Mixed Commutators vs Mixed BMO

E.Strouse

Université Bordeaux

Conference Roumaine pour la Francophonie
18 Decembre, 2017

Multiplication operators and the infinity norm

Let L^2 be the space of (almost everywhere defined) square integrable functions on the circle. We all know that the multiplication operator:

Multiplication operators and the infinity norm

Let L^2 be the space of (almost everywhere defined) square integrable functions on the circle. We all know that the multiplication operator:

$$M_\phi : L^2 \to L^2; \qquad M_\phi(g) = \phi g$$

Multiplication operators and the infinity norm

Let L^2 be the space of (almost everywhere defined) square integrable functions on the circle. We all know that the multiplication operator:

$$M_\phi : L^2 \to L^2; \quad M_\phi(g) = \phi g$$

is bounded if and only if its *symbol* ϕ is, and has norm $\|\phi\|_\infty$.

Multiplication operators and the infinity norm

Let L^2 be the space of (almost everywhere defined) square integrable functions on the circle. We all know that the multiplication operator:

$$M_\phi : L^2 \to L^2; \quad M_\phi(g) = \phi g$$

is bounded if and only if its *symbol* ϕ is, and has norm $\|\phi\|_\infty$.

Toeplitz and Hankel operators on Hilbert spaces of functions are compositions of multiplication operators and orthogonal projection on certain subspaces of L^2 - so it is natural to investigate how their norms are related to the infinity norm of their symbols.

Motivation

I will begin my talk by recalling lots of ways in which the norms of Toeplitz and Hankels are related to infinity norms. The associated techniques are what (eventually) permitted us (Stefanie Petermichl, Yumeng Ou, and myself) to get a lot of BMO-Hankel associations for some complicated BMO spaces, similarly complicated Hankel operators, then the associated commutators of Hilbert transforms and multiplication operators - and eventually some very sophisticated Calderon Zygmund operators.

Norms of Toeplitz operators - classical case

We will begin with the orthogonal decomposition $L^2 = H^2 \oplus (H^2)^\perp$ where $H^2 =$ the Hardy space.

Norms of Toeplitz operators - classical case

We will begin with the orthogonal decomposition $L^2 = H^2 \oplus (H^2)^\perp$ where $H^2 =$ the Hardy space.

We recall the most basic definition of Toeplitz operators:

$$T_\phi = P^+ \circ M_\phi$$

where P^+ is orthogonal projection onto from L^2 onto H^2 and $P^- = I - P^+$.

Norms of Toeplitz operators - classical case

We will begin with the orthogonal decomposition $L^2 = H^2 \oplus (H^2)^\perp$ where $H^2 =$ the Hardy space.

We recall the most basic definition of Toeplitz operators:

$$T_\phi = P^+ \circ M_\phi$$

where P^+ is orthogonal projection onto from L^2 onto H^2 and $P^- = I - P^+$.

It is well known that these operators are bounded if and only if the symbol is bounded (in $L^\infty(T)$) and have operator norm equal to $\|\phi\|_\infty$. (Brown-Halmos(1963))

Norms of Toeplitz operators - classical case

In fact, the orthogonal projection does not reduce the norm because, if $W = M_z$ then M_ϕ is the strong limit of the sequence $W^{*n} T_\phi P^+ W^n$; so $\|M_\phi\| \leq \|T_\phi\|$.

Norms of Toeplitz operators - classical case

In fact, the orthogonal projection does not reduce the norm because, if $W = M_z$ then M_ϕ is the strong limit of the sequence $W^{*n} T_\phi P^+ W^n$; so $\|M_\phi\| \leq \|T_\phi\|$.

The same reasoning works in several variables (with $W = M_{z_1 z_2 \cdots z_n}$) just by using $M_{z_1 z_2 \ldots z_n}$.

Norms of Toeplitz operators - classical case

In fact, the orthogonal projection does not reduce the norm because, if $W = M_z$ then M_ϕ is the strong limit of the sequence $W^{*n} T_\phi P^+ W^n$; so $\|M_\phi\| \leq \|T_\phi\|$.

The same reasoning works in several variables (with $W = M_{z_1 z_2 \cdots z_n}$) just by using $M_{z_1 z_2 \ldots z_n}$.

This kind of result does not hold, as is, for most other 'Toeplitz-type

operators.' But, in fact, we used variations of this technique several times to show that our 'mixed Hankel' operators were bounded!

A more varied selection of Toeplitz's

Truncated Toeplitz operators' multiply by a symbol then project on other shift-invariant subspaces of the Hardy space. In this case the operator norm is NOT correlated with the symbol's infinity norm. There is no unique symbol for such Toeplitz operators - and sometimes no bounded symbol exists!

A more varied selection of Toeplitz's

Truncated Toeplitz operators' multiply by a symbol then project on other shift-invariant subspaces of the Hardy space. In this case the operator norm is NOT correlated with the symbol's infinity norm. There is no unique symbol for such Toeplitz operators - and sometimes no bounded symbol exists!

There are some beautiful papers

(Baranov-Chalendar-Fricain-Mashreghi-Timotin, Bessonov, Kapustin)

showing that sometimes there is a bounded symbol for operators on these spaces, and sometimes not!

A more varied selection of Toeplitz's

Truncated Toeplitz operators' multiply by a symbol then project on other shift-invariant subspaces of the Hardy space. In this case the operator norm is NOT correlated with the symbol's infinity norm. There is no unique symbol for such Toeplitz operators - and sometimes no bounded symbol exists!

There are some beautiful papers

(Baranov-Chalendar-Fricain-Mashreghi-Timotin, Bessonov, Kapustin)

showing that sometimes there is a bounded symbol for operators on these spaces, and sometimes not!

Another case where it is very difficult to characterize the symbols of bounded Toeplitz are for operators on the Bergman spaces (again the easily treated case is for an 'analytic' symbol). The symbols are unique - but there are bounded, and even compact operators whose symbols are unbounded.

Hankel operators - definitions

Classical Hankel operators from H^2 to ${H^2}^\perp$ are defined by:

$$H_\phi = P^- \circ M_\phi$$

Hankel operators - definitions

Classical Hankel operators from H^2 to ${H^2}^\perp$ are defined by:

$$H_\phi = P^- \circ M_\phi$$

In the appropriate monomial basis, such operators have a matrix of the form:

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & a_{-4} & . \\ a_{-2} & a_{-3} & a_{-4} & . & . \\ a_{-3} & a_{-4} & . & . & . \\ a_{-4} & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

where $\phi" = " \sum_{k=-\infty}^{\infty} a_k z^k$ i.e. the a_k are the Fourier coefficients of ϕ.

Hankel operators - definitions

Classical Hankel operators from H^2 to ${H^2}^\perp$ are defined by:

$$H_\phi = P^- \circ M_\phi$$

In the appropriate monomial basis, such operators have a matrix of the form:

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdot \\ a_{-2} & a_{-3} & a_{-4} & \cdot & \cdot \\ a_{-3} & a_{-4} & \cdot & \cdot & \cdot \\ a_{-4} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $\phi" = " \sum_{k=-\infty}^{\infty} a_k z^k$ i.e. the a_k are the Fourier coefficients of ϕ. So, if ϕ is analytic $H_\phi = 0$; only the antianalytic part of ϕ matters.

Hankel operators - definitions

More general Hankel operators can be defined - if E is a subspace of L^2 we think of either $P^{E^\perp} \circ M_\phi \circ P^E$ or $P^{\bar{E}} \circ M_\phi \circ P^E$ as a generalized (big or little) Hankel.

Hankel operators - definitions

More general Hankel operators can be defined - if E is a subspace of L^2 we think of either $P^{E^\perp} \circ M_\phi \circ P^E$ or $P^{\bar{E}} \circ M_\phi \circ P^E$ as a generalized (big or little) Hankel.

In the case of several variables, or $L^2(\mathbb{D}^n)$ there are many interesting choices of E as there are many 'shift invariant' subspaces' - especially if one considers different types of shifts such as:

$$M_{z_1 z_2 \ldots z_n} \text{ or } M_{z_1} \text{ or } M_{z_2} \text{ or } M_{z_1 z_3} \text{ or } \ldots$$

Norms of Hankel operators - classical case

In fact, the characterization of boundedness of classical 'one variable' Hankels is more surprising and complicated than for Toeplitz,but complete.

Norms of Hankel operators - classical case

In fact, the characterization of boundedness of classical 'one variable' Hankels is more surprising and complicated than for Toeplitz,but complete.

H_ϕ is bounded if and only if $\phi = P^-(g)$ for $g \in$ Ł$^\infty$ and there always exists $g \in L^\infty$ such that $H_g = H_\phi$ and $\|H_\phi\| = \|g\|_\infty$.

Norms of Hankel operators - classical case

In fact, the characterization of boundedness of classical 'one variable' Hankels is more surprising and complicated than for Toeplitz,but complete.

H_ϕ is bounded if and only if $\phi = P^-(g)$ for $g \in Ł^\infty$ and there always exists $g \in L^\infty$ such that $H_g = H_\phi$ and $\|H_\phi\| = \|g\|_\infty$.

In order to give a good correspondance between function spaces and norms and Hankel operators and norms, we look for symbols 'ϕ such that *both* H_ϕ and $H_{\overline{\phi}}$ are bounded .

Norms of Hankel operators - classical case

In fact, the characterization of boundedness of classical 'one variable' Hankels is more surprising and complicated than for Toeplitz,but complete.

H_ϕ is bounded if and only if $\phi = P^-(g)$ for $g \in Ł^\infty$ and there always exists $g \in L^\infty$ such that $H_g = H_\phi$ and $\|H_\phi\| = \|g\|_\infty$.

In order to give a good correspondance between function spaces and norms and Hankel operators and norms, we look for symbols 'ϕ such that *both* H_ϕ and $H_{\overline{\phi}}$ are bounded .

This happens if and only if $\phi = P^+(g_1) + P^-(g_2)$ for g_1 and g_2 in L^∞.

Norms of Hankel operators - classical case

What makes all this interesting - especially when generalizing - is that the space of symbols ϕ such that the Hankel operators H_ϕ and $H_{\overline{\phi}}$ are bounded, or;

Norms of Hankel operators - classical case

What makes all this interesting - especially when generalizing - is that the space of symbols ϕ such that the Hankel operators H_ϕ and $H_{\overline{\phi}}$ are bounded, or;

$\{\phi : \phi = P^+(g_1) + P^-(g_2)$ for g_1 and g_2 in $L^\infty\}$

Norms of Hankel operators - classical case

What makes all this interesting - especially when generalizing - is that the space of symbols ϕ such that the Hankel operators H_ϕ and $H_{\overline{\phi}}$ are bounded, or;

$\{\phi : \phi = P^+(g_1) + P^-(g_2)$ for g_1 and g_2 in $L^\infty\}$

is the space BMO - of functions of bounded mean oscillation - which can be defined in many seemingly unrelated ways.

Norms of Hankel operators - classical case

What makes all this interesting - especially when generalizing - is that the space of symbols ϕ such that the Hankel operators H_ϕ and $H_{\overline{\phi}}$ are bounded, or;

$\{\phi : \phi = P^+(g_1) + P^-(g_2)$ for g_1 and g_2 in $L^\infty\}$

is the space BMO - of functions of bounded mean oscillation - which can be defined in many seemingly unrelated ways.

We see them as: (1) symbols of bounded Hankels - or corresponding commutators of Hilbert transforms; or (2) elements of the dual space of 'H^1_{Re}' or (3) the BMO functions whose integrals over intervals do not vary too much from their average over the interval. And the corresponding norms of the functions in these spaces are equivalent.

Norms of Hankel operators - classical case

What makes all this interesting - especially when generalizing - is that the space of symbols ϕ such that the Hankel operators H_ϕ and $H_{\overline{\phi}}$ are bounded, or;

$\{\phi : \phi = P^+(g_1) + P^-(g_2)$ for g_1 and g_2 in $L^\infty\}$

is the space BMO - of functions of bounded mean oscillation - which can be defined in many seemingly unrelated ways.

We see them as: (1) symbols of bounded Hankels - or corresponding commutators of Hilbert transforms; or (2) elements of the dual space of 'H^1_{Re}' or (3) the BMO functions whose integrals over intervals do not vary too much from their average over the interval. And the corresponding norms of the functions in these spaces are equivalent.

The several variable case is *very* different - and best discussed in the framework of commutators with Hilbert transforms.

Hilbert Transform

We begin by looking at Hilbert transforms as operators on ℓ^2 - the other side of their realization as Fourier multipliers.

Hilbert Transform

We begin by looking at Hilbert transforms as operators on ℓ^2 - the other side of their realization as Fourier multipliers.

This allows us to understand and generalize them using Hilbert space theory about boundedness and complex functions.

Hilbert Transform

We begin by looking at Hilbert transforms as operators on ℓ^2 - the other side of their realization as Fourier multipliers.

This allows us to understand and generalize them using Hilbert space theory about boundedness and complex functions.

Later we will remember that the Hilbert transform is actually a singular integral - Calderon Zygmund - operator, and generalize our results relating BMO norms and boundedness of commutators.

Hilbert Transform

If $f = \sum_{k=1}^{\infty} \hat{f}(k) z^k$ is an analytic function with $\hat{f}(0) = 0$, then the real part of f equal to $\frac{f+\overline{f}}{2}$ is the function u such that:

$$u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikt} \text{ with } \hat{u}(k) = \begin{cases} \frac{1}{2}\hat{f}(k) & \text{if } k > 0 \\ -\frac{1}{2}\hat{f}(k) & \text{if } k < 0. \end{cases}$$

Hilbert Transform

If $f = \sum_{k=1}^{\infty} \hat{f}(k) z^k$ is an analytic function with $\hat{f}(0) = 0$, then the real part of f equal to $\frac{f+\overline{f}}{2}$ is the function u such that:

$$u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikt} \text{ with } \hat{u}(k) = \begin{cases} \frac{1}{2}\hat{f}(k) & \text{if } k > 0 \\ -\frac{1}{2}\hat{f}(k) & \text{if } k < 0. \end{cases}$$

and $Im(f) = v$ with

$$v = \sum_{k=-\infty}^{\infty} \hat{v}(k) e^{ikt} \text{ with } \hat{v}(k) = \begin{cases} -\frac{i}{2}\hat{f}(k) & \text{if } k > 0 \\ \frac{i}{2}\hat{f}(k) & \text{if } k < 0. \end{cases}$$

Hilbert Transform

If $f = \sum_{k=1}^{\infty} \hat{f}(k) z^k$ is an analytic function with $\hat{f}(0) = 0$, then the real part of f equal to $\frac{f+\overline{f}}{2}$ is the function u such that:

$$u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikt} \text{ with } \hat{u}(k) = \begin{cases} \frac{1}{2}\hat{f}(k) & \text{if } k > 0 \\ -\frac{1}{2}\overline{\hat{f}(k)} & \text{if } k < 0. \end{cases}$$

and $Im(f) = v$ with

$$v = \sum_{k=-\infty}^{\infty} \hat{v}(k) e^{ikt} \text{ with } \hat{v}(k) = \begin{cases} -\frac{i}{2}\hat{f}(k) & \text{if } k > 0 \\ \frac{i}{2}\overline{\hat{f}(k)} & \text{if } k < 0. \end{cases}$$

Thus the map from a real function u to it's harmonic conjugate is easily expressed as **the Hilbert transform H** where:

Hilbert Transform

If $f = \sum_{k=1}^{\infty} \hat{f}(k) z^k$ is an analytic function with $\hat{f}(0) = 0$, then the real part of f equal to $\frac{f+\overline{f}}{2}$ is the function u such that:

$$u = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{ikt} \text{ with } \hat{u}(k) = \begin{cases} \frac{1}{2}\hat{f}(k) & \text{if } k > 0 \\ -\frac{1}{2}\overline{\hat{f}(k)} & \text{if } k < 0. \end{cases}$$

and $Im(f) = v$ with

$$v = \sum_{k=-\infty}^{\infty} \hat{v}(k) e^{ikt} \text{ with } \hat{v}(k) = \begin{cases} -\frac{i}{2}\hat{f}(k) & \text{if } k > 0 \\ \frac{i}{2}\overline{\hat{f}(k)} & \text{if } k < 0. \end{cases}$$

Thus the map from a real function u to it's harmonic conjugate is easily expressed as **the Hilbert transform H** where:

$$Hu = -iP^{+}u + iP^{-}u$$

Hilbert Transform

The Hilbert transform

$$H = -iP^{+} + iP^{-}$$

is trivially bounded on $L^2(\mathbb{T})$ thanks to Plancherel:

$$\sum_{k\geq 0} |\hat{u}(k)|^2 \leq \sum_{k} |\hat{u}(k)|^2$$

Hilbert Transform

The Hilbert transform

$$H = -iP^+ + iP^-$$

is trivially bounded on $L^2(\mathbb{T})$ thanks to Plancherel:

$$\sum_{k \geq 0} |\hat{u}(k)|^2 \leq \sum_{k} |\hat{u}(k)|^2$$

It is also bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$ but not on $L^1(\mathbb{T})$.

Hilbert Transform

The Hilbert transform

$$H = -iP^+ + iP^-$$

is trivially bounded on $L^2(\mathbb{T})$ thanks to Plancherel:

$$\sum_{k\geq 0} |\hat{u}(k)|^2 \leq \sum_{k} |\hat{u}(k)|^2$$

It is also bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$ but not on $L^1(\mathbb{T})$.

We set $H^1_{Re} = \{f \in L^1 : Hf \in L^1\}$, and

$$\|f\|_{H^1_{Re}} = \|f\|_1 + \|Hf\|_1$$

so that the Hilbert transform **will** be bounded on H^1_{Re}.

Hilbert Transform

The Hilbert transform

$$H = -iP^{+} + iP^{-}$$

is trivially bounded on $L^2(\mathbb{T})$ thanks to Plancherel:

$$\sum_{k\geq 0} |\hat{u}(k)|^2 \leq \sum_{k} |\hat{u}(k)|^2$$

It is also bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$ but not on $L^1(\mathbb{T})$.

We set $H^1_{Re} = \{f \in L^1 : Hf \in L^1\}$, and

$$\|f\|_{H^1_{Re}} = \|f\|_1 + \|Hf\|_1$$

so that the Hilbert transform **will** be bounded on H^1_{Re}. And now we are ready to be explicit about the definitions of BMO.

$BMO - 1$

BMO (bounded mean oscillation) is the Banach space of all functions $f \in L^1_{loc}(\mathbb{T})$ for which

$$\|f\|_{BMO} = \sup_I \frac{1}{|I|} \int_I |f - c_I| < \infty$$

for some c_I,where the supremum runs over all interval-arcs.

$BMO - 1$

BMO (bounded mean oscillation) is the Banach space of all functions $f \in L^1_{loc}(\mathbb{T})$ for which

$$\|f\|_{BMO} = \sup_I \frac{1}{|I|} \int_I |f - c_I| < \infty$$

for some c_I,where the supremum runs over all interval-arcs.

The amazing discovery by Charles Fefferman in 1970 was (the multivariable version) that BMO is actually the dual space of H^1_{Re}.

$BMO - 2$

If g is a function on the circle, then the functional:

$$\phi_g : H^1_{Re} \to \mathbb{C}$$

$$\phi_g(f) = \int fg$$

$BMO-2$

If g is a function on the circle, then the functional:

$$\phi_g : H^1_{Re} \to \mathbb{C}$$
$$\phi_g(f) = \int fg$$

is bounded precisely when $g \in BMO$, so that $g = g_0 + Hg_1$ with g_0 and g_1 in L^∞. The norm of the functional ϕ_g is equivalent to $\|g\|_{BMO}$
This shows that $L^\infty \subset BMO$ with $g_0, g_1 \in L^\infty$. and also that:

$BMO - 2$

If g is a function on the circle, then the functional:

$$\phi_g : H^1_{Re} \to \mathbb{C}$$
$$\phi_g(f) = \int fg$$

is bounded precisely when $g \in BMO$, so that $g = g_0 + Hg_1$ with g_0 and g_1 in L^∞. The norm of the functional ϕ_g is equivalent to $\|g\|_{BMO}$
This shows that $L^\infty \subset BMO$ with $g_0, g_1 \in L^\infty$. and also that:

$P^+(L^\infty) + L^\infty \subset BMO$

so functions of the form $P^+f + g$, for $f, g \in L^\infty$, which may have 'logarithmic infinities', are in BMO.

$BMO - 2$

If g is a function on the circle, then the functional:

$$\phi_g : H^1_{Re} \to \mathbb{C}$$
$$\phi_g(f) = \int fg$$

is bounded precisely when $g \in BMO$, so that $g = g_0 + Hg_1$ with g_0 and g_1 in L^∞. The norm of the functional ϕ_g is equivalent to $\|g\|_{BMO}$
This shows that $L^\infty \subset BMO$ with $g_0, g_1 \in L^\infty$. and also that:

$P^+(L^\infty) + L^\infty \subset BMO$

so functions of the form $P^+f + g$, for $f, g \in L^\infty$, which may have 'logarithmic infinities', are in BMO. And, as we saw before, these are exactly the symbols of bounded (1-variable) Hankels.

n-variable case - Fefferman

The n-variable form of Fefferman's results replace the Hilbert transform with n Riesz transforms R_j best understood on $L^1(\mathbb{R}^n)$:

n-variable case - Fefferman

The n-variable form of Fefferman's results replace the Hilbert transform with n Riesz transforms R_j best understood on $L^1(\mathbb{R}^n)$:

R_j is the transform which multiplies the Fourier transform of f by $\frac{-ix_j}{|x|}$.

n-variable case - Fefferman

The n-variable form of Fefferman's results replace the Hilbert transform with n Riesz transforms R_j best understood on $L^1(\mathbb{R}^n)$:

R_j is the transform which multiplies the Fourier transform of f by $\frac{-ix_j}{|x|}$.

If $H^1_{Fef} = \{f \in L^1(\mathbb{R}^n) : R_j(f) \in L^1 (j = 0, 1, \cdots n)\}$;

n-variable case - Fefferman

The n-variable form of Fefferman's results replace the Hilbert transform with n Riesz transforms R_j best understood on $L^1(\mathbb{R}^n)$:

R_j is the transform which multiplies the Fourier transform of f by $\frac{-ix_j}{|x|}$.

If $H^1_{Fef} = \{f \in L^1(\mathbb{R}^n) : R_j(f) \in L^1 (j = 0, 1, \cdots n)\}$;

Then BMO (with the sup calculated over all cubes in $\mathbb{R}^n$) is the dual of H^1_{Fef}.

n-variable case - Fefferman

The n-variable form of Fefferman's results replace the Hilbert transform with n Riesz transforms R_j best understood on $L^1(\mathbb{R}^n)$:

R_j is the transform which multiplies the Fourier transform of f by $\frac{-ix_j}{|x|}$.

If $H^1_{Fef} = \{f \in L^1(\mathbb{R}^n) : R_j(f) \in L^1(j = 0, 1, \cdots n)\}$;

Then BMO (with the sup calculated over all cubes in $\mathbb{R}^n$) is the dual of H^1_{Fef}.

and the elements of BMO are the functions of the form:

$$g_0 + \sum_{j=1}^{n} R_j(g_j) \text{ with } g_j \in L^\infty(j = 0, \cdots n)$$

n-variable Hilbert transforms

But the multi-variable case needs manys different definitions of BMOs.

n-variable Hilbert transforms

But the multi-variable case needs manys different definitions of BMOs.

The one which works best for us (so as to associate commutators and Hankel operators) is defined in terms of 'one variable Hilbert transforms'.

n-variable Hilbert transforms

But the multi-variable case needs manys different definitions of BMOs.

The one which works best for us (so as to associate commutators and Hankel operators) is defined in terms of 'one variable Hilbert transforms'.

We define these transforms for the two variable case, the n-variable case is done in the same way.

2-variable Hilbert transforms

The space $H^2 \otimes L^2$ is the closed subspace of functions in $L^2(\mathbb{T}^2)$ whose biharmonic extension to the bidisk is analytic in the first variable.
We call the space the 'right half plane' and write P_1 for the orthogonal projection onto the right half plane.

2-variable Hilbert transforms

The space $H^2 \otimes L^2$ is the closed subspace of functions in $L^2(\mathbb{T}^2)$ whose biharmonic extension to the bidisk is analytic in the first variable.
We call the space the 'right half plane' and write P_1 for the orthogonal projection onto the right half plane.

Similarly, $L^2 \otimes H^2$ denotes the closed subspace of functions in $L^2(\mathbb{T}^2)$ whose biharmonic extension to the bidisk is analytic in the second variable This is called the 'upper half plane' of $L^2(\mathbb{T}^2)$. We write P_2 for the orthogonal projection onto the upper half plane.

2-variable Hilbert transforms

The space $H^2 \otimes L^2$ is the closed subspace of functions in $L^2(\mathbb{T}^2)$ whose biharmonic extension to the bidisk is analytic in the first variable.
We call the space the 'right half plane' and write P_1 for the orthogonal projection onto the right half plane.

Similarly, $L^2 \otimes H^2$ denotes the closed subspace of functions in $L^2(\mathbb{T}^2)$ whose biharmonic extension to the bidisk is analytic in the second variable This is called the 'upper half plane' of $L^2(\mathbb{T}^2)$. We write P_2 for the orthogonal projection onto the upper half plane.

Then we define the '*jth*' variable Hilbert transform $H_j = -iP_j + iP_j^\perp$;

2-variable product BMO

The ' product BMO' is defined, for $n = 2$ by

$$\phi \in BMO(\mathbb{T}^2) \iff \phi = g_1 + H_1(g_2) + H_2(g_3) + H_1(H_2(g_4)) \qquad (*)$$

with all the $g_i \in L^\infty$.

2-variable product BMO

The ' product BMO' is defined, for $n = 2$ by

$$\phi \in BMO(\mathbb{T}^2) \iff \phi = g_1 + H_1(g_2) + H_2(g_3) + H_1(H_2(g_4)) \qquad (*)$$

with all the $g_i \in L^\infty$. The norm is defined by

$$\|\phi\|_{BMO} = \inf\{\max_j \|g_j\|_\infty\}$$

where the inf is taken over all decompositions of the form $(*)$.

Duality of product BMO

Now we can define $H^1_{Re}(\mathbb{T}^2)$ to be

$$\{f \in L^1(\mathbb{T}^2) : H_1(f), H_2(f), H_1(H_2(f)) \in L^1(\mathbb{T}^2)\}$$

Duality of product BMO

Now we can define $H^1_{Re}(\mathbb{T}^2)$ to be

$$\{f \in L^1(\mathbb{T}^2) : H_1(f), H_2(f), H_1(H_2(f)) \in L^1(\mathbb{T}^2)\}$$

Equipped with the norm defined by

$$\|f\|_{prod} = \|f\|_1 + \|H_1(f)\|_1 + \|H_2(f)\|_1 + \|H_1 H_2(f)\|_1$$

and it is fairly straightforward to see that our 'product bmo' is the dual of the space $H^1_{Re}(\mathbb{T}^2)$

Duality of product BMO

Now we can define $H^1_{Re}(\mathbb{T}^2)$ to be

$$\{f \in L^1(\mathbb{T}^2) : H_1(f), H_2(f), H_1(H_2(f)) \in L^1(\mathbb{T}^2)\}$$

Equipped with the norm defined by

$$\|f\|_{prod} = \|f\|_1 + \|H_1(f)\|_1 + \|H_2(f)\|_1 + \|H_1 H_2(f)\|_1$$

and it is fairly straightforward to see that our 'product bmo' is the dual of the space $H^1_{Re}(\mathbb{T}^2)$

The n-variable product bmo and H^1_{Re} are defined in exactly the same way.

Factorization

The difference between the one and several variable case has a lot to do with factorization. The Nehari theorem which proves that bounded one variable Hankels come from projections of L^∞ functions uses the following factorization to go from L^2 to L^1 and then to its dual L^∞.

Theorem

$$H^1(\mathbb{T}) = H^2(\mathbb{T})H^2(\mathbb{T})$$

in the sense that each $f \in H^1(\mathbb{T})$ *can be written as a product* $f = f_1 f_2$ *with* f_1 *and* $f_2 \in H^2(\mathbb{T})$ *and* $\|f\|_1 = \|f_1\|_2 \|f_2\|_2$.

Factorization doesn't generalize

This means that, in the one variable case, we can use our Hankel operator to define a linear functional on all products of H^2 functions, which then gives a functional on H^1_{Re} which can be extended to a functional on L^1 which corresponds to an L^∞ function.

Factorization doesn't generalize

This means that, in the one variable case, we can use our Hankel operator to define a linear functional on all products of H^2 functions, which then gives a functional on H^1_{Re} which can be extended to a functional on L^1 which corresponds to an L^∞ function.

This type of factorization does not work in several variables - because of the lack of 'inner' functions, one cannot factor out zeros and 'divide'.

Commutators with Hilbert transform vs Hankels

The commutator of two operators A and B is defined by:
$[A, B] = A \circ B - B \circ A.$

Commutators with Hilbert transform vs Hankels

The commutator of two operators A and B is defined by:
$[A, B] = A \circ B - B \circ A$.

The BMO - Hankel operator correspondance we have described shows that commutators of one variable Hilbert transforms with multiplication operators are the easiest operators whose norm is equivalent to the 'BMO' norm of their 'symbol'.

Commutators with Hilbert transform vs Hankels

In one variable, it is straightforward to see that, if H is the Hilbert transform and M_b is multiplication by the function b, then:

$$[M_b, H]f = M_b \cdot Hf - H(M_b f) =$$

Commutators with Hilbert transform vs Hankels

In one variable, it is straightforward to see that, if H is the Hilbert transform and M_b is multiplication by the function b, then:

$$[M_b, H]f = M_b \cdot Hf - H(M_b f) =$$

$$= -i\Big[\quad M_b(P^+ - P^-) - (P^+ - P^-)M_b \quad \Big](f)$$

Commutators with Hilbert transform vs Hankels

In one variable, it is straightforward to see that, if H is the Hilbert transform and M_b is multiplication by the function b, then:

$$[M_b, H]f = M_b \cdot Hf - H(M_b f) =$$

$$= -i\big[(P^+ + P^-)M_b(P^+ - P^-) - (P^+ - P^-)M_b(P^+ + P^-)\big](f)$$

Commutators with Hilbert transform vs Hankels

In one variable, it is straightforward to see that, if H is the Hilbert transform and M_b is multiplication by the function b, then:

$$[M_b, H]f = M_b \cdot Hf - H(M_b f) =$$

$$= -i\big[(P^+ + P^-)M_b(P^+ - P^-) - (P^+ - P^-)M_b(P^+ + P^-)\big](f)$$

$$= -i[2P^- M_b P^+ - 2P^+ M_b P^-](f) = -2i(H_b - H^*_{\bar{b}})(f)$$

.

Correlation between BMO norms and bounded commutators

So $[M_b, H]$ is bounded on L^2 iff $P^+b \in BMO$ and $P^-b \in BMO$ iff $b \in BMO$. and Nehari's theorem gives us

$$\|[M_b, H]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

Correlation between BMO norms and bounded commutators

So $[M_b, H]$ is bounded on L^2 iff $P^+ b \in BMO$ and $P^- b \in BMO$ iff $b \in BMO$. and Nehari's theorem gives us

$$\|[M_b, H]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

In fact, it is also true that

$$\|b\|_{BMO} \lesssim \|[M_b, H]\|_{2\to 2}$$

Correlation between BMO norms and bounded commutators

So $[M_b, H]$ is bounded on L^2 iff $P^+b \in BMO$ and $P^-b \in BMO$ iff $b \in BMO$. and Nehari's theorem gives us

$$\|[M_b, H]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

In fact, it is also true that

$$\|b\|_{BMO} \lesssim \|[M_b, H]\|_{2\to 2}$$

Thus, the norm of a bounded commutator with symbol b is equivalent to the BMO norm of b.

Various Commutators in two dimensions and their BMOs

For a symbol $b(x, y)$ that depends on two variables, there are several natural choices:

1) $[M_b, R_i]$ where $i = 1, 2$ and R_i is the Riesz transform in the ith direction. $\rightarrow$ 1 parameter BMO

Various Commutators in two dimensions and their BMOs

For a symbol $b(x, y)$ that depends on two variables, there are several natural choices:

1) $[M_b, R_i]$ where $i = 1, 2$ and R_i is the Riesz transform in the ith direction. $\rightarrow$ 1 parameter BMO

2) $[M_b, H_1 H_2]$ where H_i are Hilbert transforms in the $i - th$ variable, $i = 1, 2$. Simple case of a product CZO. $\rightarrow$ little BMO

Various Commutators in two dimensions and their BMOs

For a symbol $b(x, y)$ that depends on two variables, there are several natural choices:

1) $[M_b, R_i]$ where $i = 1, 2$ and R_i is the Riesz transform in the ith direction. $\rightarrow$ 1 parameter BMO

2) $[M_b, H_1 H_2]$ where H_i are Hilbert transforms in the $i - th$ variable, $i = 1, 2.$ Simple case of a product CZO. $\rightarrow$ little BMO

3) $[[M_b, H_1], H_2]$, simplest case of an iterated commutator. $\rightarrow$ product BMO

Various Commutators in two dimensions and their BMOs

For a symbol $b(x, y)$ that depends on two variables, there are several natural choices:

1) $[M_b, R_i]$ where $i = 1, 2$ and R_i is the Riesz transform in the ith direction. $\rightarrow$ 1 parameter BMO

2) $[M_b, H_1 H_2]$ where H_i are Hilbert transforms in the $i - th$ variable, $i = 1, 2$. Simple case of a product CZO. $\rightarrow$ little BMO

3) $[[M_b, H_1], H_2]$, simplest case of an iterated commutator. $\rightarrow$ product BMO Sadosky, Ferguson, Lacey established 2 and 3 in some very

sophisticated papers around the year 2000 - using innovative techniques and earlier work by many great mathematicians (Cotlar, Rochberg, Weiss, Coffman)

Commutators and multi-variable Hankels

Writing out the spaces on which the commutators project, one see that little BMO norm is equivalent to the norm of the associated BIG Hankel operators (associated with the commutator $[M_b, H_1 H_2]$). A Big Hankel operator is a composition of multiplication on several variable H^2 followed by projection on the orthogonal complement:

Commutators and multi-variable Hankels

Writing out the spaces on which the commutators project, one see that little BMO norm is equivalent to the norm of the associated BIG Hankel operators (associated with the commutator $[M_b, H_1 H_2]$). A Big Hankel operator is a composition of multiplication on several variable H^2 followed by projection on the orthogonal complement:

$$H_\phi : H^2(\mathbb{T}^n) \to (H^2(\mathbb{T}^n))^\perp$$

$$H_\phi(f) = P^\perp(\phi f)$$

Commutators and multi-variable Hankels

Writing out the spaces on which the commutators project, one see that little BMO norm is equivalent to the norm of the associated BIG Hankel operators (associated with the commutator $[M_b, H_1 H_2]$). A Big Hankel operator is a composition of multiplication on several variable H^2 followed by projection on the orthogonal complement:

$$H_\phi : H^2(\mathbb{T}^n) \to (H^2(\mathbb{T}^n))^\perp$$

$$H_\phi(f) = P^\perp(\phi f)$$

On the other hand, 'product ' or 'Chang-Fefferman 'BMO norm ($[[M_b, H_1], H_2]$) is associated with little Hankel operators, which project on the complex conjugate of H^2, a much smaller space.

More general BMOs - work with Petermichl Ou

We have been working on mixed BMO spaces of functions which are BMO in certain combinations of their variables.

More general BMOs - work with Petermichl Ou

We have been working on mixed BMO spaces of functions which are BMO in certain combinations of their variables.

For example, we call $BMO_{(12)3}$ the Banach space of functions $b \in L^1(\mathbb{T}^3)$ such that the families $(b(\cdot, x_2, \cdot))_{x_2 \in \mathbb{T}}$ and $(b(x_1, \cdot, \cdot))_{x_1 \in \mathbb{T}}$ are uniformly bounded in product BMO.

More general BMOs - work with Petermichl Ou

We have been working on mixed BMO spaces of functions which are BMO in certain combinations of their variables.

For example, we call $BMO_{(12)3}$ the Banach space of functions $b \in L^1(\mathbb{T}^3)$ such that the families $(b(\cdot, x_2, \cdot))_{x_2 \in \mathbb{T}}$ and $(b(x_1, \cdot, \cdot))_{x_1 \in \mathbb{T}}$ are uniformly bounded in product BMO.

We are looking at the characterizations of these types of spaces in terms of their preduals, commutators and Hankel-type operators.

(This involves certain types of weak factorization of their pre-duals.)

Mixed BMO and its predual

We have the following results: To characterize mixed BMO as a dual space:

Mixed BMO and its predual

We have the following results: To characterize mixed BMO as a dual space:

Theorem

A function $f \in L^1(\mathbb{T}^3)$ *satisfies*

$$\sup_{\|\varphi\|_{BMO_{(13)2}}} \left| \int_{\mathbb{T}^3} f\varphi dm \right| < \infty$$

if and only if there exist functions $f' \in H^1_{Re}(\mathbb{T} \times \mathbb{T}) \otimes L^1(\mathbb{T})$ *and* $f'' \in L^1(\mathbb{T}) \otimes H^1_{Re}(\mathbb{T} \times \mathbb{T})$ *such that* $f = f' + f''$.

Mixed BMO and its Commutators

To characterize mixed BMO in terms of commutators:

Mixed BMO and its Commutators

To characterize mixed BMO in terms of commutators:

Theorem

Let $b \in L^1(\mathbb{T}^3)$. The the following are equivalent.

1. $b \in BMO_{(12)3}$
2. *The commutators* $[H_3, [H_1, b]]$ *and* $[H_3, [H_2, b]]$ *are bounded on* $L^2(\mathbb{T}^3)$
3. *The commutator* $[H_3, [H_2 H_1, b]]$ *is bounded on* $L^2(\mathbb{T}^3)$.

Mixed BMO and it's Hankels

Now, using our commutator theorem, one can characterize the mixed BMO functions in terms of 'Hankel types' that is, operators of type $P^{\perp} M_b P$ as follows:

Mixed BMO and it's Hankels

Now, using our commutator theorem, one can characterize the mixed BMO functions in terms of 'Hankel types' that is, operators of type $P^{\perp}M_bP$ as follows:

Theorem

- *The commutators* $[H_3,[H_1,b]]$ *and* $[H_3,[H_2,b]]$ *are bounded on* $L^2(\mathbb{T}^3)$ *if and only if all eight operators* $P_iP_3bP_i^{\perp}P_3^{\perp}, P_i^{\perp}P_3bP_iP_3^{\perp}, P_iP_3bP_i^{\perp}P_3^{\perp}$ *with* $i\in\{1,2\}$ *are bounded on* $L^2(\mathbb{T}^3)$.
- *The commutator* $[H_3,[H_2H_1,b]]$ *is bounded on* $L^2(\mathbb{T}^3)$ *if and only if all four Hankels* $P_3Q_{12}bQ_{12}^{\perp}P_3^{\perp}, P_3^{\perp}Q_{12}^{\perp}bQ_{12}P_3, P_3Q_{12}^{\perp}bQ_{12}P_3^{\perp}, P_3^{\perp}Q_{12}bQ_{12}^{\perp}P_3$ *with* $Q_{12}=P_1P_2+P_1^{\perp}P_2^{\perp}$ *and* $Q_{12}^{\perp}=P_1^{\perp}P_2+P_1P_2^{\perp}$, *are bounded on* $L^2(\mathbb{T}^3)$.

Generalizations

Thus one can say that a function is in mixed BMO if and only if either group of listed Hankels are bounded.

Generalizations

Thus one can say that a function is in mixed BMO if and only if either group of listed Hankels are bounded.

This work led to an understanding of how much more general operators could be treated . After looking at all sorts of tensor products of Hilbert and Riesz operators we began looking at much more general operators - "paraproduct free bi-parameter Calderón-Zygmund operators of Journe type."

Generalizations

Thus one can say that a function is in mixed BMO if and only if either group of listed Hankels are bounded.

This work led to an understanding of how much more general operators could be treated . After looking at all sorts of tensor products of Hilbert and Riesz operators we began looking at much more general operators - "paraproduct free bi-parameter Calderón-Zygmund operators of Journe type."

What follows requires lots of long intricate calculations that are impossible to follow during a short talk. So, I'll just try to give you an idea of what must be done - and how it ressembles the first steps that I just ran through with you.

Generalizations

Definition A continuous linear mapping $T : C_0^\infty(\mathbb{R}^n) \otimes C_0^\infty(\mathbb{R}^m) \to [C_0^\infty(\mathbb{R}^n) \otimes C_0^\infty(\mathbb{R}^m)]'$ is called a *paraproduct free bi-parameter Calderón-Zygmund operator* if the following conditions are satisfied:

1. T is a Journé type bi-parameter δ-singular integral operator, i.e. there exists a pair (K_1, K_2) of δCZ-δ-standard kernels so that, for all $f_1, g_1 \in C_0^\infty(\mathbb{R}^n)$ and $f_2, g_2 \in C_0^\infty(\mathbb{R}^m)$,

$$\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \int f_1(y_1)\langle K_1(x_1, y_1) f_2, g_2\rangle g_1(x_1)\, dx_1 dy_1$$

when $\mathrm{spt} f_1 \cap \mathrm{spt} g_1 = \emptyset$;

$$\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \int f_2(y_2)\langle K_2(x_2, y_2) f_1, g_1\rangle g_2(x_2)\, dx_2 dy_2$$

when $\mathrm{spt} f_2 \cap \mathrm{spt} g_2 = \emptyset$.

Definition continued

2. T satisfies the weak boundedness property
$|\langle T(\chi_I \otimes \chi_J), \chi_I \otimes \chi_J\rangle| \lesssim |I||J|$, for any cubes $I \subset \mathbb{R}^n, J \in \mathbb{R}^m$.
3. T is paraproduct free in the sense that
$T(1 \otimes \cdot) = T(\cdot \otimes 1) = T^*(1 \otimes \cdot) = T^*(\cdot \otimes 1) = 0$.

Definition continued

2. T satisfies the weak boundedness property
$|\langle T(\chi_I \otimes \chi_J), \chi_I \otimes \chi_J \rangle| \lesssim |I||J|$, for any cubes $I \subset \mathbb{R}^n, J \in \mathbb{R}^m$.
3. T is paraproduct free in the sense that
$T(1 \otimes \cdot) = T(\cdot \otimes 1) = T^*(1 \otimes \cdot) = T^*(\cdot \otimes 1) = 0.$

A δCZ-δ-standard kernel is a vector valued standard kernel taking values in the Banach space consisting of all Calderón-Zygmund operators.

Such an operator satisfies all the hypotheses from what is called a Tb Theorem (Martikainen for 'square kernel operators') and so is L^2 bounded and can be represented as an average of bi-parameter dyadic shift operators together with dyadic paraproducts.

Definition continued

2. T satisfies the weak boundedness property
$|\langle T(\chi_I \otimes \chi_J), \chi_I \otimes \chi_J\rangle| \lesssim |I||J|$, for any cubes $I \subset \mathbb{R}^n, J \in \mathbb{R}^m$.
3. T is paraproduct free in the sense that
$T(1 \otimes \cdot) = T(\cdot \otimes 1) = T^*(1 \otimes \cdot) = T^*(\cdot \otimes 1) = 0.$

A δCZ-δ-standard kernel is a vector valued standard kernel taking values in the Banach space consisting of all Calderón-Zygmund operators.

Such an operator satisfies all the hypotheses from what is called a Tb Theorem (Martikainen for 'square kernel operators') and so is L^2 bounded and can be represented as an average of bi-parameter dyadic shift operators together with dyadic paraproducts.

Moreover, since T is paraproduct free, one can conclude from observing the proof of Martikainen's theorem, that all the dyadic shifts in the representation are cancellative (this has to do with disjointness of dyadic intervals that are moved).

Journé type bi-parameter δ-singular integral operators

This means that, for sufficiently 'nice' functions f and g our 'Journé type bi-parameter δ-singular integral operator' T satisfies:

$$\langle Tf, g\rangle = C\mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2}\sum_{i_1,j_1=0}^{\infty}\sum_{i_2,j_2=0}^{\infty}2^{-\max(i_1,j_1)}2^{-\max(i_2,j_2)}\langle S^{i_1j_1i_2j_2}f, g\rangle, \quad (1)$$

where expectation is calculated with respect to a certain parameter of the dyadic grids and the dyadic shift operator $S^{i_1j_1i_2j_2}$ evaluated at f is equal to:

Journé type bi-parameter δ-singular integral operators

This means that, for sufficiently 'nice' functions f and g our 'Journé type bi-parameter δ-singular integral operator' T satisfies:

$$\langle Tf, g\rangle = C\mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2}\sum_{i_1,j_1=0}^{\infty}\sum_{i_2,j_2=0}^{\infty} 2^{-\max(i_1,j_1)}2^{-\max(i_2,j_2)}\langle S^{i_1j_1i_2j_2}f, g\rangle, \quad (1)$$

where expectation is calculated with respect to a certain parameter of the dyadic grids and the dyadic shift operator $S^{i_1j_1i_2j_2}$ evaluated at f is equal to:

$$\sum_{K_1\in\mathcal{D}_1}\sum_{\substack{I_1,J_1\subset K_1, I_1,J_1\in\mathcal{D}_1\\ \ell(I_1)=2^{-i_1}\ell(K_1)\\ \ell(J_1)=2^{-j_1}\ell(K_1)}}\sum_{K_2\in\mathcal{D}_2}\sum_{\substack{I_2,J_2\subset K_2, I_2,J_2\in\mathcal{D}_2\\ \ell(I_2)=2^{-i_2}\ell(K_2)\\ \ell(J_2)=2^{-j_2}\ell(K_2)}} a_{I_1J_1K_1I_2J_2K_2}\langle f, h_{I_1}\otimes h_{I_2}\rangle h_{J_1}\otimes h_{J_2}$$

Calculations

And the coefficients satisfy:

$$a_{I_1 J_1 K_1 I_2 J_2 K_2} \leq \frac{\sqrt{|I_1||J_1||I_2||J_2|}}{|K_1||K_2|}$$

which guarantees the normalisation

Calculations

And the coefficients satisfy:

$$a_{I_1 J_1 K_1 I_2 J_2 K_2} \leq \frac{\sqrt{|I_1||J_1||I_2||J_2|}}{|K_1||K_2|}$$

which guarantees the normalisation

$$\|S^{i_1 j_1 i_2 j_2}\|_{L^2 \to L^2} \leq 1.$$

Moreover, since T is paraproduct free, all the Haar functions appearing above are cancellative.

Proving the equivalence of norms

Thus, the problem boils down to proving that, for any dyadic grids $\mathcal{D}_1, \mathcal{D}_2$ and fixed $i_1, j_1, i_2, j_2 \in \mathbb{N}$, one has

$$\|[b, S^{i_1 j_1 i_2 j_2}]f\|_{L^2} \lesssim (1 + \max(i_1, j_1))(1 + \max(i_2, j_2))\|b\|_{\text{bmo}}\|f\|_{L^2}. \quad (2)$$

Proving the equivalence of norms

Thus, the problem boils down to proving that, for any dyadic grids $\mathcal{D}_1, \mathcal{D}_2$ and fixed $i_1, j_1, i_2, j_2 \in \mathbb{N}$, one has

$$\|[b, S^{i_1 j_1 i_2 j_2}]f\|_{L^2} \lesssim (1 + \max(i_1, j_1))(1 + \max(i_2, j_2))\|b\|_{\text{bmo}}\|f\|_{L^2}. \qquad (2)$$

This is a long and very complicated calculation which begins by decomposing b and a L^2 test function f using Haar bases in the following way:

$$[b, S^{i_1 j_1 i_2 j_2}]f = \sum_{I_1, I_2} \sum_{J_1, J_2} \langle b, h_{I_1} \otimes h_{I_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle [h_{I_1} \otimes h_{I_2}, S^{i_1 j_1 i_2 j_2}] h_{J_1} \otimes h_{J_2}.$$

Proving the equivalence of norms

Thus, the problem boils down to proving that, for any dyadic grids $\mathcal{D}_1, \mathcal{D}_2$ and fixed $i_1, j_1, i_2, j_2 \in \mathbb{N}$, one has

$$\|[b, S^{i_1 j_1 i_2 j_2}]f\|_{L^2} \lesssim (1 + \max(i_1, j_1))(1 + \max(i_2, j_2))\|b\|_{\mathrm{bmo}}\|f\|_{L^2}. \quad (2)$$

This is a long and very complicated calculation which begins by decomposing b and a L^2 test function f using Haar bases in the following way:

$$[b, S^{i_1 j_1 i_2 j_2}]f = \sum_{I_1, I_2} \sum_{J_1, J_2} \langle b, h_{I_1} \otimes h_{I_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle [h_{I_1} \otimes h_{I_2}, S^{i_1 j_1 i_2 j_2}] h_{J_1} \otimes h_{J_2}.$$

and then doing seven pages of difficult estimations - there are four separate cases that depend on the relationship between the intervals I_k, and $J_k^{(i_k)}$

Theorem

Once the estimation is done we obtain the following:

Theorem

Let T be a paraproduct free bi-parameter Calderón-Zygmund operator, and b be a little bmo function, there holds

$$\|[b,T]\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \|b\|_{bmo(\mathbb{R}^n\times\mathbb{R}^m)},$$

where the underlying constant depends only on the characterizing constants of T.

which you can read about in:
Higher order Journé commutators and characterizations of multiparameter BMO , Advances in Mathematics 291 (2016) 24-58
(Yumeng Ou, Stefanie Petermichl, and me)

Summary and Goodbye

Altogether, we were quite pleased to find lots of characterizations of a very natural and variable isolating BMO. We have begun (with Brett) looking at the compacity or H-S properties of operators associated with similar types of spaces (define a mixed VMO etc) and hope to continue in several directions.

Summary and Goodbye

Altogether, we were quite pleased to find lots of characterizations of a very natural and variable isolating BMO. We have begun (with Brett) looking at the compacity or H-S properties of operators associated with similar types of spaces (define a mixed VMO etc) and hope to continue in several directions.

Thank you for your time and attention