Total mixed curvature of open curves in E^3

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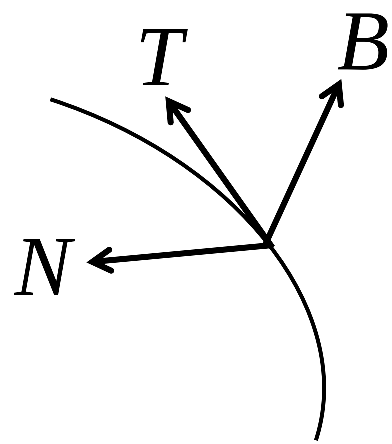
$\Sigma : x(s)$ smooth curve in E^3

s : arclength $(0 \leq s \leq L)$

T : unit tangent vector

N : principal normal vector

B : binormal vector



$$\frac{dT}{ds} = \kappa N$$

$$\frac{dN}{ds} = -\kappa T + \tau B$$

$$\frac{dB}{ds} = -\tau N$$

κ : curvature

τ : torsion

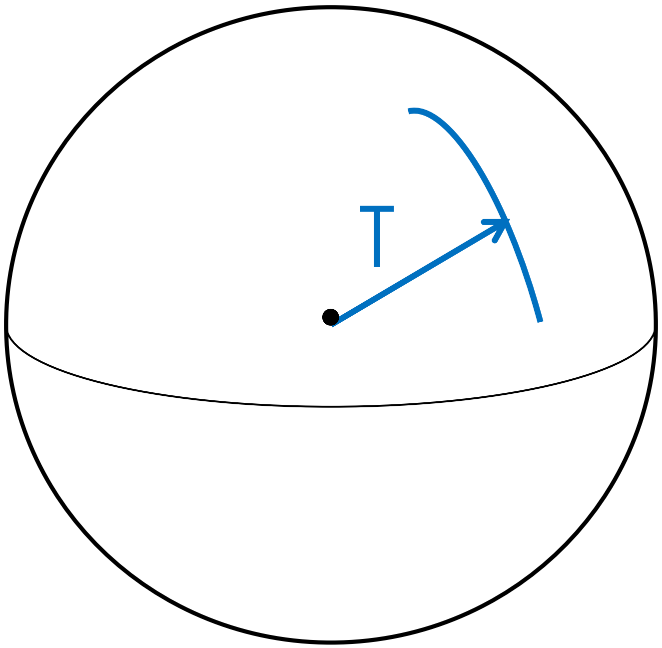
total absolute curvature $$TAC(\Sigma) = \int_0^L \kappa(s)\, ds$$

total absolute torsion $$TAT(\Sigma) = \int_0^L |\tau(s)|\, ds$$

total mixed curvature $$TMC(\Sigma) = \int_0^L \sqrt{\kappa^2 + \tau^2}\, ds$$

$$TAC(\Sigma) = \int_0^L \kappa(s)\,ds = \int_0^L \left|\frac{dT}{ds}\right| ds$$

$$= \text{length of } T(s) \text{ as a curve in } S^2$$



$$TAT(\Sigma) = \int_0^L |\tau(s)|\, ds = \int_0^L \left|\frac{dB}{ds}\right| ds$$

$$= \text{length of } B(s) \text{ as a curve in } S^2$$

$$TMC(\Sigma) = \int_0^L \sqrt{\kappa^2 + \tau^2}\, ds = \int_0^L \left|\frac{dN}{ds}\right| ds$$

$$= \text{length of } N(s) \text{ as a curve in } S^2$$

Fenchel (1929) $\int_{\Sigma} \kappa \geq 2\pi$ if Σ is closed.

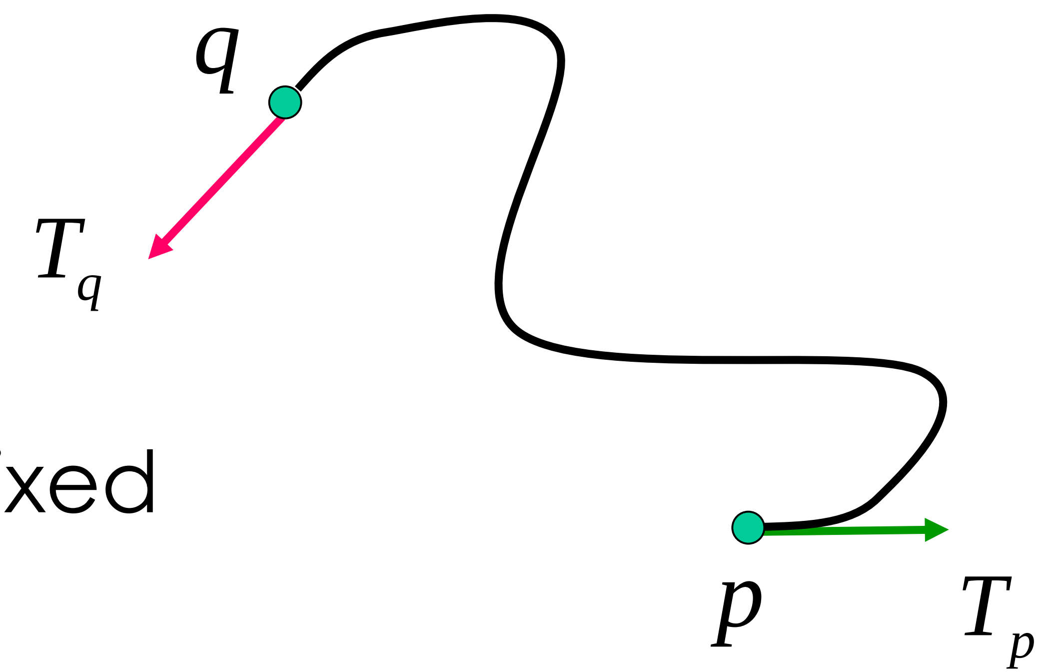
= ⟺ closed convex plane curve

Enomoto-Itoh-Sinclair (2008) determines

$$\inf\{\int_\Sigma \kappa : \Sigma \in C(p, q, T_p, T_q, L)\}$$

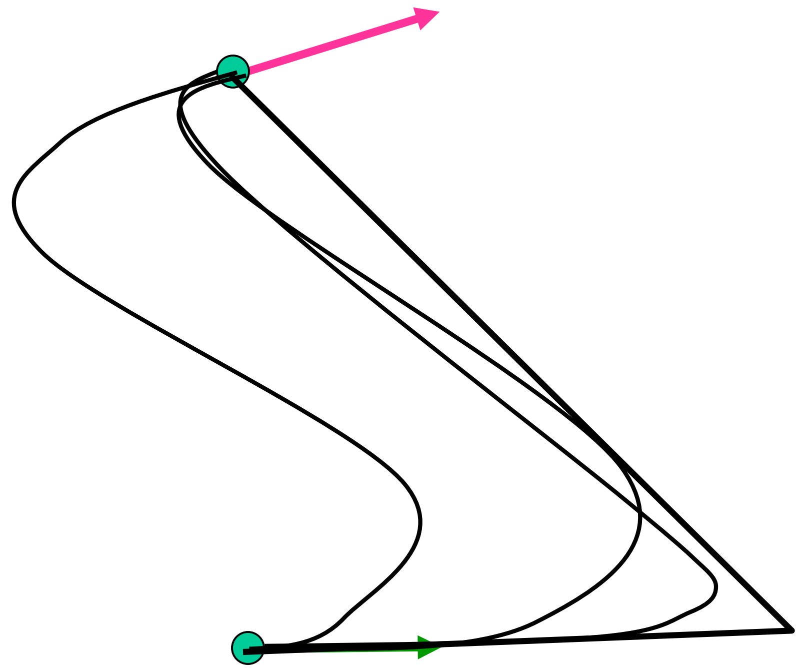
where

$C(p, q, T_p, T_q, L)$ is the set of all curves such as



p, q
T_p, T_q
length L

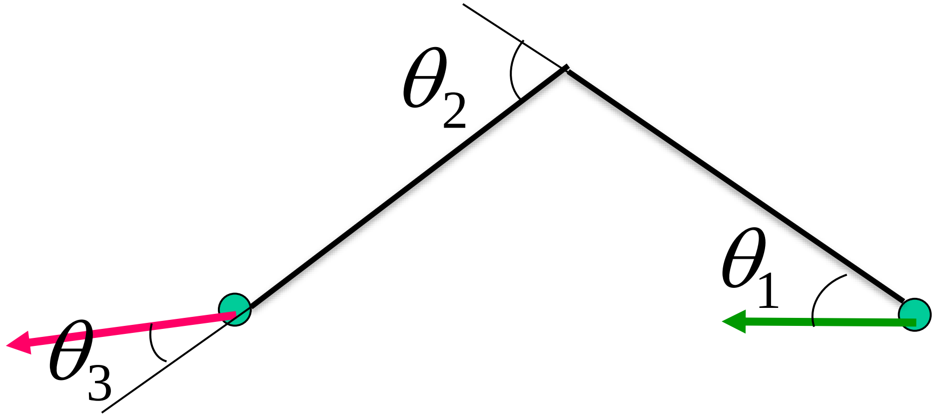
fixed



Enomoto-Itoh-Sinclair (2008)

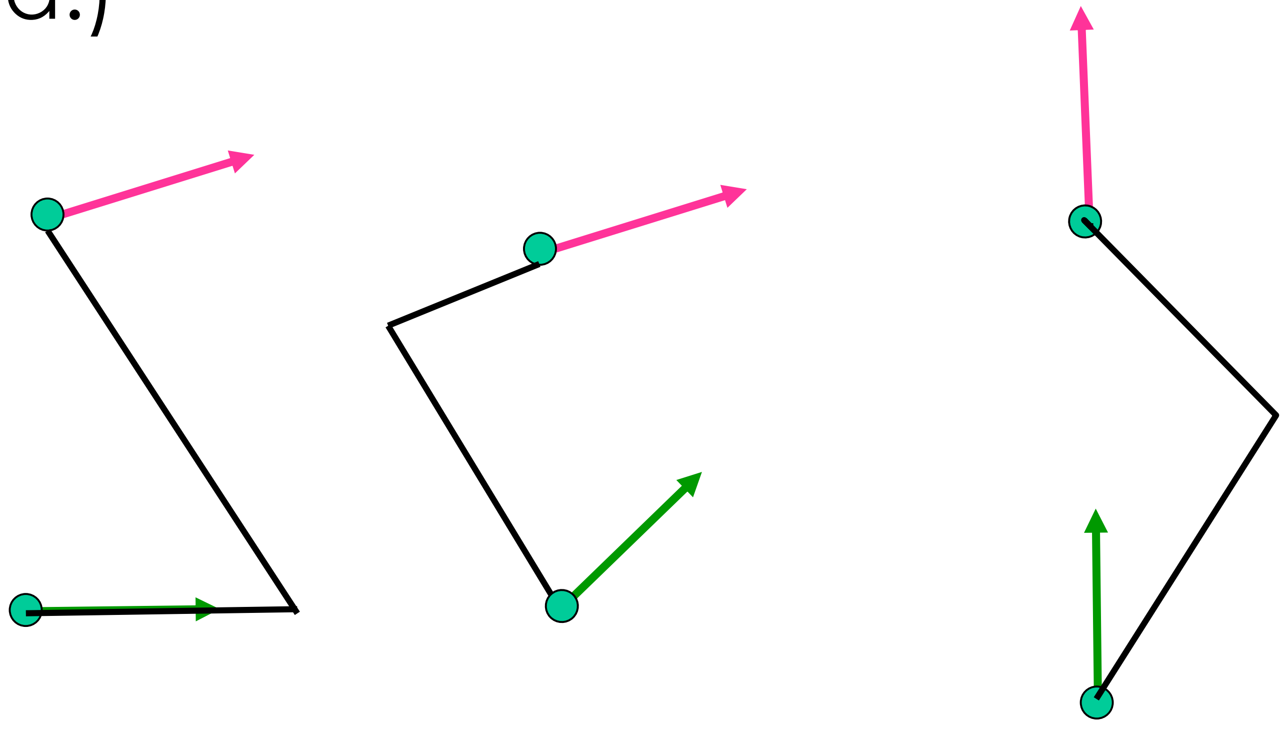
A piecewise linear curve with 2 edges

gives $\inf\{\int_\Sigma \kappa : \Sigma \in C(p,q,T_p,T_q,L)\}$ (as the limit) .



$$\inf\{\int_\Sigma \kappa : \Sigma \in C(p,q,T_p,T_q,L)\} = \theta_1 + \theta_2 + \theta_3$$

In most cases, it is the only curve that gives the infimum.
(most= except for the case when there exists a plane convex arc tangent to T_p at p, to T_q at q, including the case when the curve is closed.)

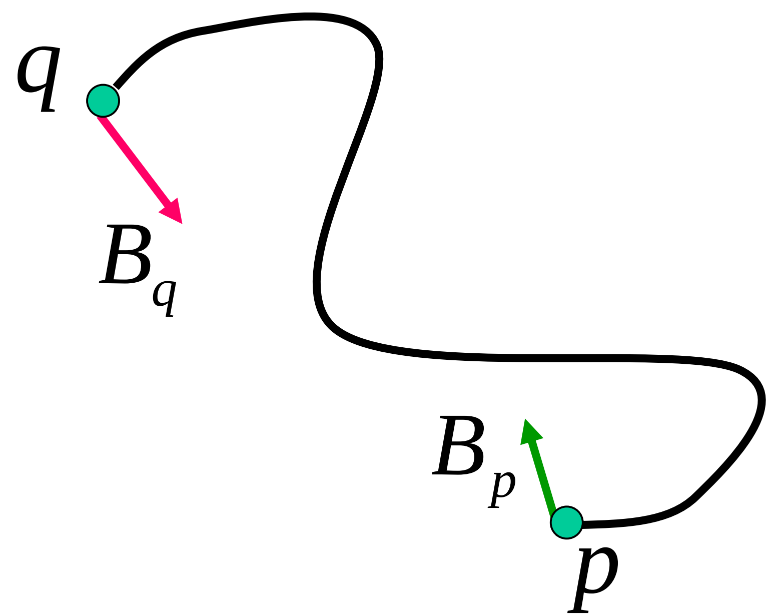


Enomoto-Itoh (2013) determines

$$\inf\{\int_\Sigma |\tau| : \Sigma \in C(p,q,B_p,B_q,L)\}$$

where

$C(p,q,B_p,B_q,L)$ is the set of all curves such as



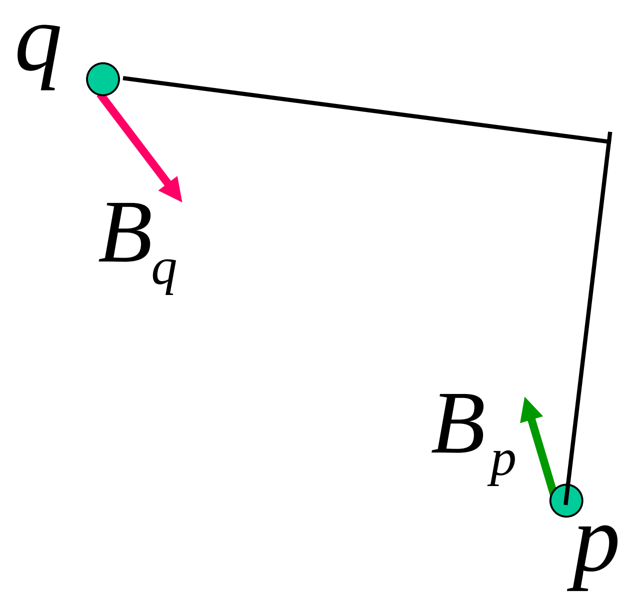
p, q
B_p, B_q
length L

} fixed

Enomoto-Itoh (2013)

$$\inf\{\int_\Sigma |\tau| : \Sigma \in C(p, q, B_p, B_q, L)\}$$

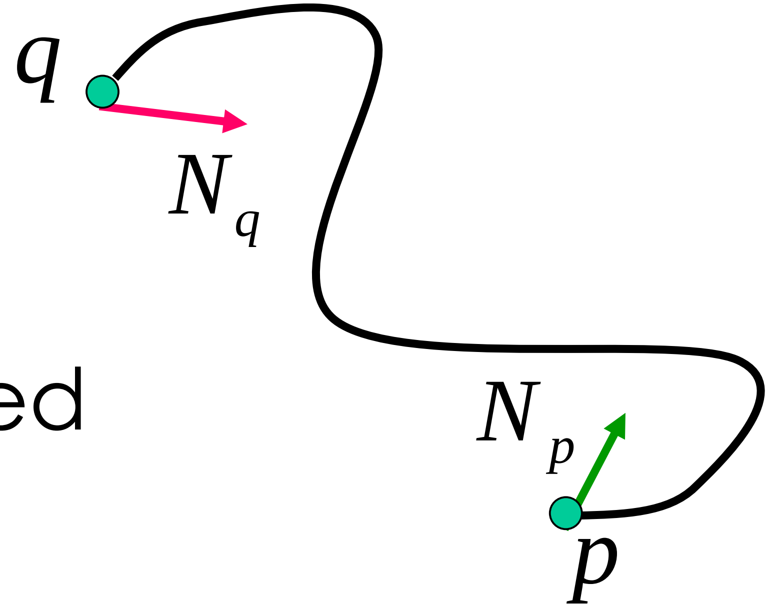
is attained by a curve (as the limit) shown below.



Enomoto-Itoh (2017)

$$\int_\Sigma \sqrt{\kappa^2+\tau^2} \geq \angle(N_p, N_q)$$

for $\Sigma \in C(p, q, N_p, N_q)$



$\left.\begin{matrix} p, q \\ N_p, N_q \end{matrix}\right\}$ fixed

Enomoto-Itoh (2017)

If $\angle(\overrightarrow{pq}, N_p) < \frac{\pi}{2}$ and $\angle(\overrightarrow{pq}, N_q) > \frac{\pi}{2}$,

there always exists $\Sigma \in C(p, q, N_p, N_q)$

such that

$$\int_{\Sigma} \sqrt{\kappa^2 + \tau^2} = \angle(N_p, N_q).$$

Such Σ is a subarc of a generalized helix.

Enomoto-Itoh (2017)

$$\inf\{\int_{\Sigma}\sqrt{\kappa^2+\tau^2} : \Sigma \in C(p,q,N_p,N_q,L)\}$$

is attained by a curve which is a union of plane curves and generalized helices.

For more details, please see

Illinois J. Math. 52 (2008).

for “total absolute curvature”

Illinois J. Math. 57 (2013)

for “total absolute torsion”

Geom. Dedicata (to appear)

for “total mixed curvature”