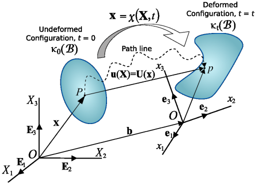
Introduction to continuum mechanics-II

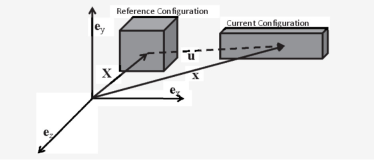
Ioan R. IONESCU
LSPM, University Paris 13 (Sorbonne-Paris-Cité)

July 3, 2017

Continuum kinematics

- $\mathcal{B}_0$ undeformed/reference configuration, $\mathcal{B}_t$ deformed/actual configuration
- Motion: $\boldsymbol{x}(t, \boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(t, \boldsymbol{X})$ with $\boldsymbol{x}(t, \mathcal{B}_0) = \mathcal{B}_t$.
- Material/Lagrangian coordinates: $\boldsymbol{X} = (X_1, X_2, X_3) \in \mathcal{B}_0$
- Spatial/Eulerian coordinates: $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathcal{B}_t$
- $\boldsymbol{u}(t, \boldsymbol{X}) = \boldsymbol{x}(t, \boldsymbol{X}) - \boldsymbol{X}$ the displacement field





Velocity-acceleration

- velocity (in Lagrange variables) $\boldsymbol{v}(t, \boldsymbol{X}) = \frac{\partial \boldsymbol{x}}{\partial t}(t, \boldsymbol{X})$
- Suppose $\boldsymbol{X} \rightarrow \boldsymbol{x}(t, \boldsymbol{X})$ one to one function, then $\exists \quad \boldsymbol{x} \rightarrow \boldsymbol{X}(t, \boldsymbol{x})$
- velocity (in Euler variables) $\boldsymbol{v}(t, \boldsymbol{x}) = \boldsymbol{v}(t, \boldsymbol{X}(t, \boldsymbol{x}))$.
- acceleration (in Lagrange variables) $\boldsymbol{a}(t, \boldsymbol{X}) = \frac{\partial^2 \boldsymbol{x}}{\partial^2 t}(t, \boldsymbol{X})$
- acceleration (in Euler variables) $\boldsymbol{a}(t, \boldsymbol{x}) = \boldsymbol{a}(t, \boldsymbol{X}(t, \boldsymbol{x}))$.

Example. Dilatation:

$x_1 = X_1 + \alpha_1 t X_1, \quad x_1 = X_2 + \alpha_2 t X_2, \quad x_3 = X_3 + \alpha_3 t X_3$

- velocity (in Lagrange variables) $\boldsymbol{v}(t, \boldsymbol{X}) = (\alpha_1 X_1, \alpha_2 X_2, \alpha_3 X_3)$
- velocity (in Euler variables) $\boldsymbol{v}(t, \boldsymbol{x}) = (\frac{\alpha_1}{1+\alpha_1 t} x_1, \frac{\alpha_2}{1+\alpha_2 t} x_2, \frac{\alpha_3}{1+\alpha_3 t} x_3)$.

Particular (material, total) derivative

Particular (total) derivative of field K (the particle is followed in its movement) : $K(t, \boldsymbol{X})$ in Lagrange description, $K(t, \boldsymbol{x}) = K(t, \boldsymbol{X}(t, \boldsymbol{x}))$ in Eulerian description

- if K is in Lagrange variables $\dfrac{dK}{dt}(t, \boldsymbol{X}) = \dfrac{\partial K}{\partial t}(t, \boldsymbol{X})$
- if K is in Euler variables $\dfrac{dK}{dt}(t, \boldsymbol{x}) = \dfrac{\partial K}{\partial t}(t, \boldsymbol{x}) + \boldsymbol{v}(t, \boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}} K(t, \boldsymbol{x})$

Examples

- $K = \boldsymbol{x}$: $\quad \dfrac{d\boldsymbol{x}}{dt}(t, \boldsymbol{x}) = \boldsymbol{v}(t, \boldsymbol{x})$
- $K = \boldsymbol{v}(t, \boldsymbol{x})$: $\quad \dfrac{d\boldsymbol{v}}{dt}(t, \boldsymbol{x}) = \boldsymbol{a}(t, \boldsymbol{x}) = \dfrac{\partial \boldsymbol{v}}{\partial t}(t, \boldsymbol{x}) + \boldsymbol{v}(t, \boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v}(t, \boldsymbol{x})$

Reynolds's transport theorem

Particular (material, total) derivative of a volume integral

Let $\omega_0 \subset \mathcal{B}_0$ and $\omega_t = \boldsymbol{x}(t, \omega_0) \subset \mathcal{B}_t$ (the subset $\omega_0 \subset \mathcal{B}_0$ is followed in its movement) and $K(t, \boldsymbol{x})$ a field in Eulerian description

$$\frac{d}{dt}\int_{\omega_t} K(t,\boldsymbol{x})\, d\boldsymbol{x} = \int_{\omega_t} \left(\frac{\partial K}{\partial t}(t,\boldsymbol{x}) + div_{\boldsymbol{x}}(K(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})) \right)\, d\boldsymbol{x}$$

$$= \int_{\omega_t} \frac{dK}{dt}(t,\boldsymbol{x}) + K(t,\boldsymbol{x}) div_{\boldsymbol{x}} \boldsymbol{v}(t,\boldsymbol{x})\, d\boldsymbol{x} = \int_{\omega_t} \frac{\partial K}{\partial t}(t,\boldsymbol{x}) + \int_{\partial\omega_t} K(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})\cdot\boldsymbol{n}\, d$$

Examples

- $K \equiv 1$: $\quad \dfrac{d}{dt} Vol(\omega_t) = \displaystyle\int_{\omega_t} div_{\boldsymbol{x}} \boldsymbol{v}(t,\boldsymbol{x})\, d\boldsymbol{x} = \int_{\partial\omega_t} \boldsymbol{v}(t,\boldsymbol{x}) \cdot \boldsymbol{n}\, dS$
- if $div_{\boldsymbol{x}} \boldsymbol{v}(t,\boldsymbol{x}) = 0$ then the volume is incompressible

Deformation of a continuous body

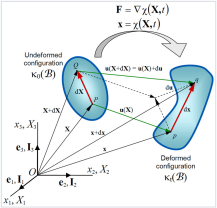


Figure 2. Deformation of a continuum body.

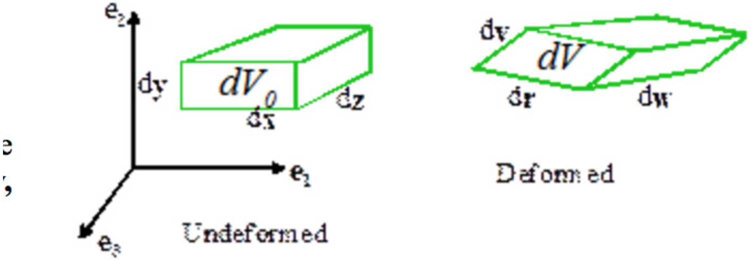
- infinitesimal element of a continuous body $d\boldsymbol{X} = (dX_1, dX_2, dX_3)$
- It is possible to show that (using Taylor expansion around a point of deformation)

$$d\boldsymbol{x} = \boldsymbol{F}(t, \boldsymbol{X})d\boldsymbol{X} = (\boldsymbol{I} + \nabla \boldsymbol{u}(t, \boldsymbol{x}))d\boldsymbol{X}$$

- $dx_i = F_{ik}dX_k = (\delta_{ik} + \dfrac{\partial u_i}{\partial X_k})dX_k$

Volume change

- Consider a differential material volume dV at some material point that goes to dv after deformation
- How to measure the volume change ?
- Reference volume: $dV_0 = d\mathbf{Z} \cdot (d\mathbf{X} \times d\mathbf{Y})$
- Deformed volume: $dv = d\mathbf{W} \cdot (d\mathbf{R} \times d\mathbf{V})$
- $J(t, \boldsymbol{X}) = det(\boldsymbol{F}(t, \boldsymbol{X}))$ Jacobien of the trasformation
- It is easy to show that the volume change: $dv = JdV$



Polar decomposition theorem

- A rotation matrix: $\boldsymbol{R}$ such that $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R}$=$\mathbf{I}$ ($det\boldsymbol{R} = I$).
- Polar decomposition theorem: For any matrix $\mathbf{F}$ with $det\mathbf{F} > \mathbf{0}$, there exists an unique rotation $\boldsymbol{R}$ and an unique positive-definite symmetric matrix $\boldsymbol{U}$ such that

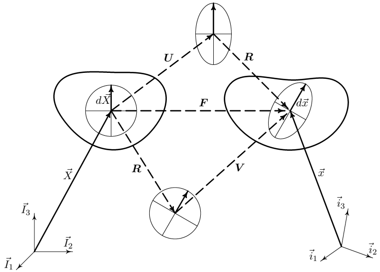
$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$$

- how to calculate it ? Calculate the Cauhy-Green strain tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and then $\boldsymbol{U} = \sqrt{\boldsymbol{C}}$, i.e. Find the eigenvalues $\{\gamma_1, \gamma_2, \gamma_3\}$ and eigenvectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ of $\boldsymbol{C}$ calculate $\mu_i = \sqrt{\gamma_i}$ and then $\boldsymbol{U}$ is the matrix with eigenvalues $\{\mu_1, \mu_2, \mu_3\}$ and the corresponding eigenvectors such that

$$\boldsymbol{U} = \mu_1\boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \mu_2\boldsymbol{u}_2 \otimes \boldsymbol{u}_2 + \mu_3\boldsymbol{u}_3 \otimes \boldsymbol{u}_3$$

$$\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$$

Polar decomposition theorem



Velocity gradient

Velocity gradient tensor $\boldsymbol{L} = \boldsymbol{L}(t, \boldsymbol{x}) = \nabla_{\boldsymbol{x}} \boldsymbol{v}(t, \boldsymbol{x}), \quad L_{ij} = \frac{\partial v_i}{\partial x_j}$

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} = (\frac{d}{dt} \boldsymbol{F}) \boldsymbol{F}^{-1}$$

$$\boldsymbol{L} = \dot{\boldsymbol{R}} \boldsymbol{R}^T + \boldsymbol{R} \dot{\boldsymbol{U}} \boldsymbol{U}^{-1} \boldsymbol{R}^T$$

Strain rate (stretching, rate of deformation) tensor $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{v})$

$$\boldsymbol{D} = \boldsymbol{D}(t, \boldsymbol{x}) = \frac{1}{2}(\nabla_{\boldsymbol{x}} \boldsymbol{v}(t, \boldsymbol{x}) + \nabla_{\boldsymbol{x}}^T \boldsymbol{v}(t, \boldsymbol{x})), \quad D_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$$

Spin tensor $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{v})$

$$\boldsymbol{W} = \boldsymbol{W}(t, \boldsymbol{x}) = \frac{1}{2}(\nabla_{\boldsymbol{x}} \boldsymbol{v}(t, \boldsymbol{x}) - \nabla_{\boldsymbol{x}}^T \boldsymbol{v}(t, \boldsymbol{x})), \quad W_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})$$

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \quad \boldsymbol{D}^T = \boldsymbol{D}, \quad \boldsymbol{W}^T = -\boldsymbol{W}$$

$$\boldsymbol{\omega}(t, \boldsymbol{x}) = curl_{\boldsymbol{x}} \boldsymbol{v}(t, \boldsymbol{x}), \quad \boldsymbol{W} \boldsymbol{c} = \frac{1}{2} \boldsymbol{\omega} \times \boldsymbol{c}, \quad \forall \boldsymbol{c}$$

Rate of Change of Length and Orientation.

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X}, \boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \Longrightarrow \frac{d}{dt}(d\boldsymbol{x}) = \boldsymbol{L} d\boldsymbol{x}$$

Orientation and length of the infinitesimal vectors

$$d\boldsymbol{x} = \boldsymbol{n} ds, d\boldsymbol{x}^1 = \boldsymbol{n}^1 ds^1, d\boldsymbol{x}^2 = \boldsymbol{n}^2 ds^2$$

Rate of change of orientation $\frac{d}{dt}(\boldsymbol{n}) = \boldsymbol{L}\boldsymbol{n} - (\boldsymbol{D}\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{n}$

$$\frac{d}{dt}(d\boldsymbol{x}^1 \cdot d\boldsymbol{x}^2) = 2\boldsymbol{D}\boldsymbol{x}^1 \cdot d\boldsymbol{x}^2$$

Rate of change of length

$$d\boldsymbol{x}^1 = d\boldsymbol{x}^2 = \boldsymbol{n} ds \Longrightarrow \frac{d}{dt}(\ln ds) = \frac{1}{ds}\frac{d}{dt}(ds) = \boldsymbol{D}\boldsymbol{n} \cdot \boldsymbol{n}$$

Rate of change of angles

$$\frac{d}{dt}(\boldsymbol{n}^1 \cdot \boldsymbol{n}^2) = 2\boldsymbol{D}\boldsymbol{n}^1 \cdot \boldsymbol{n}^2 - \left(\boldsymbol{D}\boldsymbol{n}^1 \cdot \boldsymbol{n}^1 + \boldsymbol{D}\boldsymbol{n}^2 \cdot \boldsymbol{n}^2\right) \boldsymbol{n}^1 \cdot \boldsymbol{n}^2$$

Mass conservation law

Let $\rho_0 : \boldsymbol{\mathcal{B}}_0 \to \mathbb{R}_+$, *and* $\rho(t,\cdot) : \boldsymbol{\mathcal{B}}_t \to \mathbb{R}_+$ be the **mass density** such that

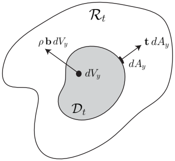
$$mass(\omega_0) = \int_{\omega_0} \rho_0(\boldsymbol{X})\, d\boldsymbol{X}, \quad mass(\omega_t) = \int_{\omega_t} \rho(t,\boldsymbol{x})\, d\boldsymbol{x}$$

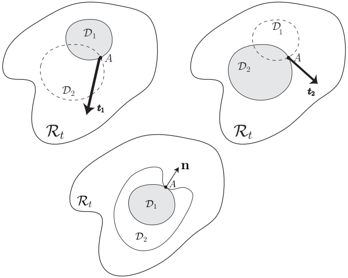
for all $\omega_0 \subset \boldsymbol{\mathcal{B}}_0$ and $\omega_t = \boldsymbol{x}(t,\omega_0) \subset \boldsymbol{\mathcal{B}}_t$.

- **Mass conservation law:** $mass(\omega_0) = mass(\omega_t)$ for all $\omega_0 \subset \boldsymbol{\mathcal{B}}_0$.
- Lagrangian description $\rho(t,\boldsymbol{x})J(t,\boldsymbol{X}) = \rho_0(\boldsymbol{X})$ for all $\boldsymbol{X} \in \boldsymbol{\mathcal{B}}_0$
- Eulerian description $\dfrac{d}{dt}\rho(t,\boldsymbol{x}) + \rho(t,\boldsymbol{x})div_{\boldsymbol{x}}\boldsymbol{v}(t,\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \boldsymbol{\mathcal{B}}_t$
- Eulerian description $\dfrac{\partial}{\partial t}\rho(t,\boldsymbol{x}) + div_{\boldsymbol{x}}(\rho(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})) = 0$

Consequence: $\dfrac{d}{dt}\displaystyle\int_{\omega_t} \rho(t,\boldsymbol{x})K(t,\boldsymbol{x})\, d\boldsymbol{x} = \int_{\omega_t} \rho(t,\boldsymbol{x})\frac{d}{dt}K(t,\boldsymbol{x})\, d\boldsymbol{x}$

Forces acting on the body





Assumptions

- **Body forces** $\rho \boldsymbol{b} d\boldsymbol{x}$: $\boldsymbol{b} = \boldsymbol{b}(t, \boldsymbol{x})$
- **Surface forces** $\boldsymbol{t} dS$ acting on $\partial\mathcal{D}_t$: the action of $\boldsymbol{\mathcal{B}}_t \setminus \mathcal{D}_t$ on $\mathcal{D}_t$ can be replaced by the distribution of the **Cauchy stress vector** $\boldsymbol{t}$
- **Cauchy's hypothesis**: $\boldsymbol{t} = \boldsymbol{t}(t, \boldsymbol{x}, \boldsymbol{n})$

The Balance of Momentum Principles

Balance principle for linear momentum (Newton's law):

$$\frac{d}{dt}\int_{\omega_t} \rho(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_{\omega_t} \rho(t,\boldsymbol{x})\boldsymbol{b}(t,\boldsymbol{x})\ d\boldsymbol{x} + \int_{\partial\omega_t} \boldsymbol{t}(t,\boldsymbol{x},\boldsymbol{n})\ dS$$

Balance principle for angular momentum (Newton's law):

$$\frac{d}{dt}\int_{\omega_t} \rho(t,\boldsymbol{x})\boldsymbol{x}\wedge\boldsymbol{v}(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_{\omega_t} \rho(t,\boldsymbol{x})\boldsymbol{x}\wedge\boldsymbol{b}(t,\boldsymbol{x})\ d\boldsymbol{x} + \int_{\partial\omega_t} \boldsymbol{x}\wedge\boldsymbol{t}(t,\boldsymbol{x},\boldsymbol{n})\ dS$$

for all $\omega_t \subset \boldsymbol{\mathcal{B}}_t$.

Consequences of balance principles and stress tensor

Consequences of linear momentum balance principle + Cauchy's hypothesis

- $\boldsymbol{n} \to \boldsymbol{t}(t, \boldsymbol{x}, \boldsymbol{n})$ is linear and there exists $\boldsymbol{\sigma}(t, \boldsymbol{x})$ **Cauchy stress tensor** such that

$$\boldsymbol{t}(t, \boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\sigma}(t, \boldsymbol{x})\boldsymbol{n}$$

- Equation of motion

$$\rho(t, \boldsymbol{x})\frac{d}{dt}\boldsymbol{v}(t, \boldsymbol{x}) = div_{\boldsymbol{x}}\boldsymbol{\sigma}(t, \boldsymbol{x}) + \rho(t, \boldsymbol{x})\boldsymbol{b}(t, \boldsymbol{x})$$

Consequence of angular momentum balance principle

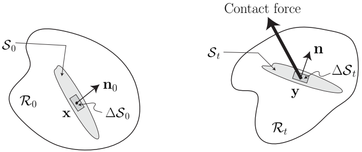
- Cauchy stress tensor is symetric

$$\boldsymbol{\sigma}^T(t, \boldsymbol{x}) = \boldsymbol{\sigma}(t, \boldsymbol{x})$$

Equation of motion in Lagrange formulation

First Piola-Kirchhoff stress tensor (non-symmetric !)

$$\boldsymbol{\Pi}(t, \boldsymbol{X}) = J(t, \boldsymbol{X})\boldsymbol{\sigma}(t, \boldsymbol{x}(t, \boldsymbol{X}))\boldsymbol{F}^{-T}(t, \boldsymbol{X})$$



Nanson's formula $\boldsymbol{n}dS = J\boldsymbol{F}^{-T}\boldsymbol{n}_0 dS_0 \Longrightarrow \boldsymbol{\sigma}(t, \boldsymbol{x})\boldsymbol{n}dS = \boldsymbol{\Pi}(t, \boldsymbol{X})\boldsymbol{n}_0 dS_0$

$$\rho_0(\boldsymbol{X})\frac{d}{dt}\boldsymbol{v}(t, \boldsymbol{X}) = div_{\boldsymbol{X}}\boldsymbol{\Pi}(t, \boldsymbol{X}) + \rho_0(\boldsymbol{X})\boldsymbol{b}(t, \boldsymbol{X})$$

Equilibrium equation $div_{\boldsymbol{X}}\boldsymbol{\Pi}(t, \boldsymbol{X}) + \rho_0(\boldsymbol{X})\boldsymbol{b}(t, \boldsymbol{X}) = 0$

Second Piola-Kirchhoff tensor (symmetric !) $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\Pi}$

Examples of stress tensors