Homomorphisms of abelian p-groups produce p-automatic recurrent sequences

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Definition

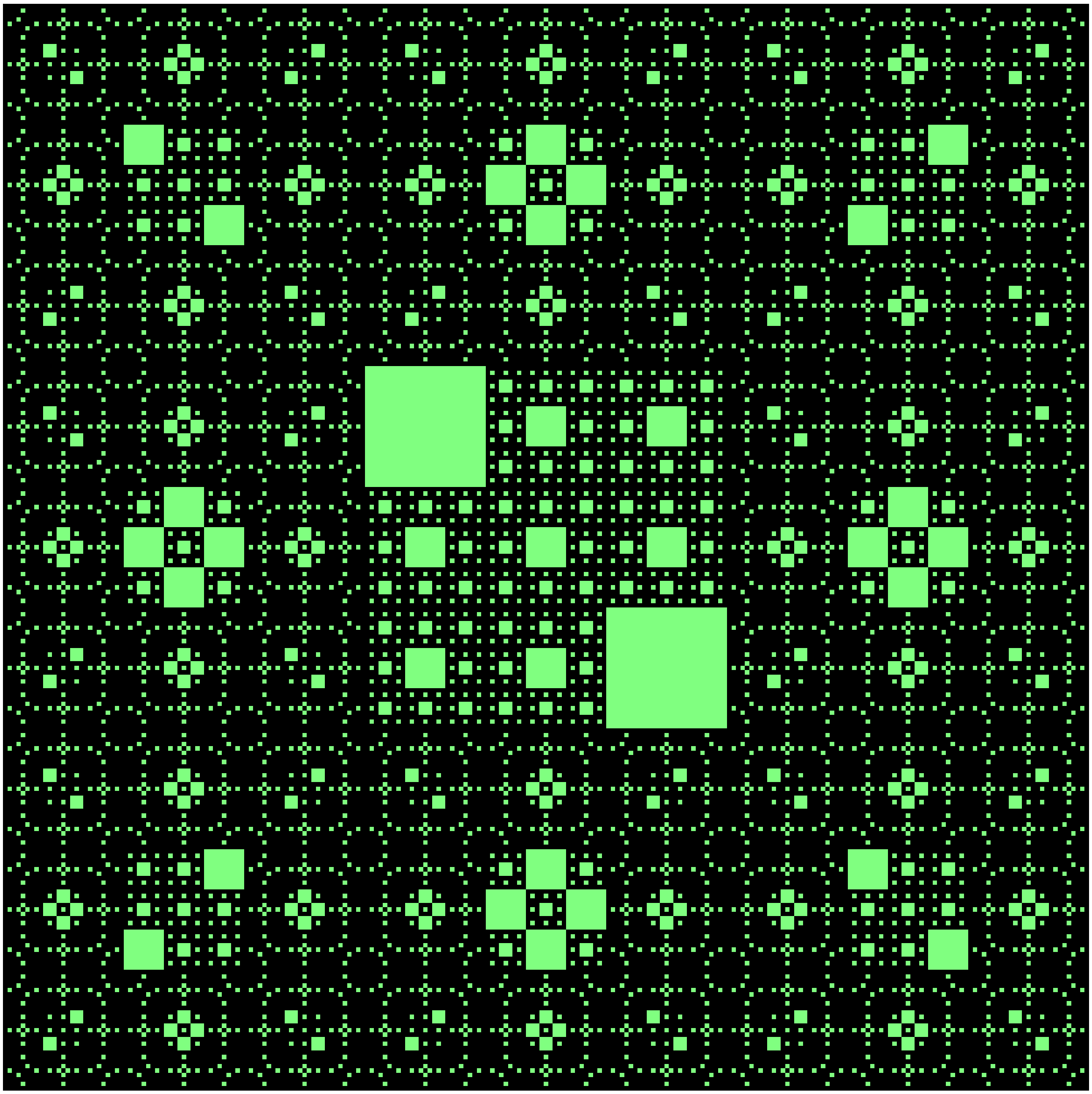
$(A, f, 1)$: A finite, $f : A^3 \to A$, $1 \in A$

Recurrent double sequence $(a(i,j))$:

- $\forall i \ \ \forall j \ \ a(i,0) = a(0,j) = 1$
- $i > 0 \ \wedge \ j > 0$:

$$a(i,j) = f(a(i-1,j), a(i-1,j-1), a(i,j-1))$$

Passoja-Lakhtakia Carpet modulo 9



$(\mathbb{Z}/9\mathbb{Z}, x + y + z, 1)$

Christol-Salon Theorem

Theorem 1 *$a : \mathbb{N}^n \to A$, A finite, p prime. Then the following are equivalent:*

1. *a is p-automatic.*

2. *$\exists$ $(B, \mathcal{E}, b_1, \Theta)$, $\Theta(b_i) \in B^{p^n}$, $\Theta(b_1)(\vec{0}) = b_1$, $b = \lim_{i \to \infty} \Theta^i(b_1)$, $g : B \to A$, $a = g(b)$.*

3. *$\forall$ embedding $\iota : A \to K$ in a sufficiently large finite field K of characteristic p, $S = \sum \iota(a(\vec{x}))\vec{X}^{\vec{x}}$ algebraic / $K(\vec{X})$.*

4. *$\exists$ embedding $\iota : A \to K$ in a sufficiently large finite field K of characteristic p; $S = \sum \iota(a(\vec{x}))\vec{X}^{\vec{x}}$ algebraic / $K(\vec{X})$.*

Denef-Lipshitz Theorem

Theorem 2 *p prime, $k > 0$, $a : \mathbb{N}^n \to \mathbb{Z}_p$, $\sum a(\vec{x})\vec{X}^{\vec{x}}$ algebraic / $\mathbb{Z}_p(\vec{X})$.*
Then $(a(\vec{x}) \mod p^k)$ p-automatic.

$\forall$ $b : \mathbb{N}^n \to \mathbb{Z}/p^k\mathbb{Z}$ *p-automatic* $\exists$ $a : \mathbb{N}^n \to \mathbb{Z}_p$
$\forall$ $\vec{x} \in \mathbb{N}^n$, $a(\vec{x}) \equiv b(\vec{x}) \mod p^k$ *and*
$\sum a(\vec{x})\vec{X}^{\vec{x}}$ *algebraic* / $\mathbb{Z}_p(\vec{X})$.

Main Result

Theorem 3 *p prime, $m \geq 1$,*

$H = \mathbb{Z}/p^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{d_s}\mathbb{Z}$,

$f : H^m \to H$ a shifted homomorphism.

$(H, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ n-dimensional recurrence,

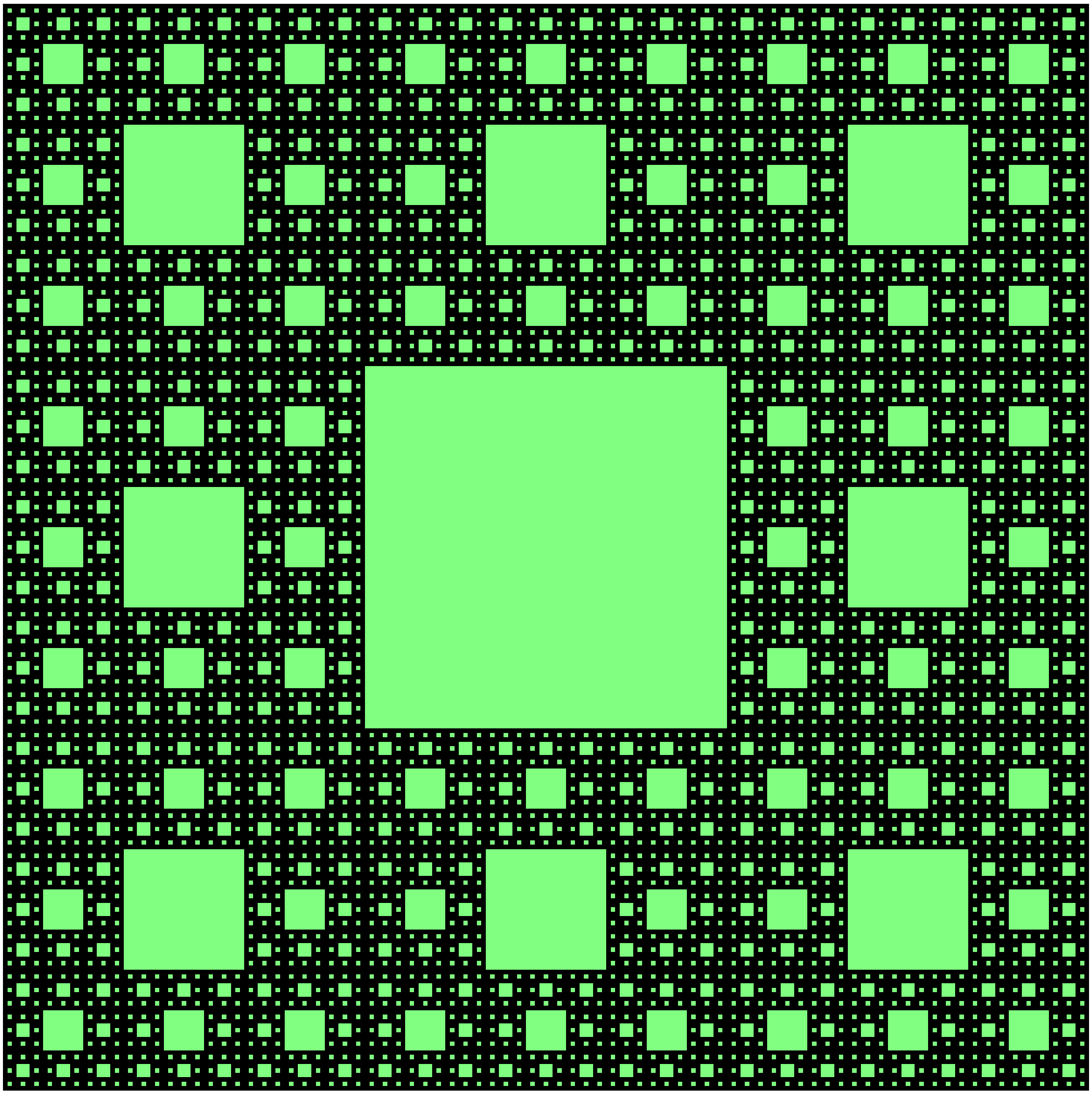
$c : C_P \to H$ satisfies $\forall\ i = 1, \ldots, n$, $\forall\ a \in \mathbb{N}$, if $(x_i = a) \cap \mathbb{N}^n \subset C_P$, $c\,|\,(x_i = a) \cap \mathbb{N}^n$ is p-automatic.

Then $(H, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ produces a p-automatic n-dimensional sequence.

Corollary 4 *The sequence can be defined by a substitution of type $p^a \to p^b$, $a < b$.*

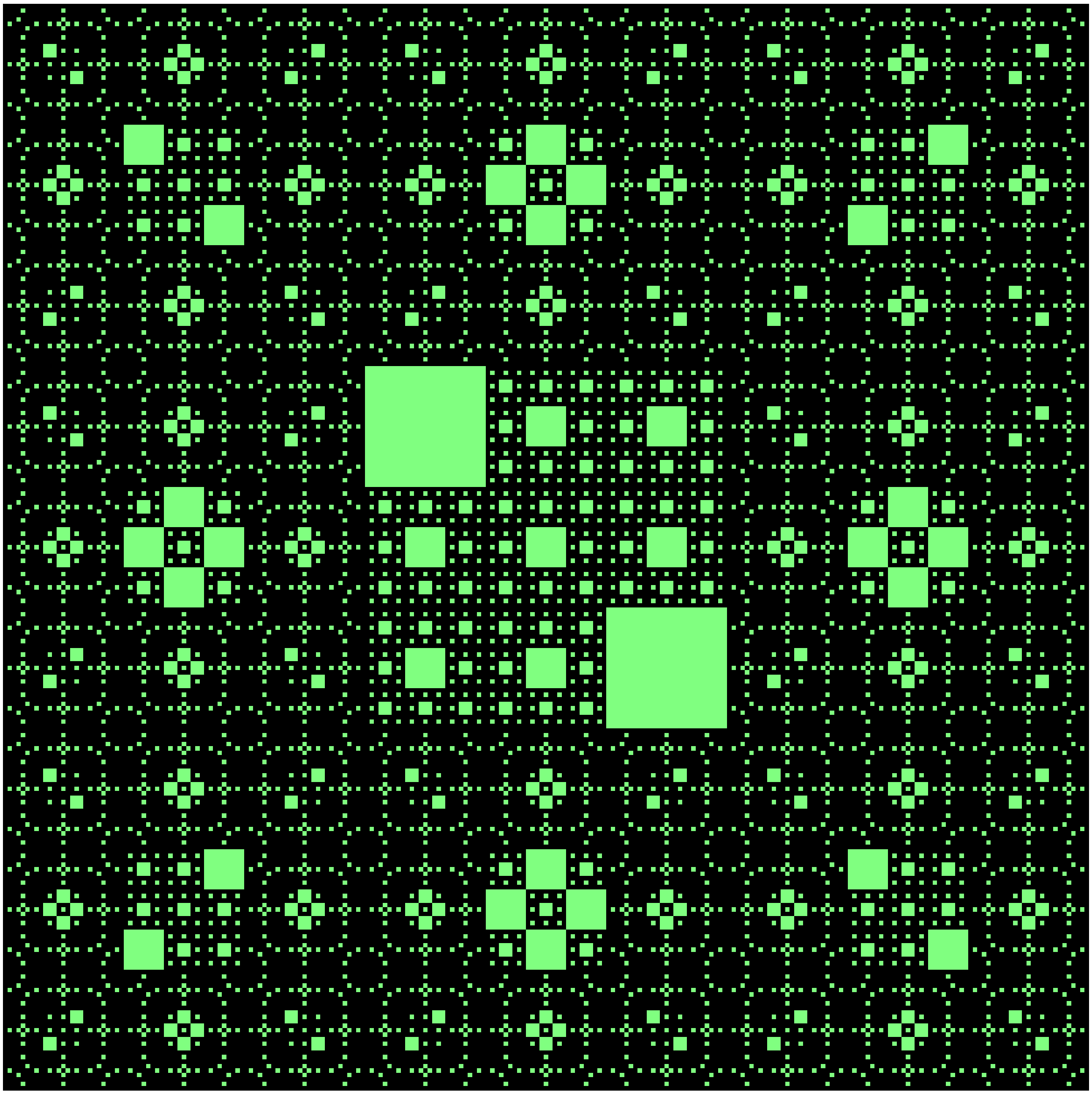
There is an algorithm able to find it.

Sierpinski's Carpet



$(\mathbb{Z}/3\mathbb{Z}, x + y + z, 1)$

Passoja-Lakhtakia Carpet modulo 9



$(\mathbb{Z}/9\mathbb{Z}, x + y + z, 1)$

Sierpinski's Carpet

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$F_n = F \otimes F \cdots \otimes F$$

$$s(F) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$S_n = s(F_n) = s(F) \otimes s(F) \cdots \otimes s(F)$$

$$L = \{(x, y) \in \mathbb{N}^2 \,|\, s(x, y) = 0\}$$

$$L = A^* \binom{1}{1} A^*$$

$$A = \{\binom{0}{0}, \binom{0}{1}, \binom{0}{2}, \binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{2}{0}, \binom{2}{1}, \binom{2}{2}\}$$

Passoja-Lakhtakia Carpet modulo 9

In this case the algorithm finds out a square substitution of type $3 \to 9$ with 57 rules.

$s : \mathbb{Z}/9\mathbb{Z} \to \{0, 1\}$ given by $s(0) = 0$ and $\forall x \neq 0$, $s(x) = 1$.

$(s(a(m, n)))$ is also 3-automatic.

There exist 3×3 matrices with $A \neq B$ and $\Sigma(A) \neq \Sigma(B)$ such that $s(A) = s(B)$ but still $s(\Sigma(A)) \neq s(\Sigma(B))$.

Happily $(a(m, n))$ is also given by another system of substitutions of type $9 \to 27$ which has also 57 rules.

By application of s, this system of substitutions collapses successfully on a consistent system of substitutions of type $9 \to 27$ with 8 rules.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\Sigma(A) = \begin{pmatrix} A & B & C \\ B & D & B \\ C & B & A \end{pmatrix} \quad \Sigma(B) = \begin{pmatrix} A & B & C \\ B & E & B \\ C & B & A \end{pmatrix}$$

$$\Sigma(C) = \begin{pmatrix} A & B & C \\ B & F & B \\ C & B & A \end{pmatrix}$$

$$\Sigma(D) = \begin{pmatrix} G & H & H \\ H & H & H \\ H & H & G \end{pmatrix} \quad \Sigma(E) = \begin{pmatrix} H & G & H \\ G & H & G \\ H & G & H \end{pmatrix}$$

$$\Sigma(F) = \begin{pmatrix} H & H & G \\ H & H & H \\ G & H & H \end{pmatrix}$$

$$\Sigma(G) = \begin{pmatrix} G & G & G \\ G & G & G \\ G & G & G \end{pmatrix} \quad \Sigma(H) = \begin{pmatrix} H & H & H \\ H & G & H \\ H & H & H \end{pmatrix}$$