Generic Properties of Lengths Spaces

Work in progress

Joël Rouyer

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Generic Lengths Spaces
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Introduction
Meaning of *generic*
Length spaces
Tangency
Cusps
Dimension
Farthest points

Imbedded in $\mathbb{R}^n$

Convex (hyper)surfaces

Abstract

- 1959: V. Klee, A generic C.S. is C^1 and strictly convex.
- 1977: P. Gruber, . . . and not C^2.
- 1979: R. Schneider, 1980,1988: T. Zamfirescu, 2012: K. Adiprasito and T. Zamfirescu, 2015: Schneider 2015. Study of directional curvature. (extrinsic property)
- 1982: T. Zamfirescu, A generic point is an *endpoint*.
- 1995: T. Zamfirescu, A generic point has a single farthest point, to whom it is joined by exactly 3 segments.
- 1988,91: P. Gruber, A generic C.S. has no (simple) closed geodesic.

Baire Categories in Geometry

Imbedded in $\mathbb{R}^n$ | Convex (hyper)surfaces | Continua | Compacta

Abstract

- 1988: J. A. Wieaker, Most compacta are homeomorphic to a cantor set.
- 1989: P. Gruber, generic dimension of compacta and continua.
- 1997: A. V. Kuz'minykh, Most compacta are totally anisometric : $d(a,b) = d(a',b') > 0 \Rightarrow \{a,b\} = \{a',b'\}$
- 1989–2005: Results on the embedding: E.S. De Blasi, P. Gruber, J. Myjak & R. Rudnick, J. A. Wieaker, T. Zamfirescu, N.V. Zhivkov.

Baire Categories in Geometry

Imbedded in $\mathbb{R}^n$	Convex (hyper)surfaces	Continua	Compacta
Abstract			compact metric spaces

- 2011: J. Rouyer.

Imbedded in $\mathbb{R}^n$	Convex (hyper)surfaces	Continua	Compacta
Abstract	Alexandrov Surface		compact metric spaces

- 2012: K. Adiprazito and T. Zamfirecu, Most points are endpoints.
- 2015: J.-I. Itoh, J. R. , C. Vîlcu, No conical points, but no Gaussian curvature.
- 2016: J. R. , C. Vîlcu, No/infinitely many simple closed geodesic, depending on the curvature bound and the connected component of the space.

Baire Categories in Geometry

Imbedded in $\mathbb{R}^n$	Convex (hyper)surfaces	Continua	Compacta
Abstract	Alexandrov Surface	Length spaces	compact metric spaces

Work in progress.

Contents

Baire Categories

Let X be a topological space.

- $R \subset X$ is rare or nowhere dense iff $\mathrm{int}\left(\mathrm{cl}\left(R\right)\right) = \emptyset$.
- $M \subset X$ is meager or of first category iff it is included in a countable union of rare sets.
- X is a Baire space iff any meager set have empty interior.
- The Baire's theorem states that any complete metric space is a Baire space.

Convention

We say that

- most $x \in X$ are ...
- or that a generic $x \in X$ is ...

to express that the set of those $x \in X$ which are not ... form a meager set in X.

Length spaces I

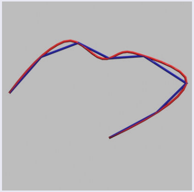
Preliminary Remark

During this talk, a *length space* is supposed to be compact. (unlike most authors)

Definition

Let $\gamma : [a, b] \to X$*. The length of* γ *is*

$$L(\gamma) = \sup_{(t_0,\dots,t_n)\in S} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$



where
$S = \{(t_0, \dots, t_n) \in \mathbb{R}^n \,|\, n \in \mathbb{N}, a = t_0 < t_1 < \dots < t_n = b\}$

Length spaces II

Theorem

Let X be a compact metric space. Denote by $\Gamma(x,y)$ the set of curves from x to y. The following statements are equivalent:

- **existence of segments:** $\forall x,y\in X \exists\gamma\in\Gamma(x,y)$ *s.t.* $d(x,y)=L(\gamma)$.
- **existence of midpoints:** $\forall x,y\in X \exists z\in X$ *s.t.*

$$d(x,z)=d(z,y)=\frac{1}{2}d(x,y).$$

- **intrinsic metric:** $\forall x,y\in X, d(x,y)=\inf_{\gamma\in\Gamma(x,y)}L(\gamma)$.

Definition

A compact metric space satisfying these properties is called a (compact) length space.
The set of length spaces is denoted by $\mathcal{L}$.

Length spaces III

Examples

(Counter)example

$\left\{x \in \mathbb{R}^2 \,\middle|\, \|x\| = 1\right\}$ endowed with the metric $d_0(x, y) = \|x - y\|$ is not a length space, but endowed with $d_1(x, y) = \arccos \langle x, y \rangle$ is a length space.

Example

$\mathbb{R}^2/\mathbb{Z}^2$ *endowed with*

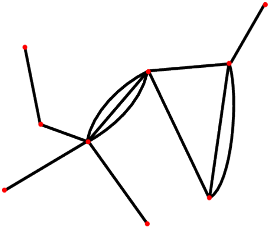
$$d\left((x_1, y_1), (x_2, y_2)\right) = \min\left(|x_1 - x_2|, 1 - |x_1 - x_2|\right) + \min\left(|y_1 - y_2|, 1 - |y_1 - Y_2|\right).$$

Example

More generally, any reversible (compact) Finsler manifold, and so any (compact) Riemannian manifold.

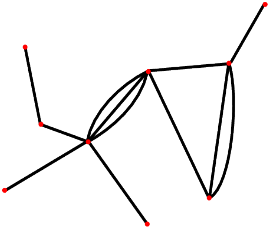
Length space IV

Finite metric graphs



- Start with a finite combinatorial graph: (V, E), $E \subset \mathcal{P}_2(V) \times \mathbb{N}$
- Assign lengths to edges: choose $\lambda : E \to]0, +\infty[$

Length space IV
Finite metric graphs



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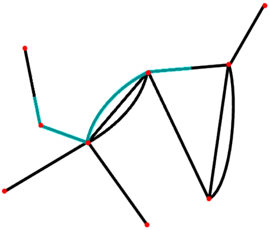
- The set of points is $G = V \cup]0, 1[\times E$

Length space IV
Finite metric graphs



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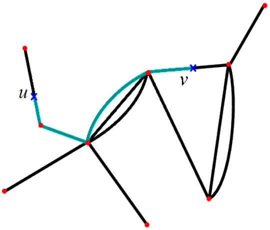
- Assign lengths to edges: choose $\lambda : E \to]0, +\infty[$
- The set of points is $G = V \cup]0, 1[\times E$
- Define the length of a simple path: $\ell_G(\gamma) = \sum_{\Delta \in E} \lambda(\delta) \ell_{]0,1[}(\gamma_E)$, where $\gamma_E = \gamma \cap E$.



- Start with a finite combinatorial graph: (V, E), $E \subset \mathcal{P}_2(V) \times \mathbb{N}$
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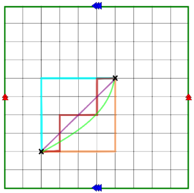
- The set of points is $G = V \cup]0,1[\times E$
- Define the length of a simple path: $\ell_G(\gamma) = \sum_{\Delta \in E} \lambda(\delta) \ell_{]0,1[}(\gamma_E)$, where $\gamma_E = \gamma \cap E$.
- Define $d(u, v) = inf_\gamma \ell(\gamma)$, where the infimum is taken over all the simple paths γ from u to v.

Any finite metric graph is a length space,
We denote by $\mathcal{G}$ the set of finite metric graphs.

Definition

A geodesic is a path which is locally a segment.



Finsler torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with $\|\ \|_1$.

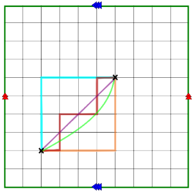
- Geodesics may branch.
- No injectivity radius.

Length space V

Geodesics in length spaces

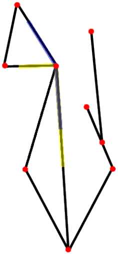
Definition

A geodesic is a path which is locally a segment.



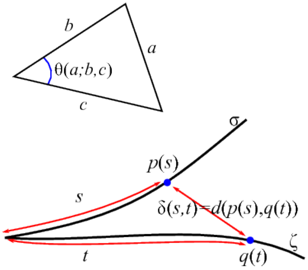
Finsler torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with $|| \; ||_1$.

A metric graph.



- Geodesics may branch.
- No injectivity radius.
- Geodesics may stop.
- Existence of *endpoints*.

Length space VI
Angles in length spaces 1



One can define *lower* and *upper angles* between segments.

$$\underline{\measuredangle}(\sigma,\zeta) = \liminf_{s,t\to 0} \theta(\delta(s,t); s,t)$$

$$\overline{\measuredangle}(\sigma,\zeta) = \limsup_{s,t\to 0} \theta(\delta(s,t); s,t)$$

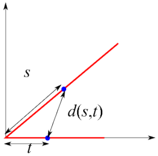
- When the two angles agree, we say that the segments make a well-defined angle.
- In Alexandrov spaces, all angle are well-defined.
- In Riemannian manifold, this notion of angles is equivalent to the usual one.

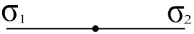
Length space VII
Angles in length spaces 2

For instance, in $(\mathbb{R}^2/\mathbb{Z}^2, \| \|_1)$, the angles between $\sigma : Y = 0$ and $\zeta : Y = aX$ are

$\measuredangle(\sigma, \zeta) = 0$,

$$\overline{\measuredangle}(\sigma, \zeta) = \arccos\left(\frac{1-a}{1+a}\right).$$





In any length space, if σ_1 and σ_2 are two parts of a same segment then

$$\measuredangle(\sigma_1, \sigma_1) = \overline{\measuredangle}(\sigma_1, \sigma_1) \quad = 0,$$
$$\measuredangle(\sigma_1, \sigma_2) = \overline{\measuredangle}(\sigma_1, \sigma_2) \quad = \pi.$$

Space of length spaces I
The Gromov-Hausdorff metric

Notation. Let Z be a metric space.

- $\mathcal{K}(Z)$ denotes the set of nonempty compact subsets of Z
- for $A \in \mathcal{K}(Z)$ and $\rho \in \mathbb{R}_+$,
  $A + \rho \overset{\text{def}}{=} \{y \in Z \,|\exists x \in A \text{ s.t. } d(x,y) \leq \rho\}$

Definition

- *for $A, B \in \mathcal{K}(Z)$, the Pompeiu-Hausdorff distance is:*

$$d_{PH}^{Z}(A,B) = \inf\{\varepsilon \,| A \subset B + \varepsilon, B \subset A + \varepsilon\}$$

- *for X, Y compact metric spaces, the Gromov-Hausdorff distance is:* $d_{GH}(X,Y) = \inf\limits_{Z,f,g} d_{PH}^{Z}(f(X), g(Y)),$

  where the infimum is taken over all metric spaces Z and all isometric embeddings $f : X \to Z$, $g : Y \to Z$.

Space of Length spaces II

Theorem

The set $\mathcal{GH}$ of all compact metric spaces, up to isometries, endowed with d_{GH} is a complete metric space.

Theorem

$\mathcal{L}$ is closed in $\mathcal{GH}$ and so, is a complete metric space.

Theorem

$\mathcal{G}$ is dense in $\mathcal{L}$.

Theorem

The set of Riemannian surfaces is dense in $\mathcal{L}$.

Definition of f-tangency I

motivation

- No differential structure
- No angle

Definition

A comparison function is smooth increasing function $f :]0, \infty[\to]0, \infty[$ *s.t.* $f(x) = o(x)$ *when* x *goes to* 0 .

Notation

The set of segments emanating from a point x *will be denoted by* Σ_x.

Definition of f-tangency II

Definition

Let f be a comparison function, $\sigma, \gamma \in \Sigma_x$

1. *σ, γ are said to be weakly f-tangent if there exists a sequence of positive numbers t_n tending to 0 such that $\sigma(t_n)\gamma(t_n) < f(t_n)$.*
2. *σ, γ are said to be f-tangent if there exists $\tau > 0$ such that for any $t \in [0, \tau]$ $\sigma(t)\gamma(t) \leq f(t)$.*
3. *σ, γ are said to be strongly f-tangent if there exists $\tau > 0$ such that for any s, $t \in [0, \tau]$ $\sigma(t)\gamma(s) \leq |s - t| + f(\min(s, t))$.*

f-tangency and angles

Proposition

For any comparison function f,

1. *If $\sigma,\gamma \in \Sigma_x$ are strongly f-tangent then $\overline{\measuredangle}(\sigma,\gamma) = 0$.*
2. *If $\sigma,\gamma \in \Sigma_x$ are weakly f-tangent then $\underline{\measuredangle}(\sigma,\gamma) = 0$.*

Theorem

Let f be a comparison function. For most $X \in \mathcal{L}$, if $\sigma, \gamma \in \Sigma_x$ are f-tangent, then either $\sigma \subset \gamma$ or $\gamma \subset \sigma$.

Corollary

In a generic length space geodesics do not bifurcate.

A generic result about angles

Theorem

In a generic length space, at any point x, any two segments $\sigma, \gamma \in \Sigma_x$ satisfy $\underline{\measuredangle}(\sigma,\gamma) = 0$ or $\overline{\measuredangle}(\sigma,\gamma) = \pi$.

Problem

- *How common/rare are the pairs $(\sigma,\gamma) \in \Sigma_x^2$ such that $\underline{\measuredangle}(\sigma,\gamma) = 0$* **and** *$\underline{\measuredangle}(\sigma,\gamma) = \pi$?*
- *How common/rare are the pairs of segments with a well-defined angle ?*

Definition of f-cusp

Definition

Let f be a comparison function.
If $x \in X \in \mathcal{L}$ is such that any two segments σ,γ emanating from x are (resp. weakly, resp. strongly) tangent we call x a (resp. weak, resp. strong) f-cusp.

Example

If $X \in G$, its (weak/strong) f-cusp are exactly its endpoints.

Cusp properties

Proposition

1. *A strong f-cusp is a f-cusp.*
2. *A f-cusp is a weak λf-cusp for any $\lambda > 1$.*

Proposition

A weak f-cusp is interior to no segment.

Theorem

Let f be a comparison function. In a generic length space,

1. *there is no f-cusp,*
2. *a generic point $x \in X$ is a weak f-cusps.*

Names are: *box dimension*, *box-counting* $\sim$, *capacity* $\sim$, *fractal* $\sim$, *Kolmogorov* $\sim$, *Minkowski* $\sim$, *Minkowski-Bouligand*, ...

Notation

- $N(X,\varepsilon) = \min\{\operatorname{card}(F) \,|F \subset X \ \forall x \in X \ d(x,F) \leq \varepsilon\}$
- $M(X,\varepsilon) = \max\left\{\operatorname{card}(F) \middle| \begin{array}{l} F \subset X \text{ and} \\ \forall x, y \in F \ x \neq y \Rightarrow xy \geq \varepsilon \end{array}\right\}$,

Theorem and definition

The *upper* and *lower box dimension* of a compact metric space X are defined as

$$\dim^B(X) = \limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \limsup_{\varepsilon\to 0} \frac{\log M(X,\varepsilon)}{-\log\varepsilon}$$

$$\dim_B(X) = \liminf_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \liminf_{\varepsilon\to 0} \frac{\log M(X,\varepsilon)}{-\log\varepsilon}.$$

Generic dimension

$$\dim_T \leq \dim_H \leq \dim_B \leq \dim^B$$

Theorem

Let X be a generic length space.

- $\dim_B(X) = 1$ *and* $\dim^B(X) = \infty$.
- $\mathcal{H}^1(X) = \infty$

Theorem

In a generic compact length space, $\forall x \in X, \forall \rho > 0, \dim_B(S_x(\rho)) = 0$.

Question

What can one say (generically) of $\dim^B S_x(\rho)$ *?*

Notation

For $x \in X \in \mathcal{L}$, $S_x(\rho)$ is the sphere centred at x with radius ρ, that is $\{y \in X \,|\, d(x, y) = \rho\}$

Farthest points

Notation

$\rho_x = max_{y \in X} d(x, y)$

$F_x = S_x(\rho_x)$

Theorem

For a generic $X \in \mathcal{L}$ and a generic $x \in X$, $\mathrm{card}(F_x) = 1$*.*

Farthest points

Notation

$\rho_x = max_{y \in X} d(x, y)$

$F_x = S_x(\rho_x)$

Theorem

For a generic $X \in \mathcal{L}$ *and a generic* $x \in X$, $\mathrm{card}(F_x) = 1$.

Embedded in $\mathbb{R}^n$	**Convex surfaces** T.Z. (1995)	**Continua** ?	**Compacta** Kuz'minykh (1997)
Abstract	**Alex. Surfaces** J.R & C.V. (2018?)	**Length spaces** J.R.(2019?)	**compact metric spaces** J.R. (2011)

Proof I

Preliminary

Theorem

On a compact manifold endowed with a generic Riemannian structure, a generic point has a single farthest point.

J. Rouyer (2003).

Lemma

The function F is upper semi-continuous, that is

$$\lim_{x \to x_0} F_x \subset F_{x_0}.$$

Proof II

Denote by $\delta(A)$ the diameter of A.

$$\begin{aligned}
\mathcal{M} &\overset{\text{def}}{=} \{X \,|\, \{x \in X \,|\, \#F_x > 1\} \text{ non meager}\} \\
&= \bigcup_p \{X \,|\, \text{int}\{x \in X \,|\, \delta(F_x) \geq 1/p\} \neq \varnothing\} \\
&= \bigcup_p \bigcup_q \left\{X \,\middle|\, \exists y \in X \; \bar{B}\left(y, \frac{1}{q}\right) \subset \{x \in X \,|\, \delta(F_x) \geq 1/p\}\right\} \\
&\overset{\text{def}}{=} \bigcup_p \bigcup_q \mathcal{M}_{pq}.
\end{aligned}$$

- $\mathcal{M}_{pq}$ has empty interior.
- It remains to prove that it is closed.

Proof III

$$\mathcal{M}_{pq} = \left\{ X \,\middle|\, \exists y \in X \ \bar{B}\left(y, \frac{1}{q}\right) \subset \{x \in X \,|\delta\left(F_x\right) \geq 1/p\} \right\}$$

- $\mathcal{M}_{pq} \ni X_n \in \mathcal{M}_{pq} \underset{GH}{\to} X \in \mathcal{L}$.
- W.l.g., we can assume that $X_n, X \subset Z$ and $X_n \underset{PH}{\to} X$.
- Take $y_n \in X_n$ s.t. $\bar{B}\left(y_n, \frac{1}{q}\right) \subset \{x \in X_n|\delta\left(F_x\right) \geq 1/p\}$
- Take a converging sub-sequence; let $y \in X$ be the limit.
- We claim that $\bar{B}\left(y, \frac{1}{q}\right) \subset \{x \in X_n|\delta\left(F_x\right) \geq 1/p\}$, and so, $X \in \mathcal{M}_{pq}$.
    - $z \in \bar{B}\left(y, \frac{1}{q}\right) \leftarrow z_n \in \bar{B}\left(y_n, \frac{1}{q}\right) \subset \left\{x \in X_n \,\middle|\, \delta(F_x) \geq \frac{1}{p}\right\}$
    - $\delta(F_{z_n}) \geq \frac{1}{p}$.
    - By semi-continuity of F, $\delta(F_z) \geq \frac{1}{q}$
    - $z \in \left\{x \in X \,\middle|\, \delta(F_x) \geq \frac{1}{p}\right\}$

Thank you very much for your attention !