IMAR Bucarest 2017

Atelier de travail : Espaces de fonctions et théorie des opérateurs

Quelques résultats récents autour des espaces de Müntz

with E. Fricain and with L. Gaillard

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1. $M_\Lambda = vect\left\{x^\lambda; \lambda \in \Lambda\right\}$ is dense in $C([0,1])$ (resp. in L^p)
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We would like to know "everything" on the Banach space $\overline{M_\Lambda}^E \subsetneq E$ when the series converges !

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Assume that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$ and $\inf\left(\lambda_{k+1} - \lambda_k\right) > 0$

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For $p = +\infty$, we have then some <u>infinite</u> dimensional (closed) subspaces of functions in $C([0,1])$, with continuous derivatives on $[0,1)$

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  even better: operators preserving M_Λ ...

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Then
X is almost isometric to a subspace of the space of convergent sequences c. i.e.

For every $\varepsilon > 0$, there exists $J_\varepsilon : X \to c$ such that

$$(1-\varepsilon)\|f\|_\infty \leqslant \|J_\varepsilon f\|_\infty \leqslant \|f\|_\infty \qquad \forall f \in X$$

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M_Λ^p is isomorphic to a subspace of ℓ_p.

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i.e.

$$\Big\| \sum_k a_k \lambda_k^{1/p} x^{\lambda_k} \Big\|_p \approx \Big(\sum_{k\geqslant 0} |a_k|^p \Big)^{\frac{1}{p}}$$

and

$$\Big\| \sum_k a_k x^{\lambda_k} \Big\|_\infty \approx \sup_N \Big| \sum_{k=0}^{N} a_k \Big|$$

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In the case $p=2$, it means that we have an *asymptotic orthonormal system*.

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For quite free, we get

Full Müntz theorem in L^2 (Szàsz 1916)

Let $\Lambda = (\lambda_n)$ be a sequence of $\mathbb{C}_{-\frac{1}{2}}$.

M_Λ is dense in $L^2\big([0,1], dx\big)$ if and only if $\displaystyle\sum \frac{\frac{1}{2}+Re(\lambda_n)}{\left|\lambda_n+\frac{1}{2}\right|^2+1} = +\infty.$

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Indeed: let $g \in \mathcal{H}^2\left(\mathbb{C}_0\right)$, recall

$$\|g\|^2_{\mathcal{H}^2(\mathbb{C}_0)} = \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \left|g(x+iy)\right|^2 dy.$$

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$$\forall z \in \mathbb{C}_0 \,, \qquad g(z) = \int_{\mathbb{R}^+} F(t) e^{-tz}\, dt \quad \text{and } \|F\|_2 = \|g\|_{\mathcal{H}^2(\mathbb{C}_0)} \,.$$

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It means that the function $f(s) = F\big(-\ln(s)\big)$ satisfies $g = \mathcal{M}(f)$ and

$$\int_0^1 |f(s)|^2\,\frac{ds}{s} = \int_{\mathbb{R}^+} \left|F(t)\right|^2 dt = \|g\|^2_{\mathcal{H}^2(\mathbb{C}_0)}.$$

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Compose the previous maps : $\mathcal{D}(f) = \mathcal{M}(\sqrt{s} f)$.... That's all...

L^2 Müntz spaces as model spaces

(from joint work with E.Fricain)

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The model space associated to this inner function Θ is

$$K_\Theta = \mathcal{H}^2(\mathbb{C}_0) \ominus \Theta\mathcal{H}^2(\mathbb{C}_0) = \left(\Theta\mathcal{H}^2(\mathbb{C}_0)\right)^\perp.$$

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We also get the asymptotically orthonormal version...

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And there are many other ways to exploit the dictionary $\mathcal{D}$... both ways...

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Ideas:

- Carleson's embedding: $M_\Lambda^p \hookrightarrow L^p(\mu)$ for some positive measure μ on $[0,1)$
- Hardy type operators: (Volterra and) Cesàro operator

$$f \in M_\Lambda \longmapsto \Gamma(f)(x) = \frac{1}{x}\int_0^x f(t)\,dt \in M_\Lambda$$

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- If $\mu((1-\delta,1)) = o(\delta)$ *(μ vanishing sublinear)* then $i_p(\mu)$ is compact.
- When Λ is a quasi-geometric sequence, then the converse is true, but it is false for arbitrary Λ.

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(Gaillard-L.) $p \geqslant 1$

Consider the properties

1. the following sequence is bounded

$$D_n(p) = \int_{[0,1)} (p\lambda_n + 1)^{\frac{1}{p}} t^{\lambda_n} \Big(\sum_k (p\lambda_k + 1)^{\frac{1}{p}} t^{\lambda_k} \Big)^{p-1} \, d\mu$$

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2. $i_p(\mu)$ is compact
3. $\displaystyle\int_{[0,1)} t^{p\lambda_n} \, d\mu = o(1/\lambda_n)$.

We have

- μ is vanishing sublinear $\Longrightarrow (1) \Longrightarrow (2) \Longrightarrow (3)$.
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Actually, the key point is that (whatever Λ) $i_p(\mu)$ is dominated by a diagonal operator:

$$\Big\| \sum_k a_k (p\lambda_k+1)^{\frac{1}{p}} t^{\lambda_k} \Big\|_{L^p(\mu)} \leqslant \Big\| (D_k^{\frac{1}{p}}(p).a_k)_{k\geqslant 0} \Big\|_{\ell^p}$$

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For $p = 1$, everything is equivalent

For $p > 1$, everything becomes equivalent when Λ is quasi-geometric

Carleson's embedding $p=2$

Similar results for Schatten classes $\mathcal{S}^q$ when $q \geqslant 2$...

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More generally

$$\|i_\mu^2\|_{\mathcal{S}^q} \approx \left(\int_0^1 \left(\int_{[0,1)} \frac{d\mu(t)}{(1-st)^{\frac{2}{q}+1}} \right)^{\frac{q}{2}} ds \right)^{\frac{1}{q}}$$

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4. More generally, understand the link between the nature of the Banach space M^p_Λ and the arithmetical nature of Λ.

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Mulțumesc frumos !