About endpoints of convex surfaces. The 13th International Conference on Discrete Mathematics : Discrete Geometrie and Convex Bodies. Bucharest

Alain Rivière[1]

[1]Université de Picardie Jules Verne
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
CNRS-UMR 7352

September, 2017.

notations, definitions

- d is an integer ≥ 2 (often think $d = 2$!).

notations, definitions

- d is an integer ≥ 2 (often think $d = 2$!).
- $\mathcal{B}$ denotes the set of all *convexe bodies* of the Euclidean space $\mathbb{E}^{d+1}$, i. e. the compact convex subsets with non empty interior.
  It is endowed with the Pompeiu-Hausdorff metric $\mathrm{d_H}$.

notations, definitions

- d is an integer ≥ 2 (often think $d = 2$!).
- $\mathcal{B}$ denotes the set of all *convexe bodies* of the Euclidean space $\mathbb{E}^{d+1}$, i. e. the compact convex subsets with non empty interior.
  It is endowed with the Pompeiu-Hausdorff metric d_H.
- The boundary $\Sigma = \partial C$ of $C \in \mathcal{B}$ is a *convex surface*, it is endowed with its inner geodesic metric.

notations, definitions

- d is an integer ≥ 2 (often think $d = 2!$).
- $\mathcal{B}$ denotes the set of all *convexe bodies* of the Euclidean space $\mathbb{E}^{d+1}$, i. e. the compact convex subsets with non empty interior.
  It is endowed with the Pompeiu-Hausdorff metric $\mathrm{d_H}$.
- The boundary $\Sigma = \partial C$ of $C \in \mathcal{B}$ is a *convex surface*, it is endowed with its inner geodesic metric.
- $\mathcal{E}_C$ or $\mathcal{E}_\Sigma$ denotes the set of all *endpoints* of Σ, that is the points which are not in the interior of some shorter path in Σ.

Some Examples

- $\mathcal{E}_\Sigma = \emptyset$ if Σ is of class C^2 (no endpoints in Σ).

Some Examples

- $\mathcal{E}_\Sigma = \emptyset$ if Σ is of class C^2 (no endpoints in Σ).
- $\mathcal{E}_\Sigma$ is always negligible : $\mathcal{H}^d(\mathcal{E}_\Sigma) = 0$. Otsu and Shioya 1994.

Some Examples

- $\mathcal{E}_\Sigma = \emptyset$ if Σ is of class C^2 (no endpoints in Σ).
- $\mathcal{E}_\Sigma$ is always negligible : $\mathcal{H}^d(\mathcal{E}_\Sigma) = 0$. Otsu and Shioya 1994.
- When $a \in \Sigma$ is *conical*, that is when the tangent cone $\mathrm{T}_a\Sigma$ contains no line, then $a \in \mathcal{E}_\Sigma$.
  The set of conical points of Σ is always countable.

Some Examples

- $\mathcal{E}_\Sigma = \emptyset$ if Σ is of class C^2 (no endpoints in Σ).
- $\mathcal{E}_\Sigma$ is always negligible : $\mathcal{H}^d(\mathcal{E}_\Sigma) = 0$. Otsu and Shioya 1994.
- When $a \in \Sigma$ is *conical*, that is when the tangent cone $\mathrm{T}_a\Sigma$ contains no line, then $a \in \mathcal{E}_\Sigma$.
  The set of conical points of Σ is always countable.
- If $\mathbb{E}^{d+1} = E \oplus F$ with $\dim F = 2$, $r > 0$ and $C = \mathrm{conv}(S_E \cup rS_F)$, then $S_E \subset \mathcal{E}_\Sigma$.
  Here $\dim_{\mathrm{H}} \mathcal{E}_\Sigma = d-2$ and $\mathcal{H}^{d-2}(\mathcal{E}_\Sigma) > 0$.

Some Examples

- $\mathcal{E}_\Sigma = \emptyset$ if Σ is of class C^2 (no endpoints in Σ).
- $\mathcal{E}_\Sigma$ is always negligible : $\mathcal{H}^d(\mathcal{E}_\Sigma) = 0$. Otsu and Shioya 1994.
- When $a \in \Sigma$ is *conical*, that is when the tangent cone $\mathrm{T}_a\Sigma$ contains no line, then $a \in \mathcal{E}_\Sigma$.
  The set of conical points of Σ is always countable.
- If $\mathbb{E}^{d+1} = E \oplus F$ with $\dim F = 2$, $r > 0$ and $C = \mathrm{conv}(S_E \cup rS_F)$, then $S_E \subset \mathcal{E}_\Sigma$.
  Here $\dim_{\mathrm{H}} \mathcal{E}_\Sigma = d-2$ and $\mathcal{H}^{d-2}(\mathcal{E}_\Sigma) > 0$.
- $a \in \Sigma$ is said *regular* when $\mathrm{T}_a\Sigma$ is isometric to $\mathbb{E}^d$.
  The set $\mathcal{R}_\Sigma$ of regular points is always strongly convex in Σ.
  Petrunyn 1998 (Milka 1983 enough for us).
  This is used to check that $S_E \subset \mathcal{E}_\Sigma$ in the above example.

Typical case

- We said that *most*, or *typical*, elements of a Baire space share a property when the exceptional set is meager (i.e. included in the union of countably many closed sets with empty interiors).

Typical case

- We said that *most*, or *typical*, elements of a Baire space share a property when the exceptional set is meager (i.e. included in the union of countably many closed sets with empty interiors).
- $\dim_H$ denotes the Hausdorff dimension and $\mathcal{H}^s$ the Hausdorff measure in dimension s.

Typical case

- We said that *most*, or *typical*, elements of a Baire space share a property when the exceptional set is meager (i.e. included in the union of countably many closed sets with empty interiors).
- $\dim_H$ denotes the Hausdorff dimension and $\mathcal{H}^s$ the Hausdorff measure in dimension s.
- For most $C \in \mathcal{B}$, most elements of ∂C are endpoints. Zamfirescu 82.

Typical case

- We said that *most*, or *typical*, elements of a Baire space share a property when the exceptional set is meager (i.e. included in the union of countably many closed sets with empty interiors).
- $\dim_H$ denotes the Hausdorff dimension and $\mathcal{H}^s$ the Hausdorff measure in dimension s.
- For most $C \in \mathcal{B}$, most elements of ∂C are endpoints. Zamfirescu 82.
- Most $C \in \mathcal{B}$ satisfy $\dim_{\mathrm{H}} \mathcal{E}_C \geq \max(d-2, d/3)$. Riv 2015/2014.

Typical case

- We said that *most*, or *typical*, elements of a Baire space share a property when the exceptional set is meager (i.e. included in the union of countably many closed sets with empty interiors).
- $\dim_H$ denotes the Hausdorff dimension and $\mathcal{H}^s$ the Hausdorff measure in dimension s.
- For most $C \in \mathcal{B}$, most elements of ∂C are endpoints. Zamfirescu 82.
- Most $C \in \mathcal{B}$ satisfy $\dim_{\mathrm{H}} \mathcal{E}_C \geq \max(d-2, d/3)$. Riv 2015/2014.
- Remarks
  - $d-2 = d/3 \Leftrightarrow d = 3$ and $d-2 > d/3 \Leftrightarrow d \geq 4$.
  - Proof of $\dim_{\mathrm{H}} \mathcal{E}_C \geq d-2$ uses exple $\mathrm{conv}(S_E \cup rSF)$.
  - Proof of $\dim_{\mathrm{H}} \mathcal{E}_C \geq d/3$ uses classical conical points.

Some questions

- Does it exist a convex body $C \in \mathcal{B}$ satisfying $\dim_{\mathrm{H}} \mathcal{E}_C > \max(d-2, d/3)$
  When $d = 2$ can we have $\mathcal{H}^{2/3}(\mathcal{E}_C) > 0$?

Some questions

- Does it exist a convex body $C \in \mathcal{B}$ satisfying $\dim_{\mathrm{H}} \mathcal{E}_C > \max(d-2, d/3)$
  When $d = 2$ can we have $\mathcal{H}^{2/3}(\mathcal{E}_C) > 0$?
- Are there convex bodies $C_0, C \in \mathcal{B}$ and $a \in \partial C \cap \partial C_0$ such that $C_0 \subset C$, $a \in \mathcal{E}_C$ but $a \notin \mathcal{E}_{C_0}$. (yes known ?)

Some questions

- Does it exist a convex body $C \in \mathcal{B}$ satisfying $\dim_{\mathrm{H}} \mathcal{E}_C > \max(d-2, d/3)$
  When $d = 2$ can we have $\mathcal{H}^{2/3}(\mathcal{E}_C) > 0$?
- Are there convex bodies $C_0, C \in \mathcal{B}$ and $a \in \partial C \cap \partial C_0$ such that $C_0 \subset C$, $a \in \mathcal{E}_C$ but $a \notin \mathcal{E}_{C_0}$. (yes known ?)
- Can we also ask C_0 to be a Euclidean ball ?

Some questions

- Does it exist a convex body $C \in \mathcal{B}$ satisfying $\dim_{\mathrm{H}} \mathcal{E}_C > \max(d-2, d/3)$
  When $d=2$ can we have $\mathcal{H}^{2/3}(\mathcal{E}_C) > 0$?
- Are there convex bodies $C_0, C \in \mathcal{B}$ and $a \in \partial C \cap \partial C_0$ such that $C_0 \subset C$, $a \in \mathcal{E}_C$ but $a \notin \mathcal{E}_{C_0}$. (yes known ?)
- Can we also ask C_0 to be a Euclidean ball ?
- Can we also ask C_0 to be *flat* at a ($a \in U_0(\partial C_0)$), that is containing for every $R > 0$ a neighborhood of a in a Euclidean ball B of radius R?

Some questions

- Does it exist a convex body $C \in \mathcal{B}$ satisfying $\dim_{\mathrm{H}} \mathcal{E}_C > \max(d-2, d/3)$
  When $d = 2$ can we have $\mathcal{H}^{2/3}(\mathcal{E}_C) > 0$?
- Are there convex bodies $C_0, C \in \mathcal{B}$ and $a \in \partial C \cap \partial C_0$ such that $C_0 \subset C$, $a \in \mathcal{E}_C$ but $a \notin \mathcal{E}_{C_0}$. (yes known ?)
- Can we also ask C_0 to be a Euclidean ball ?
- Can we also ask C_0 to be *flat* at a ($a \in U_0(\partial C_0)$), that is containing for every $R > 0$ a neighborhood of a in a Euclidean ball B of radius R?
- Can we also ask that $d = 2$ and C to be of revolution around an axe containing a ? (no known ?)

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $C \in \mathcal{B}$ and $\varepsilon > 0$, let $\mathcal{M}_{C,\varepsilon}$ be the set of all the points of ∂C which are the middle of some shorter path of ∂C with length 2ε, and $\mathcal{E}_{C,\varepsilon} = \partial C \setminus \mathcal{M}_{C,\varepsilon}$.
  Then we have $\mathcal{E}_C = \bigcap_{\varepsilon>0} \mathcal{E}_{C,\varepsilon}$

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $C \in \mathcal{B}$ and $\varepsilon > 0$, let $\mathcal{M}_{C,\varepsilon}$ be the set of all the points of ∂C which are the middle of some shorter path of ∂C with length 2ε, and $\mathcal{E}_{C,\varepsilon} = \partial C \setminus \mathcal{M}_{C,\varepsilon}$.
  Then we have $\mathcal{E}_C = \bigcap_{\varepsilon>0} \mathcal{E}_{C,\varepsilon}$
- We can restrict our study to $\mathcal{B}_0 = \{C \in \mathcal{B} \mid 0 \in \mathrm{int}\, C\}$

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $C \in \mathcal{B}$ and $\varepsilon > 0$, let $\mathcal{M}_{C,\varepsilon}$ be the set of all the points of ∂C which are the middle of some shorter path of ∂C with length 2ε, and $\mathcal{E}_{C,\varepsilon} = \partial C \setminus \mathcal{M}_{C,\varepsilon}$.
  Then we have $\mathcal{E}_C = \bigcap_{\varepsilon > 0} \mathcal{E}_{C,\varepsilon}$
- We can restrict our study to $\mathcal{B}_0 = \{C \in \mathcal{B} \mid 0 \in \mathrm{int}\, C\}$
- For $\|x\| = 1$ we set $\{\Phi_C(x)\} = \mathbb{R}_+ x \cap \partial C$.

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $C \in \mathcal{B}$ and $\varepsilon > 0$, let $\mathcal{M}_{C,\varepsilon}$ be the set of all the points of ∂C which are the middle of some shorter path of ∂C with length 2ε, and $\mathcal{E}_{C,\varepsilon} = \partial C \setminus \mathcal{M}_{C,\varepsilon}$.
  Then we have $\mathcal{E}_C = \bigcap_{\varepsilon>0} \mathcal{E}_{C,\varepsilon}$
- We can restrict our study to $\mathcal{B}_0 = \{C \in \mathcal{B} \mid 0 \in \operatorname{int} C\}$
- For $\|x\| = 1$ we set $\{\Phi_C(x)\} = \mathbb{R}_+ x \cap \partial C$.
- We look for a compact set K of the unit sphere, with $\mathrm{dim_H}\, K = d-2$, and such that the following G_δ is dense in $\mathcal{B}_0$:
  $$G_K = \{C \in \mathcal{B}_0 \mid \Phi_C(K) \subset \mathcal{E}_C\} = \bigcap_{\varepsilon>0} G_{K,\varepsilon}$$
  where $G_{K,\varepsilon} = \{C \in \mathcal{B}_0 \mid \Phi_C(K) \subset \mathcal{E}_{C,\varepsilon}\}$.

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $\mathbb{E}^{d+1} = E \oplus F$ like in the exemple, we can find $K \subset S_E$ with $\dim_{\mathrm{H}} K = d-2$, and with K *strongly radially porous*. This means that for each $\varepsilon > 0$ and $n \geq 1$, K has a finite covering by pairwise disjointed balls $B_i(c_i, r_i)$ and such that $K \cap B(a_i, r_i) \subset B(a_i, r_i/n)$

About the proof of $\dim_H \mathcal{E}_C \geq d-2$

- For $\mathbb{E}^{d+1} = E \oplus F$ like in the exemple, we can find $K \subset S_E$ with $\dim_{\mathrm{H}} K = d-2$, and with K *strongly radially porous*. This means that for each $\varepsilon > 0$ and $n \geq 1$, K has a finite covering by pairwise disjointed balls $B_i(c_i, r_i)$ and such that $K \cap B(a_i, r_i) \subset B(a_i, r_i/n)$
- Using our exemple and the strong porosity, we can check the wanted density.
  In this point, Riv2015 is rather clumsy !

About the proof of density of G_K

- First we have a dense set of smooth $C \in \mathcal{B}_0$ such that for some $0 < r < R$ and all $a \in \partial C$, there are closed balls containing a in their boundary spheres, and such that $B(c, r) \subset C \subset B(c', R)$.

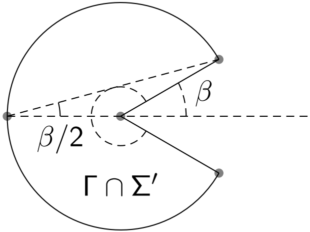
About the proof of density of G_K

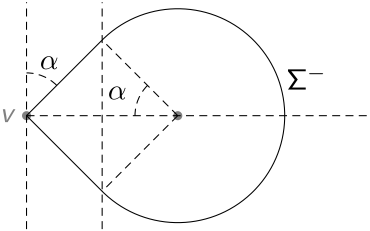
- First we have a dense set of smooth $C \in \mathcal{B}_0$ such that for some $0 < r < R$ and all $a \in \partial C$, there are closed balls containing a in their boundary spheres, and such that $B(c,r) \subset C \subset B(c',R)$.
- Given such a C and $\varepsilon > 0$, we find a finite family of half spaces H_i such that if $C' = C \cap \bigcap H_i$, then $d_H(C,C') < \varepsilon$ and $C' \cap \Phi_C(K) = \emptyset$, because of the strong porousity of K.

About the proof of density of G_K

- First we have a dense set of smooth $C \in \mathcal{B}_0$ such that for some $0 < r < R$ and all $a \in \partial C$, there are closed balls containing a in their boundary spheres, and such that $B(c,r) \subset C \subset B(c',R)$.
- Given such a C and $\varepsilon > 0$, we find a finite family of half spaces H_i such that if $C' = C \cap \bigcap H_i$, then $d_H(C, C') < \varepsilon$ and $C' \cap \Phi_C(K) = \emptyset$, because of the strong porousity of K.
- Then we get $C'' \in G_K$ with $d_H(C, C'') < \varepsilon$ by substituting to each H_i some C_i congruent to our exemple associated with $\mathbb{E}^{d+1} = E \oplus F_i$.

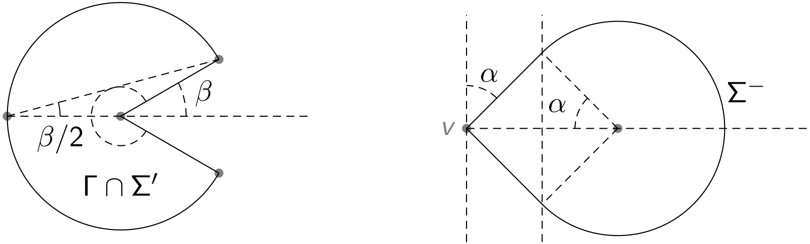
Why $d/3$?





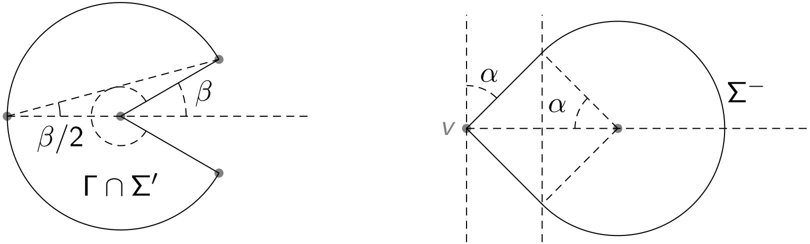
- Our modified sphere is the boundary Σ' of the largest convex set among all those whose boundaries contain the truncated sphere Σ^- (of radius 1).

Why $d/3$?



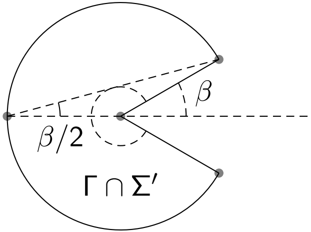
- Our modified sphere is the boundary Σ' of the largest convex set among all those whose boundaries contain the truncated sphere Σ^- (of radius 1).
- For a small α, the smallest possible distance r_α from the vertex v to a shorter path γ, in Σ' and between points of Σ^- satisfies $r_\alpha \sim \frac{\pi}{4}\alpha^3$ and $\varepsilon_a = 4\tan\alpha \sim 4\alpha$

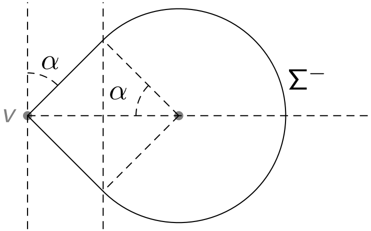
Why $d/3$?



- Because of this we choose a function $h > 0$ with $h(t) = \mathrm{o}(t^3)$ (like $t^3/|\ln t|$ near zero), and then K *h-radially porous*, that is: for all $x \in K$, there is a sequence of balls such that for each n we have $\mathrm{B}_K(x, r_n) \subset \mathrm{B}_K(x, h(r_n))$, and with the radius sequence (r_n) decreasing of null limit.

Why $d/3$?





- Because of this we choose a function $h > 0$ with $h(t) = \mathrm{o}(t^3)$ (like $t^3/|\ln t|$ near zero), and then K *h-radially porous*, that is: for all $x \in K$, there is a sequence of balls such that for each n we have $\mathrm{B}_K(x, r_n) \subset \mathrm{B}_K(x, h(r_n))$, and with the radius sequence (r_n) decreasing of null limit.
- We can also ask $\dim_{\mathrm{H}} K = d/3$.