The reachable space of the heat equation and spaces of analytic functions in a square

A. Hartmann, K. Kellay and M. Tucsnak
IMB Université de Bordeaux

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We consider the the $1-d$ heat equation

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geqslant 0,\ x \in (0,\pi), \\ w(t,0) = u_0(t),\ \ w(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi), \end{cases} \tag{1}$$

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For $\tau > 0$, the input-to-state maps $(\Phi_\tau)_{\tau>0}$ is given by

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau,\cdot) \qquad (\tau > 0,\ u_0,\ u_\pi \in L^2[0,\tau]),$$

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Describe the reachable set

$$\mathrm{Ran}\,\Phi_\tau = \left\{ w(\tau,\cdot) : w \text{ solving the heat equation with } u_0,\ u_\pi \in L^2[0,\tau] \right\}$$

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$$\left\{\sum c_n \sin(nx) : \sum_n |c_n|^2 n e^{n\pi} < \infty\right\} \subset \mathrm{Ran}\,\Phi_\tau$$

So

$$\left\{\psi \in \mathrm{Hol}(S_\varepsilon) : \psi^{(2k)}(0) = \psi^{(2k)}(\pi) = 0\right\} \subset \mathrm{Ran}\,\Phi_\tau$$

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$$\operatorname{Hol}(D_\varepsilon) \subset \operatorname{Ran}\Phi_\tau$$

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Main results

Let

$$D = \{s = x + iy \in \mathbb{C} : |y| < x \text{ and } |y| < \pi - x\}$$

$$A^2(D) := \left\{ f \in \mathrm{Hol}(D) : \int_D |f(x+iy)|^2 dxdy < \infty \right\}$$

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Theorem

For every $\tau > 0$ *we have*

$$E^2(D) \subsetneq \mathrm{Ran}\,\Phi_\tau$$

Fourier series expansion

Using the decomposition of the solution w in the standard Fourier basis $(\sin(nx))_{n\geqslant 1}$ of $L^2[0,\pi]$, we get

$$(\Phi_\tau u)(x) = \frac{2}{\pi}\sum_{n\geqslant 1} n\left[\int_0^\tau e^{n^2(\sigma-\tau)}u_0(\sigma)\,d\sigma\right]\sin(nx)$$
$$+\frac{2}{\pi}\sum_{n\geqslant 1} n(-1)^{n+1}\left[\int_0^\tau e^{n^2(\sigma-\tau)}u_\pi(\sigma)\,d\sigma\right]\sin(nx) \qquad (\tau>0,\ x\in(0,\pi)),$$

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Lemma

$$(\Phi_\tau u)(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau-\sigma,x)u_0(\sigma)\,d\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau-\sigma,x)u_\pi(\sigma)\,d\sigma,$$

where

$$K_0(\sigma,x) = -\sqrt{\frac{1}{\pi\sigma}}\sum_{m\in\mathbb{Z}} e^{-\frac{(x+2m\pi)^2}{4\sigma}} \quad \textit{and} \quad K_\pi(\sigma,x) = -K_0(\sigma,\pi-x) \qquad (\sigma>0, x\in\mathbb{R}),$$

Proof.

Poisson summation formula □

Proof of the Proposition : $\mathrm{Ran}\,\Phi_\tau \subset A^2(D)$

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$$\Phi_\tau u(x) = (\phi_\tau^0 u)(x) + \int_0^\tau \frac{\partial \widetilde{K}_0}{\partial x}(\tau-\sigma,x)u_0(\sigma)\,d\sigma + (\phi_\tau^\pi u)(x) + \int_0^\tau \frac{\partial \widetilde{K}_\pi}{\partial x}(\tau-\sigma,x)u_\pi(\sigma)\,d\sigma$$

where

$$(\phi_\tau^0 u)(s) := \frac{1}{2\sqrt{\pi}}\int_0^\tau \frac{s e^{-\frac{s^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{\frac{3}{2}}} f(\sigma)\sigma\,d\sigma \quad \text{and} \quad (\phi_\tau^\pi u)(s) = (\phi_\tau^0 u)(\pi-s)$$

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Since there exist $a,\ b>0$ such that for every $k\in\mathbb{Z}\setminus\{-1,0\}$ we have

$$\left|(s+k\pi)e^{-\frac{(s+k\pi)^2}{4(\tau-\sigma)}}\right|^2 \leqslant ak^2 e^{\frac{-bk^2}{(\tau-\sigma)}} \qquad (s\in D).$$

It suffices to show that $\phi_\tau^0 u$ and $\phi_\tau^\pi u$ can be be extended to a function in $A^2(D)$

A result of Aikawa, Hayashi and Saitoh

Let

$$\Delta = \left\{ s \in \mathbb{C} \mid -\frac{\pi}{4} < \arg s < \frac{\pi}{4} \right\}.$$

Let ω be a positive measurable function on Δ, then

$$A^2(\Delta, \omega) = \left\{ f \in \mathrm{Hol}(D) : \int_D |f(x+iy)|^2 \omega(x+iy) dxdy < \infty \right\}$$

Theorem

For $s \in \Delta$, $\tau > 0$ and $f \in L^2[0,\tau]$ we set

$$(P_\tau f)(s) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{s e^{-\frac{s^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{\frac{3}{2}}} f(\sigma)\sqrt{\sigma}\, d\sigma.$$

Then P_τ defines an isometric isomorphism from $L^2[0,\tau]$ onto $A^2(\Delta, \omega_0)$, where $\omega_0(s) = \frac{e^{\frac{\mathrm{Re}(s^2)}{2\tau}}}{\tau}$.

Corollary

$\phi_\tau^0 \in \mathcal{L}(L^2[0,\tau], A^2(D))$

A resul of Levin-Lyubarski

Theorem

Let $\Lambda = \{(2k+1)(1 \pm i) : k \in \mathbb{Z}\}$ *and let*

$$H(\lambda) = \frac{\pi}{2} \begin{cases} \mathrm{Im}\,\lambda & \textit{if } \arg\lambda \in (\pi/4, 3\pi/4) \\ -\mathrm{Re}\,\lambda & \textit{if } \arg\lambda \in (3\pi/4, 5\pi/4), \\ -\mathrm{Im}\,\lambda & \textit{if } \arg\lambda \in (5\pi/4, 7\pi/4, \\ \mathrm{Re}\,\lambda & \textit{if } \arg\lambda \in (5\pi/4, \pi/4). \end{cases}$$

then the family $(e_\lambda)_{\lambda \in \Lambda}$ *is a Riesz basis in* $E^2(D)$ *where*

$$e_\lambda(s) = e^{\lambda s} e^{-\lambda\pi/2} e^{-H(\lambda)}, \qquad \lambda \in \Lambda, s \in D.$$

Lemma

let $\tau > 0$ *and* $\varphi \in E^2(D)$*. Then there exists* $\varphi_1 \in A^2(\Delta, \omega_0)$ *and* $\varphi_2 \in A^2(\pi - \Delta, \omega_\pi)$ *such that*

$$\varphi(s) = \varphi_1(s) + \varphi_2(s) \qquad\qquad (s \in D).$$

here $\omega_\pi(s) = \omega_0(\pi - s)$*, for* $s \in \pi - \Delta$*.*

Proof.

Let $\varphi \in E^2(D)$, $\varphi = \sum_{\lambda\in\Lambda} a_\lambda e_\lambda$

$$\varphi_1(s) = F(s) \sum_{\substack{\lambda\in\Lambda \\ \mathrm{Re}\,\lambda<0}} a_\lambda e_\lambda(s) \qquad s \in \Delta$$

$$\varphi_2(s) = F(\pi - s) \sum_{\substack{\lambda\in\Lambda \\ \mathrm{Re}\,\lambda>0}} a_\lambda e_\lambda(s), \qquad s \in \pi - \Delta$$

we use Hilbert's inequality to prove required estimates. □

Proof of the main result

Let

$$\widetilde{K}_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m \neq 0} e^{-\frac{(x+2m\pi)^2}{4\sigma}} \quad \text{and} \quad \widetilde{K}_\pi(\sigma, x) = \widetilde{K}_0(\sigma, \pi - x)$$

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We decompose these functions as

$$\tilde{K}_0(\sigma,s) = A(\sigma,s)+B(\sigma,s), \quad \text{and} \quad \tilde{K}_\pi(\sigma,s) = C(\sigma,s)+D(\sigma,s),$$

where

$$A(\sigma,s) = -\sqrt{\frac{1}{\pi\sigma}}\sum_{m\geqslant 1} e^{-\frac{(s+2m\pi)^2}{4\sigma}} \qquad \text{and} \qquad C(\sigma,s) = A(\sigma,\pi-s)$$

Denote

$$R_{A,\tau}u(s) = \int_0^\tau \frac{\partial A}{\partial s}(\tau-\sigma,s)\sqrt{\sigma}u(\sigma)d\sigma,$$

and similarly introduce $R_{B,\tau}$, $R_{C,\tau}$ and $R_{D,\tau}$.

Lemma

$$\lim_{\tau\to 0+}\left\|\begin{bmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau}\end{bmatrix}\right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi))} = 0\,.$$

Proof of the main result

Let $f \in L^2[0,\tau]$ and let $Q_\tau f(s) = P_\tau f(\pi - s)$ for $s \in \pi - \Delta$. We have

$$M_\tau := \begin{bmatrix} P_\tau + R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & Q_\tau + R_{D,\tau} \end{bmatrix} \in \mathcal{L}\left((L^2([0,\pi]))^2, A^2(\Delta,\omega_0) \times A^2(\pi - \Delta,\omega_\pi)\right).$$

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Since

$$\begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \in \mathcal{L}\left((L^2([0,\pi]))^2, A^2(\Delta,\omega_0) \times A^2(\pi-\Delta,\omega_\pi)\right)$$

is invertible

$$\left\| \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\| = 1 \quad \text{and} \quad \lim_{\tau \to 0+} \left\| \begin{bmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau} \end{bmatrix} \right\| = 0\,.$$

M_τ is invertible for τ small. The result follows by applying Lemma of decomposition in $E^2(D)$.

Let $\varphi \in E^2(D)$. Then there exists a decomposition

$$\varphi(s) = \varphi_1(s) + \varphi_2(s) \qquad (s \in D).$$

where $\varphi_1 \in A^2(\Delta, \omega_0)$ and $\varphi_2 \in A^2(\pi - \Delta, \omega_\pi)$. Since $M_{\tau^*}^{-1}$ is invertible for some small τ^* there exists $u_0, u_\pi \in L^2[0, \tau^*]$ sucth that

$$M_{\tau^*} \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

So $E^2(D) \subset \mathrm{Ran}\,\Phi_{\tau^*}$ and $\mathrm{Ran}\,\Phi_\tau$ is independent of $\tau > 0$.