Resolvent expansion for the discrete one-dimensional Schrödinger operator

Arne Jensen

Department of Mathematical Sciences
Aalborg University

Bucharest July 2014

Background

The results presented here are joint work with Kenichi Ito, Kobe University, Japan. The talk is based on the following preprint:

- K. Ito, A. Jensen, *A complete classification of threshold properties for one-dimensional Schrödinger operators*, `arXiv:1312.1396`

The techniques used come from a series of papers with Gheorghe Nenciu on resolvent expansions.

Introduction

We consider sequences $x\colon \mathbb{Z} \to \mathbb{C}$ and the operator on sequences

$$(H_0 x)[n] = -(x[n+1] + x[n-1] - 2x[n]).$$

Restricted to $\mathcal{H} = \ell^2(\mathbb{Z})$ it is a bounded selfadjoint operator with

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0,4].$$

Its resolvent is denoted by $R_0(z) = (H_0 - z)^{-1}$, $z \in \mathbb{C} \setminus [0,4]$. Let $H = H_0 + V$, V a compact selfadjoint operator on $\mathcal{H}$. Then $\sigma_{\mathrm{ess}}(H) = [0,4]$.

Introduction

We consider sequences $x\colon \mathbb{Z} \to \mathbb{C}$ and the operator on sequences

$$(H_0x)[n] = -(x[n+1] + x[n-1] - 2x[n]).$$

Restricted to $\mathcal{H} = \ell^2(\mathbb{Z})$ it is a bounded selfadjoint operator with

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0,4].$$

Its resolvent is denoted by $R_0(z) = (H_0 - z)^{-1}$, $z \in \mathbb{C} \setminus [0,4]$. Let $H = H_0 + V$, V a compact selfadjoint operator on $\mathcal{H}$. Then $\sigma_{\mathrm{ess}}(H) = [0,4]$.

Goal: Analyze the thresholds 0 and 4 of H in terms of $R(z) = (H - z)^{-1}$.

Introduction

We only present results for the threshold 0. Results for 4 follow directly due to the following well known observation.

Define $(Jx)[n] = (-1)^n x[n]$. Then J is bounded selfadjoint and unitary on $\mathcal{H}$. Let $V_J = JVJ^{-1}$. We have

$$J(H_0 + V)J^{-1} = -(H_0 + V_J - 4)$$

and thus

$$JR(z)J^{-1} = J(H_0 + V - z)^{-1}J^{-1} = -(H_0 + V_J - (z-4))^{-1}.$$

Introduction

We introduce for $s \in \mathbb{R}$

$$\mathcal{L}^s = \ell^{1,s}(\mathbb{Z}) = \Big\{x\colon \mathbb{Z} \to \mathbb{C};\ \|x\|_{1,s} = \sum_{n\in\mathbb{Z}} (1+n^2)^{s/2}|x[n]| < \infty\Big\},$$

$$(\mathcal{L}^s)^* = \ell^{\infty,-s}(\mathbb{Z}) = \Big\{x\colon \mathbb{Z} \to \mathbb{C};\ \|x\|_{\infty,-s} = \sup_{n\in\mathbb{Z}} (1+n^2)^{-s/2}|x[n]| < \infty\Big\}.$$

The superscript s is dropped when $s=1$: $\mathcal{L} = \mathcal{L}^1$, $\mathcal{L}^* = (\mathcal{L}^1)^*$.
We denote the set of all bounded operators from a general Banach space $\mathcal{K}$ to another $\mathcal{K}'$ by $\mathcal{B}(\mathcal{K},\mathcal{K}')$, and replace $\mathcal{B}$ by $\mathcal{C}$ when considering those for the compact operators.

We define $\mathcal{B}^s = \mathcal{B}(\mathcal{L}^s, (\mathcal{L}^s)^*)$.

Introduction

Assumption

Let $V \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and assume that there exist a real number $\beta \geq 1$, a Hilbert space $\mathcal{K}$, an injective operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^\beta) \cap \mathcal{C}(\mathcal{K}, \mathcal{L})$ and a self-adjoint unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that

$$V = vUv^* \in \mathcal{B}((\mathcal{L}^\beta)^*, \mathcal{L}^\beta) \cap \mathcal{C}(\mathcal{L}^*, \mathcal{L}).$$

This is a version of the classical factored perturbation technique, used extensively by both Kato and Kuroda in the 60-ies and 70-ies. Here we incorporate decay conditions into the assumptions.

Introduction

Assumption

Let $V \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and assume that there exist a real number $\beta \geq 1$, a Hilbert space $\mathcal{K}$, an injective operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^{\beta}) \cap \mathcal{C}(\mathcal{K}, \mathcal{L})$ and a self-adjoint unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that

$$V = vUv^* \in \mathcal{B}((\mathcal{L}^{\beta})^*, \mathcal{L}^{\beta}) \cap \mathcal{C}(\mathcal{L}^*, \mathcal{L}).$$

This is a version of the classical factored perturbation technique, used extensively by both Kato and Kuroda in the 60-ies and 70-ies. Here we incorporate decay conditions into the assumptions.

This class includes sums of multiplicative perturbations and finite rank perturbations. It is a much larger class than previously considered in this context.

Main results

Resolvent expansion

We state the main result on resolvent expansion in a simplified form.

For $z \in \mathbb{C} \setminus [0, \infty)$ we take the determination of the square root with $\operatorname{Im} \sqrt{z} > 0$.

Theorem

Suppose $\beta \geq 4$ in the Assumption, and let $N \in [-1, \beta - 5]$ be any integer. Then, as $z \to 0$ in $\mathbb{C} \setminus [0, \infty)$, the resolvent $R(z)$ has the asymptotic expansion in the uniform topology of $\mathcal{B}^{N+4}$:

$$R(z) = \sum_{j=-2}^{N} z^{j/2} G_j + \mathcal{O}(z^{(N+1)/2}), \quad G_j \in \mathcal{B}^{j+3},$$

and the coefficients G_j can be computed explicitly.

Main results

Resolvent expansion

The main new result here is the complete classification of all possibilities for the coefficients.

This amounts to a complete classification of solutions to

$$(H_0 + V)x = 0$$

Main results

We now classify the possibilities in the expansion, based on the generalized zero eigenspace. Define

$$\widetilde{\mathcal{E}} = \{\Psi \in (\mathcal{L}^{\beta})^*;\ H\Psi = 0\}, \quad \widetilde{d} = \dim \widetilde{\mathcal{E}},$$

We can show that the eigenspace is finite-dimensional, and the eigenfunctions have special asymptotics at infinity.
Define the sequences $\mathbf{1}, \boldsymbol{\sigma} \in (\mathcal{L}^0)^*$ and $\mathbf{n}, |\mathbf{n}| \in \mathcal{L}^*$ by

$$\mathbf{1}[n] = 1, \quad \boldsymbol{\sigma}[n] = \begin{cases} \pm 1 & \text{if } \pm n > 0, \\ 0 & \text{if } n = 0, \end{cases} \quad \mathbf{n}[n] = n, \quad |\mathbf{n}|[n] = |n|,$$

respectively.

Main results

Theorem

Suppose $\beta \geq 1$ in the Assumption. Then,

$$\widetilde{\mathcal{E}} \subset \mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2}, \quad \widetilde{d} < \infty.$$

The classification of the singular part of the resolvent expansion uses the following (canonical) subspaces.

$$\mathcal{E} = \widetilde{\mathcal{E}} \cap \left(\mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2}\right), \quad d = \dim \mathcal{E};$$
$$E = \widetilde{\mathcal{E}} \cap \mathcal{L}^{\beta-2}, \quad d_0 = \dim E.$$

We have $E \subset \mathcal{E} \subset \widetilde{\mathcal{E}}$ and $d_0 \leq d \leq \widetilde{d} \leq d_0 + 4$.

Main results

Classification of threshold $\lambda = 0$

Definition

The threshold $\lambda = 0$ is said to be

1. a *regular point*, if $\mathcal{E} = E = \{0\}$;
2. an *exceptional point of the first kind*, if $\mathcal{E} \supsetneqq E = \{0\}$;
3. an *exceptional point of the second kind*, if $\mathcal{E} = E \supsetneqq \{0\}$;
4. an *exceptional point of the third kind*, if $\mathcal{E} \supsetneqq E \supsetneqq \{0\}$.

Our major new result is the equivalence of this classification with a classification based on the coefficients G_{-2} and G_{-1} in the resolvent expansion.

Main results

Theorem

Suppose $\beta \geq 4$ *in the Assumption. Then* $\widetilde{d} = d_0 + 2$, *and there exist bases* $\Psi_j \in E$, $j = 1, \ldots, d_0$, *and* $\Psi_j \in \mathcal{E}/E$, $j = d_0 + 1, \ldots, d$, *such that*

$$G_{-2} = -\sum_{j=1}^{d_0} \langle \Psi_j, \cdot \rangle \Psi_j, \quad G_{-1} \equiv i \sum_{j=d_0+1}^{d} \langle \Psi_j, \cdot \rangle \Psi_j \mod \langle E, \cdot \rangle E,$$

where $\langle E, \cdot \rangle E \subset \mathcal{B}((\mathcal{L}^{\beta-2})^*, \mathcal{L}^{\beta-2})$ *is the subspace spanned by the operators of the form* $\langle \Psi, \cdot \rangle \Psi'$ *with* $\Psi, \Psi' \in E$. *Furthermore, one can choose* $\Psi_j \in E$, $j = 1, \ldots, d$, *to be orthonormal, and hence* $-G_{-2}$ *is the orthogonal projection onto* E.

Main results

Different formulation of the previous classification results:

- Regular case: $G_{-2} = 0$, $G_{-1} = 0$.
- Exceptional point first kind: $G_{-2} = 0$, $G_{-1} \neq 0$, $\mathrm{Rank}\, G_{-1} \leq 2$.
- Exceptional point second kind: $G_{-2} \neq 0$. G_{-1} may be zero or nonzero.
- Exceptional point third kind: $G_{-2} \neq 0$, $G_{-1} \neq 0$, $\mathrm{Rank}\, G_{-1} \leq 2$.

Main results

Comments:

- The regular case is the generic case.

Main results

Comments:

- The regular case is the generic case.
- An exceptional point of the first kind is also called the threshold resonance case, and $x \in \mathcal{E}/E$ threshold resonance functions. At most two linearly independent threshold resonance functions exist.

Main results

Comments:

- The regular case is the generic case.
- An exceptional point of the first kind is also called the threshold resonance case, and $x \in \mathcal{E}/E$ threshold resonance functions. At most two linearly independent threshold resonance functions exist.
- An exceptional point of the second kind corresponds to existence of zero eigenfunctions. Any finite value of d_0 may occur.

Main results

Comments:

- The regular case is the generic case.
- An exceptional point of the first kind is also called the threshold resonance case, and $x \in \mathcal{E}/E$ threshold resonance functions. At most two linearly independent threshold resonance functions exist.
- An exceptional point of the second kind corresponds to existence of zero eigenfunctions. Any finite value of d_0 may occur.
- An exceptional point of the third kind is a combination of those of the first and second kind.

Main results

Comments:

- The regular case is the generic case.
- An exceptional point of the first kind is also called the threshold resonance case, and $x \in \mathcal{E}/E$ threshold resonance functions. At most two linearly independent threshold resonance functions exist.
- An exceptional point of the second kind corresponds to existence of zero eigenfunctions. Any finite value of d_0 may occur.
- An exceptional point of the third kind is a combination of those of the first and second kind.
- For V multiplicative, i.e. $(Vx)[n] = V[n]x[n]$ for some decaying function $V : \mathbb{Z} \to \mathbb{C}$, only regular points and exceptional points of the first kind occur. In this case $\mathrm{Rank}\, G_{-1} \leq 1$. In particular, zero eigenvalues do not occur.

Examples of perturbations

We now give some examples of perturbations. A general example:

Proposition

Let $\beta \geq 1$ be any real number, and $v_j \in \mathcal{L}^\beta$, $j = 1, 2, \ldots,$ be at most a countable number of linearly independent vectors with

$$\sum_j \|v_j\|^2_{\mathcal{L}^\beta} < \infty.$$

Then for any $\sigma_j \in \{\pm 1\}$ the operator series

$$V = \sum_j \sigma_j \langle v_j, \cdot \rangle v_j$$

converge in the uniform topology of $\mathcal{B}((\mathcal{L}^\beta)^*, \mathcal{L}^\beta)$ and satisfy the Assumption with the same β.

Examples of perturbations

For a multiplicative V, $(Vx)[n] = V[n]x[n]$, the condition is $\sum_n (1+n^2)^\beta |V[n]| < \infty$. Note that in the discrete case these are the only local perturbations, i.e. $\operatorname{supp} Vx \subseteq \operatorname{supp} x$.

For multiplicative potentials examples of zero resonances can be constructed using the von Neumann-Wigner technique (1929). We are looking for a multiplicative potential V such that there is a sequence $x \in (\mathcal{L}^0)^*$ satisfying

$$-(x[n+1] + x[n-1] - 2x[n]) + V[n]x[n] = 0.$$

We find such V by first choosing x and then taking V accordingly to

$$V[n] = \frac{x[n+1] + x[n-1]}{x[n]} - 2.$$

Examples of perturbations

Local perturbation with threshold resonance

Example 1.

$$x[n] = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{otherwise,} \end{cases} \qquad V[n] = \begin{cases} -1 & \text{if } n = 0, \\ 1 & \text{if } n = \pm 1, \\ 0 & \text{otherwise;} \end{cases}$$

Example 2.

$$x[n] = \begin{cases} 3 & \text{if } n = 0, \\ 2 & \text{if } n = \pm 1, \\ 1 & \text{otherwise,} \end{cases} \qquad V[n] = \begin{cases} -2/3 & \text{if } n = 0, \\ 0 & \text{if } n = \pm 1, \\ 1 & \text{if } n = \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

Examples of perturbations

Examples of threshold eigenvalues

Let us define the potential V by

$$V = -\sum_{j=1}^{N} \langle v_j, \cdot \rangle v_j; \quad v_j[n] = \begin{cases} \sqrt{2} & \text{if } n = 3j, \\ -1/\sqrt{2} & \text{if } n = 3j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the linearly independent sequences $\Psi_j \in \mathcal{L}$, $j = 1, \ldots, N$, given by

$$\Psi_j[n] = \begin{cases} 1 & \text{if } n = 3j, \\ 0 & \text{otherwise} \end{cases}$$

all satisfy $H\Psi_j = (H_0 + V)\Psi_j = 0$ and $\Psi_j \in \mathcal{L}^\beta$ for all $\beta > 0$. Thus we have examples of zero eigenvalues with $d_0 = N$, for any $N \geq 1$.

Examples of perturbations

An example of both a threshold eigenvalue and threshold resonances

We define the following sequences:

$$\phi_j[n] = \begin{cases} -1, & n = 4j, \\ 1, & n = 4j+1, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 0, 1, 2,$$

$$u_j[n] = \begin{cases} 1 & n \leq 4j, \\ -1 & n > 4j, \end{cases} \qquad j = 0, 1, 2.$$

Then we define

$$Vx = -\sum_{j=0}^{2} \langle x, \phi_j \rangle \phi_j.$$

With these definitions we have

$$(H_0 + V)u_j = 0, \quad j = 0, 1, 2.$$

Examples of perturbations

An example of both a threshold eigenvalue and threshold resonances

If we define

$$w[n] = \begin{cases} 1 & n = 1, 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases}$$

then we have $u_2 = \frac{1}{2}u_0 + \frac{1}{2}u_1 + w$.
It is easy to see that this example can be modified to provide a threshold eigenvalue of any finite multiplicity, besides the two linearly independent resonance functions.

Strategy

For $z \in \mathbb{C} \setminus [0, \infty)$ we fix $\mathrm{Im}\sqrt{z} > 0$ and then introduce

$$\kappa = -i\sqrt{z}; \quad z = -\kappa^2.$$

This is convenient, since we will be able to work with selfadjoint operators. We write $R(z) = R(\kappa)$ etc.

Central object of study is

$$M(\kappa) = U + v^* R_0(\kappa) v.$$

It is a symmetrized version of $1 + R_0(z)V$.
Central idea is that the study of $R(\kappa)$ is reduced to the study of $M(\kappa)$ via the relations:

$$R(\kappa) = R_0(\kappa) - R_0(\kappa) v M(\kappa)^{-1} v^* R_0(\kappa),$$
$$M(\kappa)^{-1} = U - U v^* R(\kappa) v U.$$

Strategy

From

$$M(\kappa) = U + v^* R_0(\kappa) v.$$

and

$$R(\kappa) = R_0(\kappa) - R_0(\kappa) v M(\kappa)^{-1} v^* R_0(\kappa)$$

follows the strategy:

- Obtain an asymptotic expansion of $R_0(\kappa)$ around $\kappa = 0$
- Show that the asymptotic expansion of $M(\kappa)$ leads to invertibility and asymptotic expansion of $M(\kappa)^{-1}$.
- Combine these result to obtain the expansion of $R(\kappa)$.

Asymptotic expansion of $R_0(\kappa)$

The Fourier transform $\mathcal{F}\colon \mathcal{H} \to L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, our conventions are: For $x \in \mathcal{H}$ and $f \in L^2(\mathbb{T})$

$$(\mathcal{F}x)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{-in\theta} x[n], \quad (\mathcal{F}^{-1}f)[n] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{in\theta} f(\theta)\, d\theta.$$

We have

$$\mathcal{F}(H_0 x)(\theta) = (2 - 2\cos\theta)(\mathcal{F}x)(\theta) = \Big(4\sin^2\frac{\theta}{2}\Big)(\mathcal{F}x)(\theta),$$

Thus

$$\mathcal{F}(R_0(z)x)(\theta) = \frac{(\mathcal{F}x)(\theta)}{4\sin^2(\theta/2) - z}, \quad z \in \mathbb{C} \setminus [0,4].$$

Asymptotic expansion of $R_0(\kappa)$

For $z \in \mathbb{C} \setminus [0,4]$ sufficiently close to 0 change the variable from z to ϕ through the correspondence

$$z = 4\sin^2\frac{\phi}{2}, \quad \operatorname{Im}\phi > 0.$$

Then $R_0(z)$ is given by convolution with the function

$$R_0(z;n) = \frac{ie^{i\phi|n|}}{2\sin\phi}.$$

This leads to the asymptotic expansion we need.

Asymptotic expansion of $R_0(\kappa)$

Proposition

Let $N \geq -1$ be any integer. Then, as $\kappa \to 0$ with $\operatorname{Re}\kappa > 0$, the resolvent $R_0(\kappa)$ has the expansion in $\mathcal{B}^{N+2}$:

$$R_0(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j^0 + \mathcal{O}(\kappa^{N+1}), \quad G_j^0 \in \mathcal{B}^{j+1},$$

and the coefficients G_j^0 are given explicitly as convolution operators with polynomials $G_j^0(n)$ of degree $j+1$ in $|n|$. For instance,

$$G_{-1}^0(n) = \frac{1}{2}, \quad G_0^0(n) = -\frac{1}{2}|n|, \quad G_1^0(n) = \frac{1}{4}|n|^2 - \frac{1}{16},$$
$$G_2^0(n) = -\frac{1}{12}|n|^3 + \frac{1}{12}|n|, \quad G_3^0(n) = \frac{1}{48}|n|^4 - \frac{5}{96}|n|^2 + \frac{3}{256}.$$

Expansion of $M(\kappa)$

Proposition

Suppose $\beta \geq 1$ in the Assumption, and let $N \in [-1, \beta - 2]$ be any integer. Then, as $\kappa \to 0$ with $\operatorname{Re}\kappa > 0$, the operator $M(\kappa)$ has the expansion in $\mathcal{B}(\mathcal{K})$:

$$M(\kappa) = \sum_{j=-1}^{N} \kappa^j M_j + \mathcal{O}(\kappa^{N+1}),$$

where the coefficients $M_j \in \mathcal{B}(\mathcal{K})$ are given by

$$M_0 = U + v^* G_0^0 v, \quad M_j = v^* G_j^0 v \text{ for } j \neq 0.$$

Inversion procedure

Framework

Let $\mathcal{K}$ be a Hilbert space and $A(\kappa)$ a family of bounded operators on $\mathcal{K}$ with $\kappa \in D \subset \mathbb{C} \setminus \{0\}$. Suppose that

1. The set $D \subset \mathbb{C} \setminus \{0\}$ is invariant under the complex conjugation and accumulates on $0 \in \mathbb{C}$.
2. For each $\kappa \in D$ the operator $A(\kappa)$ satisfies $A(\kappa)^* = A(\overline{\kappa})$ and has a bounded inverse $A(\kappa)^{-1} \in \mathcal{B}(\mathcal{K})$.
3. As $\kappa \to 0$ in D, the operator $A(\kappa)$ has an expansion in the uniform topology of the operators at $\mathcal{K}$:

$$A(\kappa) = A_0 + \kappa \widetilde{A}_1(\kappa); \quad \widetilde{A}_1(\kappa) = \mathcal{O}(1). \tag{1}$$

4. The spectrum of A_0 does not accumulate at $0 \in \mathbb{C}$ as a set.

Inversion procedure

If A_0 is invertible in $\mathcal{B}(\mathcal{K})$, the Neumann series provides a formula for the expansion of $A(\kappa)^{-1}$:

$$A(\kappa)^{-1} = \sum_{j=0}^{\infty} (-1)^j \kappa^j A_0^{-1} \big[\widetilde{A}_1(\kappa) A_0^{-1}\big]^j.$$

Assume A_0 not invertible in $\mathcal{B}(\mathcal{K})$. By assumption the operator $A(\kappa)^{-1}$ is defined for $\kappa \in D$, however the expansion around $\kappa = 0$ may now contain powers of κ^{-1}.

Inversion procedure

Some terminology

The *pseudoinverse* $a^\dagger$ of a complex number $a \in \mathbb{C}$ is

$$a^\dagger = \begin{cases} 0 & \text{if } a = 0 \\ a^{-1} & \text{if } a \neq 0 \end{cases}$$

If $\mathcal{K}' \subset \mathcal{K}$ is a closed subspace, we identify $\mathcal{B}(\mathcal{K}')$ with its embedding in $\mathcal{B}(\mathcal{K})$ in the standard way. For an operator $A \in \mathcal{B}(\mathcal{K}') \subset \mathcal{B}(\mathcal{K})$ we say that A is *invertible in* $\mathcal{B}(\mathcal{K}')$ if there exists an operator $A^\dagger \in \mathcal{B}(\mathcal{K}')$ such that $A^\dagger A = AA^\dagger = 1_{\mathcal{K}'}$, which we identify with the orthogonal projection onto $\mathcal{K}' \subset \mathcal{K}$. $A^\dagger$ is the pseudoinverse of the operator.

Inversion procedure

Proposition

Let $A(\kappa)$ be the family introduced above. Let Q be the orthogonal projection onto $\operatorname{Ker} A_0$, and define the operator $a(\kappa) \in \mathcal{B}(Q\mathcal{K})$ by

$$a(\kappa) = \frac{1}{\kappa}\{1_{Q\mathcal{K}} - Q(Q + A(\kappa))^{-1}Q\}$$
$$= \sum_{j=0}^{\infty}(-1)^j \kappa^j Q\widetilde{A}_1(\kappa)\big[(Q + A_0)^{-1}\widetilde{A}_1(\kappa)\big]^j Q.$$

Then $a(\kappa)$ is bounded in $\mathcal{B}(Q\mathcal{K})$ as $\kappa \to 0$ in D. Moreover, if for each $\kappa \in D$ sufficiently close to 0 the operator $a(\kappa)$ is invertible in $\mathcal{B}(Q\mathcal{K})$, then

$$A(\kappa)^{-1} = (Q + A(\kappa))^{-1} + \frac{1}{\kappa}(Q + A(\kappa))^{-1}a(\kappa)^{\dagger}(Q + A(\kappa))^{-1}.$$

Inversion procedure

We now apply this result to $A(\kappa) = M(\kappa)$ assuming $\operatorname{Ker} M_0 \neq \{0\}$. In this case we can assume a higher expansion $\widetilde{A}_1(\kappa) = A_1 + \kappa \widetilde{A}_2(\kappa)$, $\widetilde{A}_2(\kappa) = \mathcal{O}(\kappa)$. Then we get

$$a(\kappa) = a_0 + \kappa \widetilde{a}_1(\kappa); \quad a_0 = QM_1Q, \quad \widetilde{a}_1(\kappa) = \mathcal{O}(1).$$

If the leading operator a_0 is **invertible** in $\mathcal{B}(Q\mathcal{K})$, then substitution of the Neumann series for $a(\kappa)^\dagger$ into the above formula yields the expansion of $A(\kappa)^{-1}$.

Inversion procedure

We now apply this result to $A(\kappa) = M(\kappa)$ assuming $\operatorname{Ker} M_0 \neq \{0\}$. In this case we can assume a higher expansion $\widetilde{A}_1(\kappa) = A_1 + \kappa \widetilde{A}_2(\kappa)$, $\widetilde{A}_2(\kappa) = \mathcal{O}(\kappa)$. Then we get

$$a(\kappa) = a_0 + \kappa \widetilde{a}_1(\kappa); \quad a_0 = QM_1Q, \quad \widetilde{a}_1(\kappa) = \mathcal{O}(1).$$

If the leading operator a_0 is **invertible** in $\mathcal{B}(Q\mathcal{K})$, then substitution of the Neumann series for $a(\kappa)^\dagger$ into the above formula yields the expansion of $A(\kappa)^{-1}$.

Otherwise, if a_0 is **not invertible**, by applying the Proposition to $a(\kappa)$ again we obtain the expansion of $a(\kappa)^\dagger$, and find that $A(\kappa)^{-1}$ has at least a κ^{-2} singularity in its expansion. We can repeat this argument. The iteration procedure stops when applied to $M(\kappa)$ after a few iterations, since the operator $M(\kappa)^{-1}$ can have at worst a κ^{-2} singularity due to the selfadjointness of H_0.

Analysis of $\operatorname{Ker} M_0$ and conclusion of the argument

The next step is to analyze the space $\operatorname{Ker} M_0$ and the connection between $x \in \operatorname{Ker} M_0$ and solutions to $H\Psi = 0$.

This analysis is long and somewhat complicated, so I will not present it.

Putting all components together leads to the resolvent expansion, with more precise conditions on β, and with explicit expressions for the leading coefficients.

Comments on the literature

There are some related results analyzing the resolvent of $H = H_0 + V$ for V multiplicative, and obtaining the leading term in the asymptotic expansion.

- D. E. Pelinovsky and A. Stefanov, J. Math. Phys. **49**, 113501, 2008 (only regular case)
- Scipio Cuccagna, J. Math. Anal. Appl. **354** (2009) 594605.

Coefficient G_0^0

Coefficient G_0^0 with kernel convolution by $-\frac{1}{2}|n|$ has representations

$$(G_0^0 x)[n] = -\frac{n}{2}\langle \mathbf{1}, x\rangle + \frac{1}{2}\langle \mathbf{n}, x\rangle - \sum_{k\geq n}(k-n)x[k]$$

$$= \frac{n}{2}\langle \mathbf{1}, x\rangle - \frac{1}{2}\langle \mathbf{n}, x\rangle - \sum_{k\leq n}(n-k)x[k].$$

Quasi-symmetric eigenspace

The following space is needed in detailed analysis

$$\widetilde{\mathcal{E}}_{\mathrm{qs}} = \widetilde{\mathcal{E}} \cap \left(\mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2}\right), \quad \widetilde{d}_{\mathrm{qs}} = \dim \widetilde{\mathcal{E}}_{\mathrm{qs}}.$$

It follows directly from the definition that $E \subset \widetilde{\mathcal{E}}_{\mathrm{qs}}$ and $d_0 \leq \widetilde{d}_{\mathrm{qs}} \leq d_0 + 2$.

Example of cases: Bases of spaces

	$\widetilde{\mathcal{E}}/E$	$\mathcal{E}/E$	E	$\widetilde{\mathcal{E}}_{\rm qs}/E$	type
Case i.	$\{\Psi_1^0, \Psi_2^0\}$	$\{\Psi_1^0\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case ii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$0 < \# < \infty$	$\{\Psi_5\}$	exceptional II
Case iii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$0 < \# < \infty$	$\emptyset$	exceptional II
Case iv.	$\{\Psi_1^0, \Psi_6\}$	$\{\Psi_1^0\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case v.	$\{\Psi_5, \Psi_6\}$	$\emptyset$	$0 < \# < \infty$	$\{\Psi_5\}$	exceptional II
Case vi.	$\{\Psi_5, \Psi_6\}$	$\emptyset$	$0 < \# < \infty$	$\emptyset$	exceptional II
Case vii.	$\{\Psi_1^0, \Psi_4\}$	$\{\Psi_1^0, \Psi_4\}$	$0 < \# < \infty$	$\{\Psi_4\}$	exceptional III
Case viii.	$\{\Psi_3, \Psi_2^0\}$	$\{\Psi_3\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case ix.	$\{\Psi_5, \Psi_4\}$	$\{\Psi_4\}$	$0 < \# < \infty$	$\{\Psi_5, \Psi_4\}$	exceptional III
Case x.	$\{\Psi_5, \Psi_4\}$	$\{\Psi_4\}$	$0 < \# < \infty$	$\{\Psi_4\}$	exceptional III
Case xi.	$\{\Psi_3, \Psi_6\}$	$\{\Psi_3\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case xii.	$\{\Psi_3, \Psi_4\}$	$\{\Psi_3, \Psi_4\}$	$0 < \# < \infty$	$\{\Psi_4\}$	exceptional III