Envelopes of α-sections

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Collaborators

This talk is based on a joint work with

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- $\alpha \in]0, 1[$
  α-section of K = an oriented line $\Delta \subset E$ cutting K in two parts,
  K^- (to the right) of area $|K^-| = \alpha |K|$, and
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- K_α is the *α-core* of K = the intersection of all K^+.
- m_α = the envelope of all α-sections of K.

Two examples

If K is a disk then $m_\alpha = \partial K_\alpha$ is a circle.

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If K is a polygon then m_α is made of arcs of hyperbolae, $\forall \alpha \in]0, 1[$.

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Study secants between parallel supporting lines to K, whose distances to the corresponding lines make a ratio of $\alpha/(1-\alpha)$.

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Contributors:

- P.C. Hammer, 1951;
- V. Klee, 1953;
- T. Zamfirescu, 1967.

Related topic: common tangents and transversals

Contributors:

- S.E. Cappell, J.E. Goodman, J. Pach, R. Pollack, M. Sharir, R. Wenger, 1994;
- I. Bárány, A. Hubard, J. Jeronimo, 2008;
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Theorem

For any well-separated family of k strictly convex bodies in $\mathbb{R}^d$, $k \leq d$, the space of all α-sections is diffeomorphic to $\mathbb{S}^{d-k}$.

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Contributors:

- V. F. Lazutkin, 1973;
- J. Moser, 1973;
- E. Gutkin, A. Katok, 1995;
- S. Tabachnikov, 1995;
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Theorem

If ℓ is the envelope of α-sections of a convex set bounded by a curve κ, for some α, then κ is a caustic for the outer billiard of table $L = \mathrm{conv}\ell$.

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- C. Dupin, 1822;
- C. Schütt, E. Werner, 1990, 1994; E. Werner, 2004:
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Theorem

$K \subset \mathbb{R}^d$ with boundary of class $\mathcal{C}^{\geq 4}$; K_δ is homothetic to K, for some sufficiently small $\delta > 0$, if and only if K is an ellipsoid.

Related topic: fair partitioning of convex bodies

Fair / balanced / equi- partions of convex bodies (measures) by use of

- k-fans;
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Contributors:

- I. Bárány, J. Matousek, 2001;
- T. Sakai, 2002;
- S. Bereg, 2009;
- I. Bárány, P. Blagojević, A. Szúcs, 2010;
- P. V. M. Blagojević, G.M. Ziegler, 2014;
- R. N. Karasev, A. Hubard, B. Aronov, 2014;

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Theorem

Given an integer $n \geq 2$, there exists a fair partition of any pizza (K, L) into n parts if and only if n is even.

Main result: m_α for symmetric K

If $\alpha > 1/2$ then $K_\alpha = \emptyset$.

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Theorem

If K is symmetric then

- $m_\alpha = \partial K_\alpha$ *for all* $\alpha \in \left]0, \frac{1}{2}\right[$;
- m_α *is of class* $\mathcal{C}^1$ *for all* $\alpha \in \left]0, \frac{1}{2}\right[$ *if and only if* K *is strictly convex.*

Main result: m_α for non-symmetric K

We cannot have $m_\alpha = \partial K_\alpha$ for all $\alpha \in \left]0, \frac{1}{2}\right[$, because m_α exists for all α, but $K_\alpha = \emptyset$ for α close enough to $\frac{1}{2}$.

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If K is non-symmetric then there exists $\alpha_B \in \left[0, \frac{1}{2}\right[$ s.t.

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Corollary

K is non-symmetric iff there exists a triangle containing more than half of K (in area), with one side in K and the other two disjoint from $\mathrm{int} K$.

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There exists $\alpha_K \in \left[\frac{4}{9}, \frac{1}{2}\right]$ *s.t.*

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Hint: G is the mid-point of at least three secants of K.

K non-symmetric $\Rightarrow \alpha_B < \alpha_K$, K symmetric $\Rightarrow \alpha_B = \alpha_K = \frac{1}{2}$.

Thank you for your attention!