Incompressibility Estimates in the Laughlin Phase

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References

N. Rougerie, JY, *Incompressibility Estimates for the Laughlin Phase*, arXiv:1402.5799; to appear in CMP.

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IQHE and FQHE

The Integer Quantum Hall Effect can essentially be understood in terms of single particle physics and the Pauli principle for the electrons. In a magnetic field B there is a maximal number, $\frac{B}{2\pi}$, of fermions per unit area that can be accommodated in a given Landau level, corresponding to a filling factor 1.

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This talk concerns estimates for the 1-partice density, and also some higher marginals of the N-particle density, in correlated many-body states in the lowest Landau level (LLL), related to the Laughlin state. The results apply also to bosons, that may show features of the FQHE under rapid rotation.

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The upper bounds on the density are a manifestation of the incompressibility of the quantum fluid.

Landau levels

Magnetic Hamiltonian:

$$H = -\tfrac{1}{2}\left(\nabla_\perp + \mathrm{i}\,\mathbf{A}(\mathbf{r})\right)^2$$

with $\mathbf{A}(\mathbf{r}) = \frac{B}{2}(-y, x)$. Choose units so that $B = 2$. (For rotating systems $B = 2\Omega$.)

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Complex coordinates: $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, $\partial = \frac{d}{dz}$, $\bar{\partial} = \frac{d}{d\bar{z}}$.

We can write

$$H = a^\dagger a + \tfrac{1}{2}$$

with

$$a = \left(\bar{\partial} + \tfrac{1}{2}z\right), \qquad a^\dagger = \left(-\partial + \tfrac{1}{2}\bar{z}\right), \qquad [a, a^\dagger] = 1.$$

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The Spectrum of H is $\varepsilon_n = 2(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$.

H commutes with another set of creation and annihilation operators

$$b = (\partial + \tfrac{1}{2}\bar{z}) \qquad \text{and} \qquad b^\dagger = \left(-\bar{\partial} + \tfrac{1}{2}z\right)$$

and hence also with the angular momentum operator

$$\mathcal{L} = b^\dagger b - a^\dagger a = z\partial - \bar{z}\bar{\partial}.$$

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The joint eigenfunctions of H and $\mathcal{L}$ have the form

$$\psi_{n,l}(z,\bar{z}) = P_{n,l}(z,\bar{z})e^{-|z|^2/2}$$

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The LLL, characterized by $a\,\psi = (\bar{\partial} + \frac{1}{2}z)\psi = 0$, is generated by

$$\psi_{0,l}(z, \bar{z}) = (\pi l!)^{-1/2}\, z^l e^{-|z|^2/2}.$$

Maximum density droplet

The density $|\psi_{0,l}(z,\bar{z})|^2$ is concentrated around the maximum at radius

$$r_l = \sqrt{l}.$$

Thus, if $l \gg 1$, then l is the number of orthonormal states whose wave functions can be accommodated within a disc of area $\pi r_l^2 = \pi l$, so the density of states per unit area in the LLL is π^{-1}. The same holds for the higher Landau levels.

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The "Maximum density droplet" (MDD) in the LLL is the Slater determinant

$$\Psi_{\mathrm{MDD}} = (N!)^{-1/2} \psi_{0,0} \wedge \cdots \wedge \psi_{0,N-1}$$

The 1-particle density $\sum_l |\psi_{0,l}|^2$ is essentially π^{-1} up to radius $\sqrt{N}$, i.e., the filling factor is 1.

The Laughlin wave function(s)

The wave function of the MDD is, apart from the gaussian factor, a Vandermonde determinant and can be written as

$$\Psi_{\mathrm{MDD}} = (N!)^{-1/2} \prod_{i<j} (z_i - z_j) e^{-\sum_{i=1}^{N} |z_i|^2/2}.$$

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The Laughlin wave function(s), on the other hand, have the form

$$\Psi^{(\ell)}_{\mathrm{Laugh}} = C_{N,\ell} \prod_{i<j} (z_i - z_j)^{\ell} e^{-\sum_{i=1}^{N} |z_i|^2/2}$$

with ℓ odd ≥ 3 and $C_{N,\ell}$ a normalization constant. For Bosons ℓ is even and ≥ 2.

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In his 1983 paper Laughlin claims that the 1-particle density of $\Psi^{(\ell)}_{\rm Laugh}$ within its support is close to $(\ell\pi)^{-1}$.

Methaphoric picture; plasma analogy

Methaphoric picture of the N-particle density (not due to Laughlin!):

The particles move in a correlated way, as tightly packed as the factors $(z_i - z_j)^\ell$ allow, like huddling emperor penguins during an Antarctic winter. Each “penguin” claims on the average an area of radius $\sqrt{\ell}$.

Laughlin's argument for the density $(\ell\pi)^{-1}$ is more mathematical. It is based on the "plasma analogy":

The N-particle density $|\Psi^{(\ell)}_{\rm Laugh}|^2$ can be interpreted as the Boltzmann-Gibbs factor at temperature $T = N^{-1}$ of classical 2D jellium, i.e., a 2D Coulomb gas in a uniform neutralizing background. A mean field approximation leads to the claimed density.

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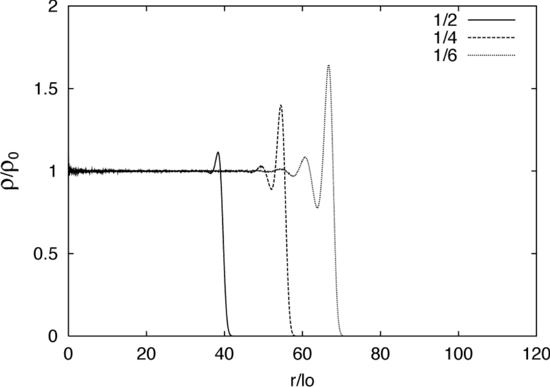
Numerical calculations (O. Ciftja) show, however, that the density may be considerably larger than $(\ell\pi)^{-1}$ close to the edge. The result can thus only hold in a suitable weak sense in the limit $N \to \infty$.



1/2
1/4
1/6
2
1.5
1
0.5
0
ρ/ρ0
0
20
40
60
80
100
120
r/lo

A proof of Laughlin's claim amounts to a study of the mean field limit of the jellium model, including error estimates.

We denote $(z_1, \dots, z_N)$ by Z for short and consider the scaled N particle probability density (normalized to 1)

$$\mu^{(N)}(Z) = N^N \left| \Psi^{(\ell)}_{\text{Laugh}}(\sqrt{N} Z) \right|^2 .$$

The density as a Boltzmann-Gibbs factor

We can write

$$\mu^{(N)}(Z) \quad = \quad \mathcal{Z}_N^{-1} \exp\left(\sum_{j=1}^{N} -N|z_j|^2 + 2\ell \sum_{i<j} \log|z_i - z_j| \right)$$

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$$\begin{aligned}\mu^{(N)}(Z) &= \mathcal{Z}_N^{-1}\exp\left(\sum_{j=1}^{N} -N|z_j|^2 + 2\ell\sum_{i<j}\log|z_i - z_j|\right)\\ &= \mathcal{Z}_N^{-1}\exp\left(-\frac{1}{T}\mathcal{H}_N(Z)\right),\end{aligned}$$

with $T = N^{-1}$ and

$$\mathcal{H}_N(Z) = \sum_{j=1}^{N}|z_j|^2 - \frac{2\ell}{N}\sum_{i<j}\log|z_i - z_j|.$$

Mean field limit

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The probability measure $\mu^{(N)}(Z)$ minimizes the free energy functional

$$\mathcal{F}(\mu) = \int \mathcal{H}_N(Z)\mu(Z) + T \int \mu(Z) \log \mu(Z)$$

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The $N \to \infty$ limit is in this interpretation a mean field limit where at the same time $T \to 0$. It is thus not unreasonable to expect that for large N, in a suitable sense

$$\mu^{(N)} \approx \rho^{\otimes N}$$

with a one-particle density ρ minimizing a mean field free energy functional.

Mean field limit (cont.)

The mean field free energy functional is defined as

$$\mathcal{E}_N[\rho] = \int_{\mathbb{R}^2} |z|^2 \rho - \ell \int \int \rho(z) \log|z - z'| \rho(z') + N^{-1} \int_{\mathbb{R}^2} \rho \log \rho$$

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It has a minimizer ρ_N among probability measures on $\mathbb{R}^2$ and this minimizer should be a good approximation for the scaled 1-particle probability density of the trial wave function, i.e.,

$$\mu_N^{(1)}(z) = \int_{\mathbb{R}^{2(N-1)}} \mu^{(N)}(z, z_2, \dots, z_N) \mathrm{d}^2 z_2 \dots \mathrm{d}^2 z_N .$$

The Mean Field Limit Theorem

Such mean field limits have been studied, using compactness arguments, by many people, including Kiessling, Spohn, Messer, Caglioti, Lions, Marchioro, Pulvirenti,.... . In joint work of N. Rougerie, S. Serfaty and JY quantitative estimates on the approximation of $\mu_N^{(1)}$ by ρ_N are proved:

Theorem (Comparison of true density and mean field density)

There exists a constant $C > 0$ such that for large enough N and any $U \in H^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$

$$\left| \int_{\mathbb{R}^2} \left(\mu_N^{(1)} - \rho_N \right) U \right| \leq C(\log N/N)^{1/2} \|\nabla U\|_{L^1} + CN^{-1} \|\nabla^2 U\|_{L^\infty}.$$

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Ingredients of the Proof

The proof is based on upper and lower bounds for the free energy. For the upper bound one uses ${\rho_N}^{\otimes N}$ as a trial measure.

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The estimate on the density follows essentially from the fact that the positive Coulomb energy $D(\mu^{(1)} - \rho^{\mathrm{MF}}, \mu^{(1)} - \rho^{\mathrm{MF}})$ is squeezed between the upper and lower bounds to the free energy.

Properties of the Mean Field Density

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with "$=$" where $\hat{\rho} > 0$ and "$>$" where $\rho = 0$.

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Applying the Laplacian gives

$$1 - \ell\pi\,\hat{\rho}(z) = 0$$

where $\hat{\rho} > 0$. Hence $\hat{\rho}$ takes the constant value $(\ell\pi)^{-1}$ on its support.

General incompressibility estimates

Consider now general states of the form

$$\Psi(z_1, \dots, z_N) = \phi(z_1, \dots, z_N) \prod_{i<j} (z_i - z_j)^\ell e^{-\sum_{j=1}^N |z_j|^2/2}$$

with ℓ even ≥ 2 for bosons, or ℓ odd ≥ 3 for fermions, and $\phi(z_1, \dots, z_N)$ a symmetric holomorphic function.

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These states constitute the kernel $\ker\mathcal{I}_N$ of

$$\mathcal{I}_N=\sum_{i<j}\delta(z_i-z_j)$$

that is well defined (even a bounded operator) in the LLL. In fact, "$\delta(z_i-z_j)$" is equivalent to

$$(\delta_{ij}\varphi)(z_i,z_j)=\frac{1}{2\pi}\varphi\left(\tfrac{1}{2}(z_i+z_j),\tfrac{1}{2}(z_i+z_j)\right).$$

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For other repulsive potentials peaked at the origin, e.g. $|z_i - z_j|^{-1}$, the interaction is not zero, but suppressed by the $(z_i - z_j)^\ell$ factors.

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and that the density $\mu^{(1)}$ satisfies the "incompressibility bound"

$$\mu^{(1)}(z) \le \frac{1}{\ell\pi}.$$

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But:

- The bound for $\mu^{(1)}$ will certainly not hold point-wise for finite N.
- A bound for arbitrary ‘correlation factors’ ϕ is out of reach at present.

It is, however, possible to derive such bounds for ϕ of the form

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\ldots,N\}} f_2(z_i, z_j) \ldots \prod_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}} f_n(z_{i_1}, \ldots, z_{i_n}).$$

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We have done this explicitly for the special case

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\dots,N\}} f_2(z_i, z_j),$$

but extensions to arbitrary fixed n (or n not growing too fast with N) are possible.

For such factors we prove that the potential energy is, indeed, bounded below by the bathtub energy in the limit $N \to \infty$.

The proof proceeds in two steps:

- Comparison of the free energy with the energy defined by the functional (MF energy functional without entropy term)

$$\hat{\mathcal{E}}[\rho] = \int_{\mathbb{R}^2} \left(|z|^2 - \frac{2}{N} \log |g_1(z)| \right) \rho(z) dz + \int_{\mathbb{R}^4} \rho(z) \big(-\ell \log |z - z'| - \log |g_2(z, z')| \big) \rho(z') dz dz'$$

where $g_1(z) = f_1(\sqrt{N} z)$, $g_2(z, z') = f_2(\sqrt{N} z, \sqrt{N} z')$.

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$$\hat{\mathcal{E}}[\rho] = \int_{\mathbb{R}^2} \left(|z|^2 - \frac{2}{N} \log |g_1(z)| \right) \rho(z) dz + \int_{\mathbb{R}^4} \rho(z) \big(-\ell \log |z - z'| - \log |g_2(z, z')| \big) \rho(z') dz dz'$$

where $g_1(z) = f_1(\sqrt{N} z)$, $g_2(z, z') = f_2(\sqrt{N} z, \sqrt{N} z')$.

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- Showing that the density of the minimizer of the MF functional satisfies the incompressibility bound.

Bound on the MF density

The variational equation for the modified MF functional is

$$|z|^2 - \frac{2}{N}\log|g_1(z)| - 2\int \log|g_2(z,z')|\rho(z')dz' - 2\ell\rho * \log|z| - C = 0$$

on the support of ρ, and applying $\frac{1}{4}\Delta$ gives

$$1 - (1/2N)\Delta\log|g_1(z)| - \tfrac{1}{2}\int \Delta_z \log|g_2(z,z'|)\rho(z')dz' - \ell\pi\,\rho(z) = 0.$$

But $\Delta\log|g_1(z)| \geq 0$ and $\Delta_z \log|g_2(z,z'| \geq 0$, so

$$\rho(z) \leq \frac{1}{\ell\pi}.$$

Justification of the mean field approximation

The first step, however, i.e., the justification of the mean field approximation, is less simple.

Note that $-\log|g_2(z,z')|$ can be much more intricate than $-\log|z-z'|$ and is not of positive type in general so the previous arguments for the mean field limit do not apply.

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Instead we use a theorem of Diaconis and Freedman. This is a quantitative version of the Hewitt-Savage theorem.

The latter says essentially that the n-th marginals of a symmetric probability measure on a S^N can, for $N\to\infty$, be approximated by a convex combination of pure tensor products, $\rho^{\otimes n}$.

The Diaconis-Freedman Theorem

Theorem (Diaconis-Freedman)

Let S be a measurable space and $\mu \in \mathcal{P}_s(S^N)$ be a probability measure on S^N invariant under permutation of its arguments. There exists a probability measure $P_\mu \in \mathcal{P}(\mathcal{P}(S))$ such that, denoting

$$\tilde{\mu} = \int_{\rho \in \mathcal{P}(S)} \rho^{\otimes N} dP_\mu(\rho),$$

we have

$$\left\| \mu^{(n)} - \tilde{\mu}^{(n)} \right\|_{\mathrm{TV}} \leq \frac{n(n-1)}{N}.$$

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$$\left\| \mu^{(n)} - \tilde{\mu}^{(n)} \right\|_{\mathrm{TV}} \leq \frac{n(n-1)}{N}.$$

In addition, the marginals of $\tilde{\mu}$ are explicitly given by those of μ:

$$\tilde{\mu}^{(n)}(x_1, \ldots, x_n) = \frac{1}{N^n} \sum_{j=1}^{n} \sum_{1 \leq i_1 \neq \ldots \neq i_j \leq N} \mu^{(j)}(x_{i_1}, \ldots, x_{i_j})\, \delta_{x_{i_{j+1}} = \ldots = x_{i_n}}.$$

Incompressibility Bounds

With the aid of the DF theorem we prove:

Theorem (Weak incompressibility Bound)

Let Ψ be a wave function in the Laughlin phase with a correlation factor of the type described and $\mu^{(1)}$ its scaled 1-particle probability density. Let V be a smooth potential with $\inf_{|x|\geq R} V(x) \to \infty$ for $R \to \infty$. Then

$$\liminf_{N\to\infty} \int \mu^{(1)}(z)V(z)dz \geq \inf\left\{\int V\rho : 0 \leq \rho \leq (\ell\pi)^{-1}, \int \rho = 1\right\}.$$

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Moreover, for a radially monotone, or Mexican hat shaped potential this bound is optimal (within the class of states considered) and saturated by the Laughlin state, resp. by 'quasi hole' states.

Incompressibility Bounds (cont.)

The weak incompressibility bound is a corollary of the following quantitative estimate of the potential energy.

Theorem (Quantitative incompressibility estimate)

Let $\mu^{(1)}$ be be as in the previous Theorem. Pick some test one-body potential U such that $U,\ \Delta U \in L^\infty(\mathbb{R}^2)$. For any N large enough and ε small enough there exists an absolutely continuous probability measure $\rho \in L^1(\mathbb{R}^2)$ satisfying

$$\rho \leq \frac{1}{\pi\ell} + \varepsilon \frac{\sup|\Delta U|}{4\pi\ell}$$

such that

$$\int_{\mathbb{R}^2} U\mu^{(1)} \geq \int_{\mathbb{R}^2} U\rho - C(N\varepsilon)^{-1}\mathrm{Err}(f_1, f_2)$$

where $\mathrm{Err}(f_1, f_2)$ can be estimated in terms of the degrees of the polynomials f_1 and f_2.

Regularization

The explicit form of the marginals of the measure $\tilde{\mu}$ is important for deriving this estimate.

However, the presence of the δ functions in the formula for $\tilde{\mu}$ requires a regularization of $\log|z - z'|$ around $z = z'$ and likewise of $\log|g_2(z, z')|$ around the points where $g_2(z, z') = 0$.

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This is done by replacing the $\log$ with a regularized

$$\log_\alpha |z| = \int \log|z - w|\, \delta_\alpha(w) dw$$

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where δ_α the density of a unit charge smeared uniformly over a disc of radius $\alpha > 0$.

Newton's theorem guarantees that $\log = \log_\alpha$ outside of the disc and one has

$$\Delta \log_\alpha |z| = 2\pi\delta_\alpha(z).$$

Moreover, $\Delta \log|f(z)| \geq 0$ and $\Delta \log_\alpha |f(z)| \geq 0$ for all analytic functions f.

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Bound on the MF density (cont.); higher correlation factors

Since $\Delta \log |g_1(z)| \geq 0$ and $\Delta_z \log_\alpha |g_2(z, z')| \geq 0$, the MF minimizer satisfies

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The inclusion of higher correlation factors, $g_n(z_{i_1}, \ldots, z_{i_n})$, $n \geq 3$, leads to an n particle interaction potential $\log |g_n(z_{i_1}, \ldots, z_{i_n})|$ that can, in principle, be treated in an analogous way. Diaconis-Friedmann is, however, only applicable for $n \ll \sqrt{N}$. Moreover, the number of terms increases like $\binom{N}{n}$ which leads to some complications for the comparison of the mean field density with the true density.

Conclusions

Using a notion of incompressibility based on a lower bound to the potential energy in terms of a 'bath tub' energy, we have shown that this property holds for a large class of states derived from the Laughlin state(s).

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Open problem:

Show that for general external potentials V the variational problem of minimizing the energy within the constrained class of functions $\mathrm{Ker}(\mathcal{I}_N)$ is, in the limit $N \to \infty$, solved by wave functions of the form

$$\prod_{j=1}^{N} f_1(z_j)\Psi^{(\ell)}_{\mathrm{Laugh}}(z_1, \dots, z_N).$$

with $\ell = 2$ for bosons and $\ell = 1$ for fermions. For sufficiently strong Coulomb repulsion $\ell = 3$ should be favored for fermions.

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