quasi-nuclear) operator. Therefore, the projective limit $\mathcal{S}(\mathbb{D}^N)$ is a nuclear Fréchet space.

$\mathcal{S}(\mathbb{D}^N) = \lim\,\mathrm{proj}_{\alpha \geq 0}\, \mathcal{D}_\alpha(\mathbb{D}^N)$

The algebra $\mathcal{P}_N$ of polynomials in N indeterminates is contained in $\mathcal{S}(\mathbb{D}^N)$, in particular, $\mathcal{S}(\mathbb{D}^N)$ is dense in $H^2(\mathbb{D}^N)$, but $\mathcal{S}(\mathbb{D}^N)$ does not coincide with $\mathcal{P}_N$.

Also, $\mathcal{S}(\mathbb{D}^N) \subset A(\mathbb{D}^N)$, where $A(\mathbb{D}^N)$ denotes the *disk algebra* on the polydisc, more precisely, it consists of all functions $f \in H(\mathbb{D}^N)$ that have continuous extensions to the boundary $\partial\mathbb{D}^N$.

$\mathcal{S}^*(\mathbb{D}^N) = \operatorname{lim\,ind}_{\alpha\geq 0} \mathcal{D}_{-\alpha}$

The conjugate dual spaces $\mathcal{D}_{-\alpha} = \mathcal{D}_\alpha^*$, indexed by the directed set $\alpha \geq 0$, make an inductive system of Hilbert spaces and let

$$\mathcal{S}^*(\mathbb{D}^N) = \operatorname*{lim\,ind}_{\alpha\geq 0} \mathcal{D}_{-\alpha},$$

where

$$\mathcal{S}^*(\mathbb{D}^N) = \bigcup_{\alpha\geq 0} \mathcal{D}_{-\alpha},$$

endowed with the strongest topology that makes all the embeddings $\mathcal{D}_{-\alpha} \hookrightarrow \mathcal{S}^*(\mathbb{D}^N)$ continuous.

The Rigging $\mathcal{S}(\mathbb{D}^N) \hookrightarrow H^2(\mathbb{D}^N) \hookrightarrow \mathcal{S}^*(\mathbb{D}^N)$

We got a scale that produces a rigging of $H^2(\mathbb{D}^N)$

$$\cdots \hookrightarrow \mathcal{D}_n \hookrightarrow \cdots \mathcal{D}_1 \hookrightarrow H^2(\mathbb{D}^N) \hookrightarrow \mathcal{D}_{-1} \hookrightarrow \cdots \hookrightarrow \mathcal{D}_{-n} \hookrightarrow \cdots.$$

In particular,

$$\mathcal{S}^*(\mathbb{D}^N) = \liminf_{\alpha<0} \mathcal{D}_\alpha = \liminf_{n\in\mathbb{N}} \mathcal{D}_{-n}$$

where $\mathcal{D}_{-n} = \mathcal{D}_n^*$ is the Dirichlet type space with index $(-n, \ldots, -n) \in \mathbb{Z}_-^N$ for all $n \in \mathbb{N}$ and is complete (by a classical theorem of L. Schwartz). Note that $H^2(\mathbb{D}^N)$ is also dense in $\mathcal{S}^*(\mathbb{D}^N)$.

The Bergman Spaces $A^2_\alpha(\mathbb{D}^N)$

The spaces $\mathcal{D}_\alpha$ for $\alpha < 0$ are, to a certain extent, of Bergman type.

More precisely, let $\mathrm{d}\,A_1$ denote the normalized area measure on the unit disc $\mathbb{D}$ and let $\mathrm{d}\,A_N = \mathrm{d}\,A_1 \times \cdots \times \mathrm{d}\,A_1$ be the corresponding product measure on $\mathbb{D}^N$. Then, a function $f \in H(\mathbb{D}^N)$ is in $\mathcal{D}_\alpha$, for a fixed $\alpha < 0$ in $\mathbb{R}^N$, if and only if

$$\int_{\mathbb{D}^N} |f(z)|^2 \frac{1}{(1-|z_1|^2)^{1+\alpha_1}} \cdots \frac{1}{(1-|z_N|^2)^{1+\alpha_N}} \,\mathrm{d}\,A_N(z) < \infty,$$

and, in this case, the norm $\|\cdot\|_\alpha$ on $\mathcal{D}_\alpha$ is equivalent to the norm $\|\cdot\|_{\mathrm{a},\alpha}$ defined by

$$\|f\|^2_{\mathrm{a},\alpha} = \int_{\mathbb{D}^N} |f(z)|^2 \frac{1}{(1-|z_1|^2)^{1+\alpha_1}} \cdots \frac{1}{(1-|z_N|^2)^{1+\alpha_N}} \,\mathrm{d}\,A_N(z).$$

The Bergman Spaces $A^2_\alpha(\mathbb{D}^N)$

Let $A^2_\alpha(\mathbb{D}^N)$ denote the Hilbert space of all functions f holomorphic in $\mathbb{D}^N$ such that

$$\|f\|^2_{\mathrm{a},\alpha} = \int_{\mathbb{D}^N} |f(z)|^2 \frac{1}{(1-|z_1|^2)^{1+\alpha_1}} \cdots \frac{1}{(1-|z_N|^2)^{1+\alpha_N}} \,\mathrm{d}\, A_N(z) < \infty.$$

with norm $\|\cdot\|_{\mathrm{a},\alpha}$ defined as before.

So, for $\alpha < 0$, the Bergman space $A^2_\alpha(\mathbb{D}^N) = \mathcal{D}_\alpha$ as topological spaces, but not isometrically.

The Bergman Spaces $A^2_\alpha(\mathbb{D}^N)$

A calculation shows that, for all $f \in A^2_\alpha(\mathbb{D}^N)$, we have

$$\|f\|^2_{\mathrm{a},\alpha} = 2^N \sum_{k\in\mathbb{Z}^N_+} \frac{1}{-\alpha\binom{k-\alpha}{-\alpha}} |a_k|^2, \text{ for } f(z) = \sum_{k\in\mathbb{Z}^N_+} a_k z^k,\ z \in \mathbb{D}^N,$$

where we denote $-\alpha\binom{k-\alpha}{-\alpha} = (-\alpha_1)\binom{k_1-\alpha_1}{-\alpha_1}\cdots(-\alpha_n)\binom{k_N-\alpha_N}{-\alpha_N}$ and use the generalized binomial coefficients.

From here it follows that there exists $C_\alpha > 0$ such that

$$\|f\|_{\mathrm{a},\alpha} \leq C_\alpha \|f\|_2 \text{ for all } f \in H^2(\mathbb{D}^N)$$

hence $H^2(\mathbb{D}^N)$ is continuously embedded into $A^2_\alpha(\mathbb{D}^N)$.

$\mathcal{S}(\mathbb{D}^N) \hookrightarrow H^2(\mathbb{D}^N) \hookrightarrow \mathcal{S}^*(\mathbb{D}^N)$

$\mathcal{S}^*(\mathbb{D}^N)$ can be realized as the inductive limit of Bergman type spaces $A^2_\alpha(\mathbb{D}^N)$ for $\alpha < 0$,

$$\mathcal{S}^*(\mathbb{D}^N) = \liminf_{\alpha<0} A^2_\alpha(\mathbb{D}^N),$$

since for all $\alpha < 0$ the Dirichlet type space $\mathcal{D}_\alpha$ coincides with the Bergman space $A^2_\alpha(\mathbb{D}^N)$ as topological spaces of holomorphic functions, even though their norms are not identical, but still equivalent.

In conclusion,

$$\mathcal{S}(\mathbb{D}^N) \hookrightarrow H^2(\mathbb{D}^N) \hookrightarrow \mathcal{S}^*(\mathbb{D}^N)$$

is a rigging of the Hardy space $H^2(\mathbb{D}^N)$ obtained through a scale of Dirichlet type spaces and Bergman spaces

$$\cdots \hookrightarrow \mathcal{D}_n \hookrightarrow \cdots \mathcal{D}_1 \hookrightarrow H^2(\mathbb{D}^N) \hookrightarrow A^2_{-1} \hookrightarrow \cdots \hookrightarrow A^2_{-n} \hookrightarrow \cdots$$

Triplets of Dirichlet Type Spaces: The General Case

Theorem

For any $\alpha, \beta \in \mathbb{R}^N$ *we can define a triplet of closely embedded Hilbert spaces* $(\mathcal{D}_\beta; \mathcal{D}_\alpha; \mathcal{D}_{2\alpha-\beta})$ *with the following properties:*

(a) *The closed embeddings* $j_\pm$ *of* $\mathcal{D}_\beta$ *in* $\mathcal{D}_\alpha$ *and, respectively, of* $\mathcal{D}_\alpha$ *in* $\mathcal{D}_{2\alpha-\beta}$*, have maximal domains* $\mathcal{D}_\alpha \cap \mathcal{D}_\beta$ *and, respectively,* $\mathcal{D}_\alpha \cap \mathcal{D}_{2\alpha-\beta}$.

(b) *The adjoint* j_+^* *is defined by* $j_+^* f = T_{\alpha-\beta} f$ *for all* $f \in \mathrm{Dom}(j_+^*) = \mathcal{D}_\alpha \cap \mathcal{D}_{2\alpha-\beta}$.

Triplets of Dirichlet Type Spaces: The General Case

Theorem (Continuation)

(c) *The kernel operator* $A = j_+ j_+^*$*, considered as a positive selfadjoint operator in* $\mathcal{D}_\alpha$*, is defined by* $Af = T_{\alpha-\beta} f$ *for all* $f \in \mathrm{Dom}(A) = \mathcal{D}_\alpha \cap \mathcal{D}_{2\alpha-\beta} \cap \mathcal{D}_{3\alpha-2\beta}$ *and is an "integral operator with kernel* K^β*", in the sense that, for all* $f \in \mathrm{Dom}(A)$*, we have*

$$(Af)(z) = \langle f, K_z^\beta \rangle_\alpha, \quad z \in \mathbb{D}^N. \tag{4.1}$$

When viewed as an operator from $\mathcal{D}_{2\alpha-\beta}$ *in* $\mathcal{D}_\beta$*,* A *admits a unique extension to a unitary operator* $\widetilde{A}\colon \mathcal{D}_{2\alpha-\beta} \to \mathcal{D}_\beta$*.*

Triplets of Dirichlet Type Spaces: The General Case

Theorem (Continuation)

(d) *The Hamiltonian operator* $H = A^{-1}$ *is a positive selfadjoint operator in* $\mathcal{D}_\alpha$ *defined by* $Hf = T_{\beta-\alpha} f$ *for all* $f \in \operatorname{Dom}(H) = \mathcal{D}_\alpha \cap \mathcal{D}_\beta \cap \mathcal{D}_{2\beta-\alpha}$. *When viewed as an operator from* $\mathcal{D}_\beta$ *in* $\mathcal{D}_{2\alpha-\beta}$, H *admits a unique extension to a unitary operator* $\widetilde{H} \colon \mathcal{D}_\beta \to \mathcal{D}_{2\alpha-\beta}$, *and* $\widetilde{H} = \widetilde{A}^{-1}$.

(e) *The canonical unitary identification of* $\mathcal{D}_{2\alpha-\beta}$ *with* $\mathcal{D}_\beta^*$ *is the operator* Θ *defined by*

$$(\Theta g)f = \langle T_{\alpha-\beta} f, g\rangle_\beta, \quad f \in \mathcal{D}_{2\alpha-\beta},\ g \in \mathcal{D}_\beta.$$

In addition, $\sigma(A) \setminus \{0\} = \{(k+1)^{\alpha-\beta} \mid k \in \mathbb{Z}_+^N\}$ *and* $\sigma(H) \setminus \{0\} = \{(k+1)^{\beta-\alpha} \mid k \in \mathbb{Z}_+^N\}$. *Moreover, the two embeddings* $j_\pm$ *are simultaneously continuous and this happens if and only if* $\alpha \leq \beta$.

Generation of Triplets of Hilbert Spaces: Factoring the Hamiltonian

Theorem

Let H be a positive selfadjoint operator in the Hilbert space $\mathcal{H}$, that admits a bounded inverse $A = H^{-1}$. Then there exists $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$, with $\operatorname{Ran}(T) = \mathcal{G}$ *and $H = T^* T$. In addition, $S = T^{-1} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. Then:*

(i) *The Hilbert space $\mathcal{H}_+ := \mathcal{D}(T) := \mathcal{R}(S)$ is embedded in $\mathcal{H}$ with its embedding i_T having range dense in $\mathcal{H}$, and its kernel operator $A = i_T i_T^*$ coincides with H^{-1}.*

(ii) *$\mathcal{H}$ is embedded in the Hilbert space $\mathcal{H}_- = \mathcal{R}(T^*)$ with its embedding $j_{T^*}^{-1}$ having range dense in $\mathcal{R}(T^*)$. The kernel operator $B = j_{T^*}^{-1} j_{T^*}^{-1*}$ of this embedding is unitary equivalent with $A = H^{-1}$.*

Generation of Triplets of Hilbert Spaces: Weak Solutions

Theorem (continued)

(iii) *The operator* $V = i_T^* | \operatorname{Ran}(T^*)$, *that is,*

$$\langle i_T x, y \rangle_{\mathcal{H}} = (x, Vy)_T, \quad x \in \operatorname{Dom}(T),\ y \in \operatorname{Ran}(T^*), \tag{5.1}$$

extends uniquely to a unitary operator $\widetilde{V}$ *between the Hilbert spaces* $\mathcal{R}(T^*)$ *and* $\mathcal{D}(T)$.

(iv) *The operator* H, *when viewed as a linear operator with domain dense in* $\mathcal{D}(T)$ *and range in* $\mathcal{R}(T^*)$, *extends uniquely to a unitary operator* $\widetilde{H} \colon \mathcal{D}(T) \to \mathcal{R}(T^*)$, *and* $\widetilde{H} = \widetilde{V}^{-1}$.

Generation of Triplets of Hilbert Spaces: Dual Space

Theorem (continued)

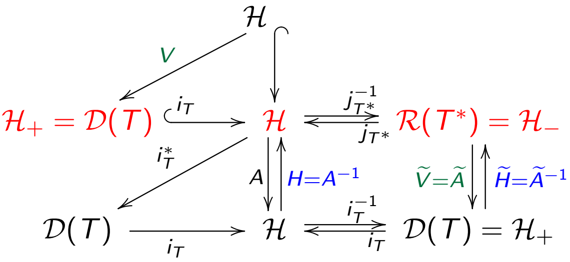
(v) *The operator* $\Theta\colon \mathcal{R}(T^*) \to \mathcal{D}(T)^*$ *defined by*

$$(\Theta\alpha)(x) := (\widetilde{V}\alpha, x)_T, \quad \alpha \in \mathcal{R}(T^*),\ x \in \mathcal{D}(T), \tag{5.2}$$

provides a canonical and unitary identification of the Hilbert space $\mathcal{R}(T^*)$ *with the conjugate space* $\mathcal{D}(T)^*$*, in particular, for all* $y \in \operatorname{Dom}(T^*)$

$$\|y\|_{T^*} = \sup\Big\{\frac{|\langle y, x\rangle_{\mathcal{H}}|}{|x|_T} \mid x \in \operatorname{Dom}(T) \setminus \{0\}\Big\}. \tag{5.3}$$

Generation of Triplets of Hilbert Spaces: The General Picture



H	Hamiltonian (unbounded)
$A = H^{-1}$	Kernel Operator
$H = T^*T$	Factor Operator (unbounded)
$A = SS^*$	Factor Operator (bounded)

Further

Joint Work with Petru Cojuhari, Cracow, Poland

- Triplets of Closely Embedded Hilbert Spaces
- Dirichlet Type Spaces on the Polydisc with Indefinite Index
- Weak Solutions for "Elliptic Like" Boundary Value Problems
- Weyl Decompositions of Triplets of Hilbert Spaces
- Indefinite Variant: Triplets of Krein Spaces
- Triplets of Krein Spaces associated to Dirac Operators
- Triplets of Hilbert Spaces Associated to Noncommutative Radon-Nikodým Derivatives and Lebesgue Decompositions
- Triplets of Hilbert Spaces in Quantum Probability