On a stochastic model of damage and rupture

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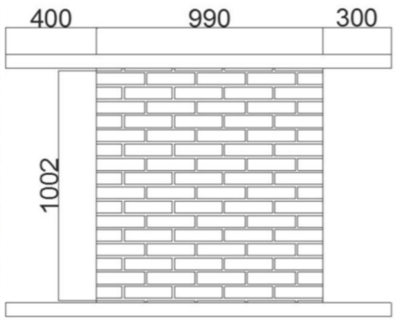
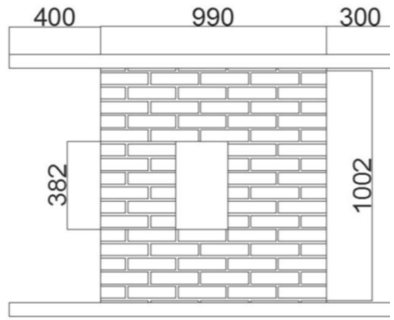
PHC Galileo with Milan

Summary

- Context
- Some results in the deterministic case
- Recalls on some mathematical tools
- Analysis of the stochastic problem

Some problems



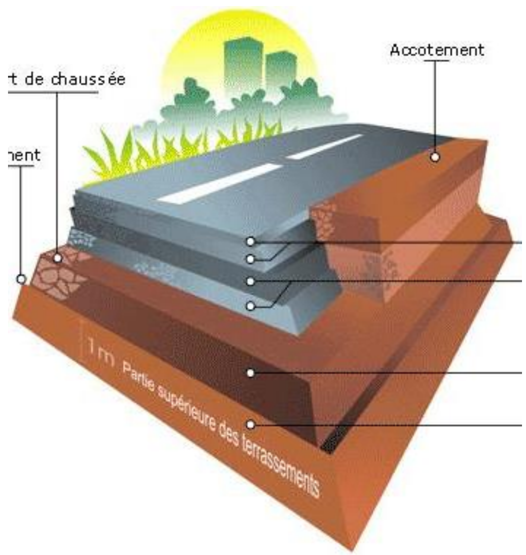


Modelling of masonry structures

Fouchal et al 2009, 2014
Rekik-Lebon 2010, 2012
Pelissou-Lebon 2009, Sacco-Lebon 2012
Raffa et al. 2016, 2017

Quasi-brittle rupture

Interfaces in pavements
Coll. Univ. Limoges



Roughness

Scale mm/cm

Some problems

Strength
Fragile rupture

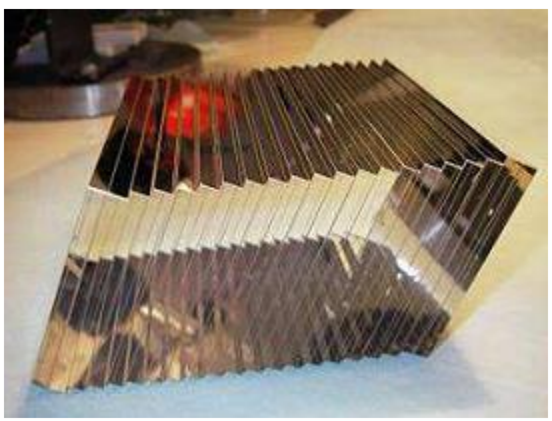
GFRP/Sikadur-30

Coll. Univ. Salerno

Cycling
Damage

GFRP/Kerabuild Eco Epobond

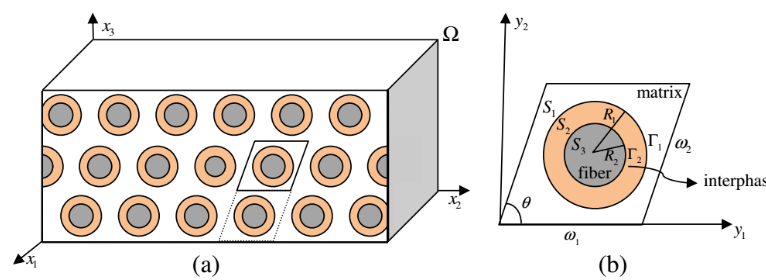
Coll. Univ. Salerno



Shocks
Dynamics

Glass/Glass

Coll. CNES, THALES

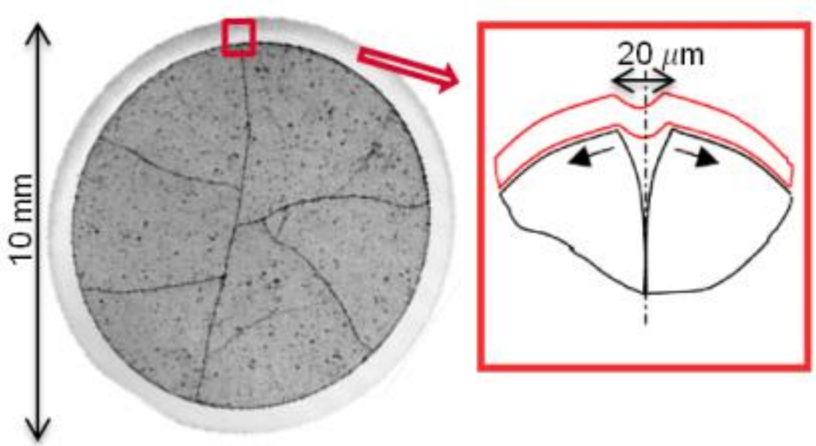


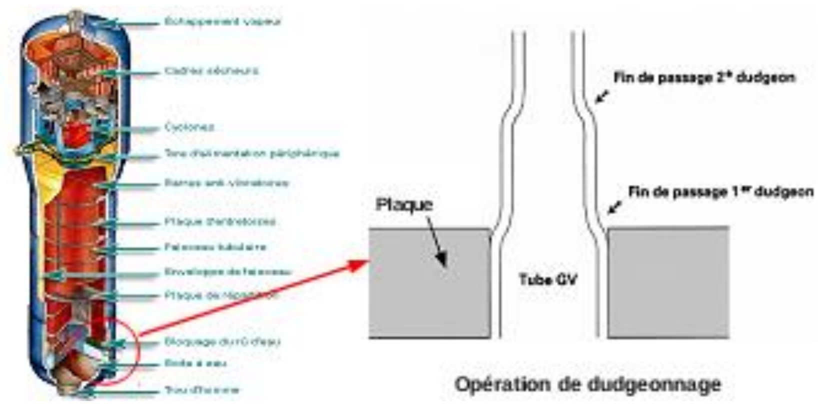
Scale μm, nm

$BaTiO_3$/Terfenol-D/$CoFe_2O_4$

Coll. Univ. La Havane

Some problems





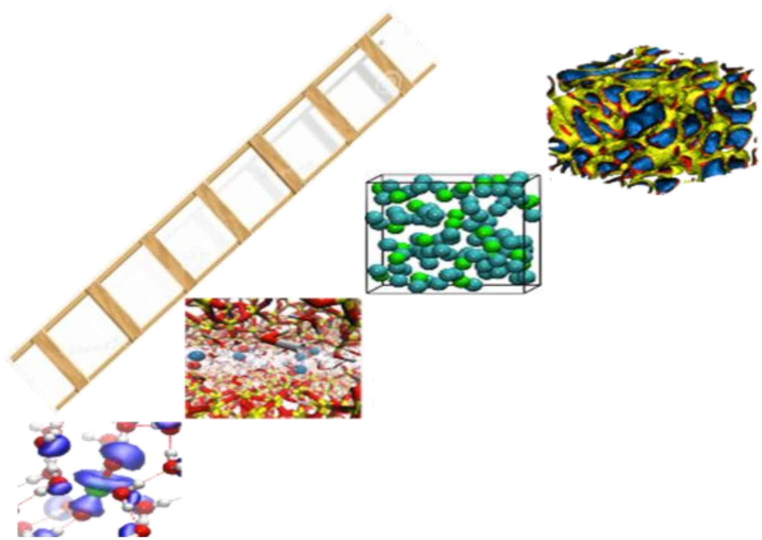
Contact UO2/Zirconium
Coll. CEA

Contact Steel/Steel
Coll. EDF

Scale mm

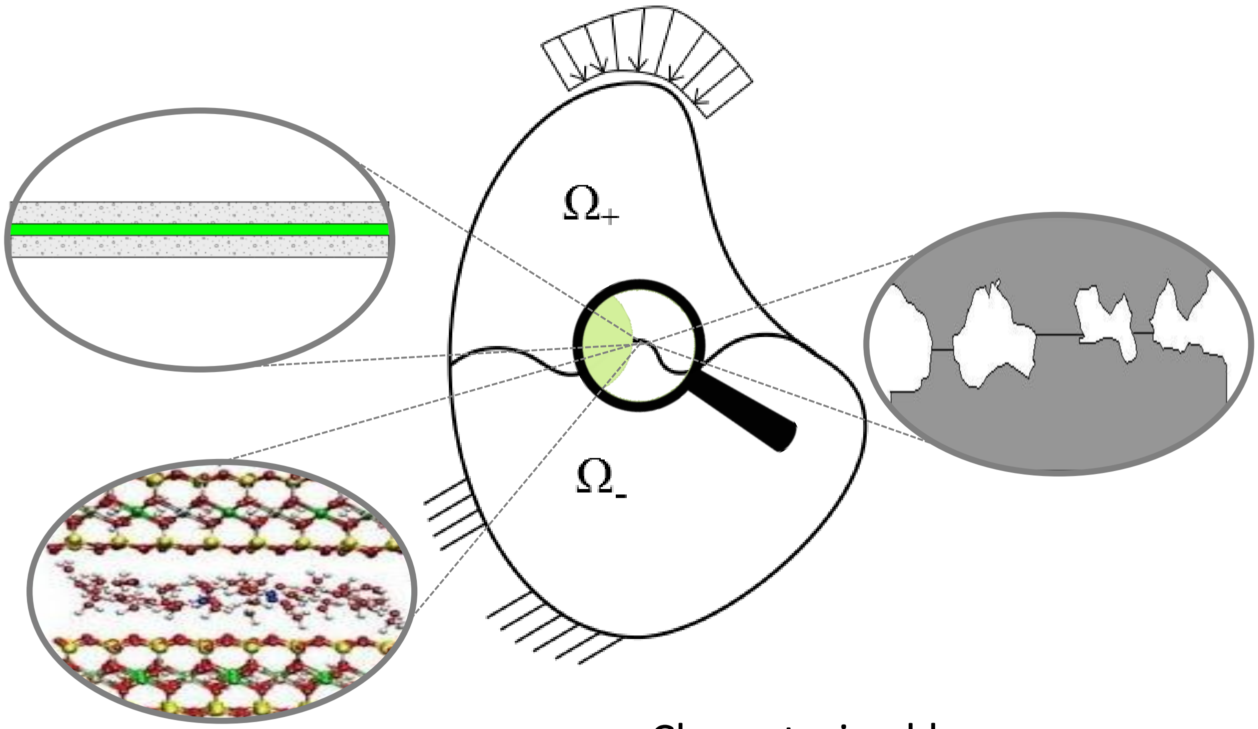
Scale μm

- Multi-scale problems (from nm to structure)
- Various physical problems
- A large number of models



Objective of the lecture

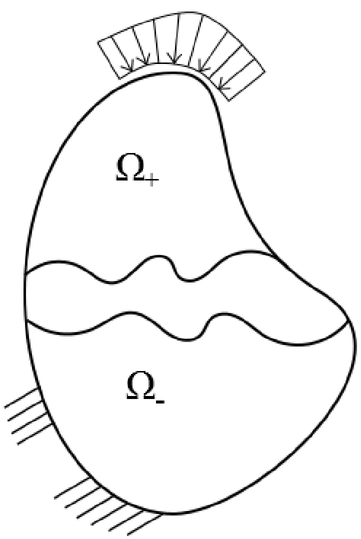
- Thin films, contact areas, "molecular" interaction areas,...

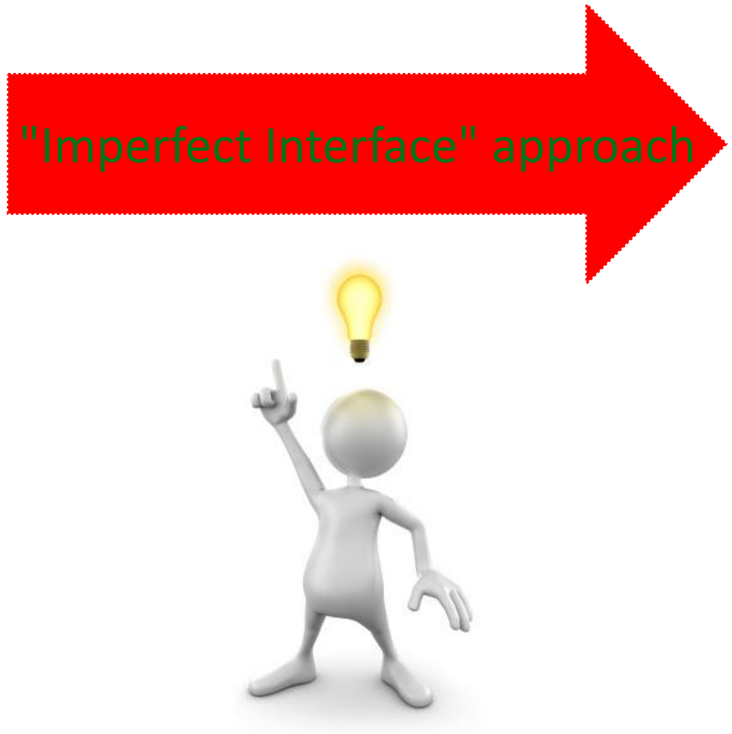


Characterized by
- low thickness
- low stiffness
- cracks, **damage**,...

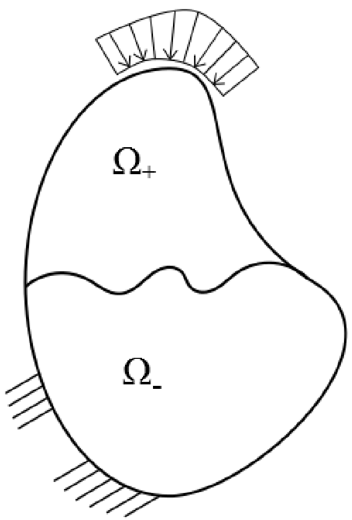
Context: "Imperfect Interface" approach

Interphase





Interface

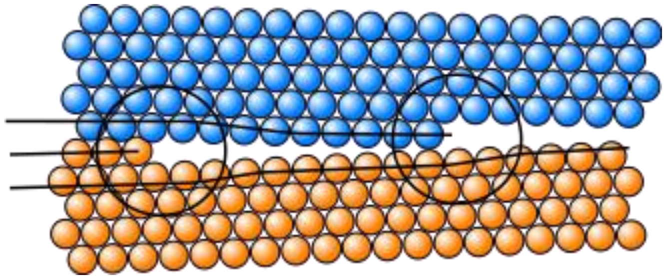


At least one parameter: ε thickness of the interphase

Idea: study the problem when $\varepsilon \to 0$

« Bonding » problems

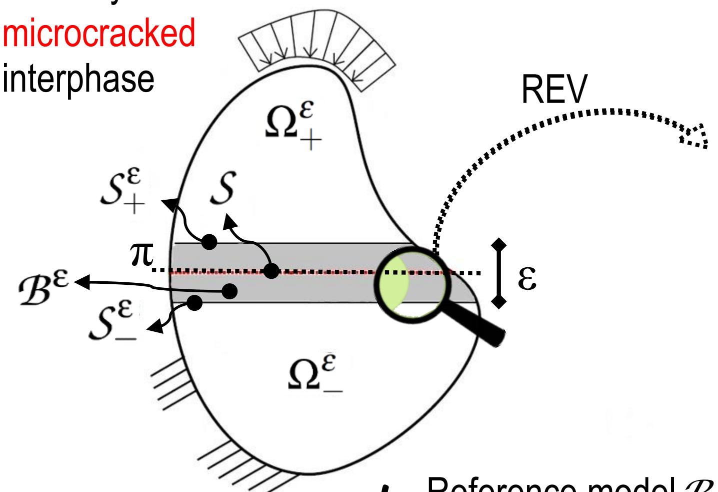
- The *perfect* interface is defined, from a mechanical point of view, as a surface through which the displacements and the stress vectors are continuous.
- This assumption of perfect interface is **inappropriate** in many engineering problems.

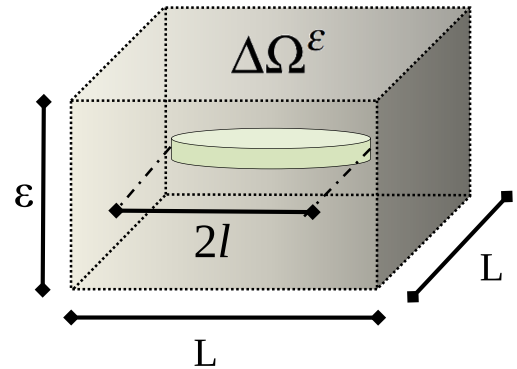


Imperfect Interface: Jumps
- In the displacements
- and/or In the stress vector

Context: Microcracked media

Auxiliary model $\mathscr{A}$:
microcracked
interphase

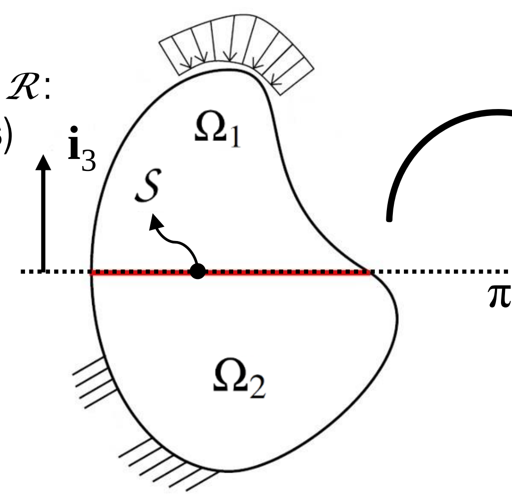




Micromechanical homogenization with the hypothesis of non-interaction between microcrackss, "stress-based approach "

Eshelby J. D.,1957,1961
Kachanov M., 1994
Tsukrov I. and Kachanov M., 1998
Sevostianov I. and Kachanov M. 2012, 2014, 2015

Reference model $\mathscr{R}$:
interface (springs)



Interface constitutive equation
<u>**soft linear or non-linear**</u>

$$\boldsymbol{\sigma}\,\mathbf{i}_3 = \frac{\mathbb{B}^\varepsilon_{\cdot 3\cdot 3}}{\varepsilon}[[\mathbf{u}]]$$

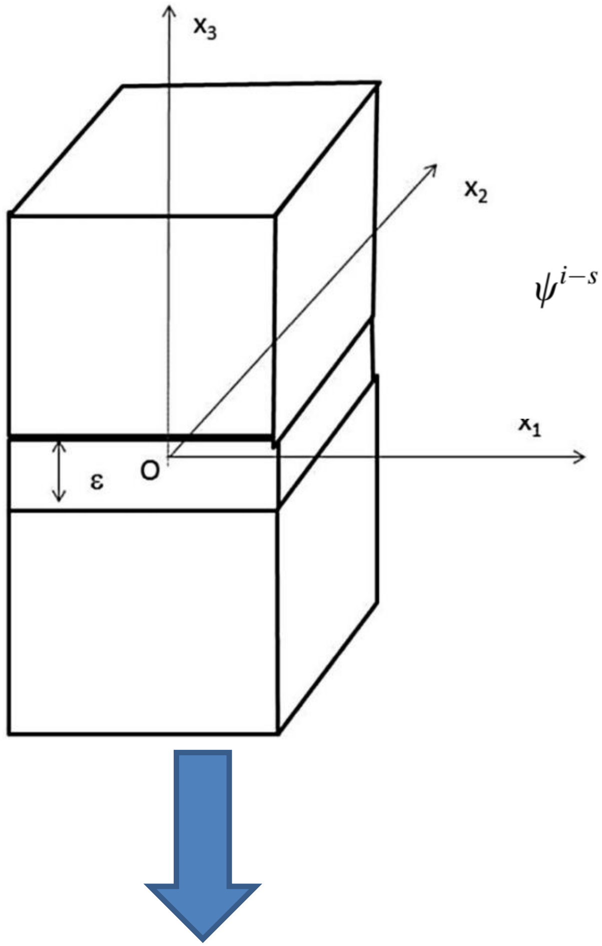
$$\mathbf{P}\,\mathbf{i}_3 = \frac{A^\varepsilon_{3333}}{2\,\varepsilon^3}[[\mathbf{u}]]^2[[\mathbf{u}]]$$

Matching asymptotic expansions in small perturbation or finite strains (St. Venant-Kirchhoff material)

Raffa, M. L. et al., 2016
Rizzoni, R. et al., 2015
Rekik, A. and Lebon, F., Adv. Eng. Software, 2012
Rekik, A. and Lebon, F., Int. J. Solid Struct., 2010
Fouchal, F. et al., Open Civ. Eng. J., 2014

Attention : dependancy en *l*

Bonetti E. et al. 2017



$$\sigma^{\varepsilon}_{ij} = a^{\pm}_{ijhk} e_{hk}(u^{\varepsilon})$$

Free energy of the microcracked media

$$\psi^{i}(e(u^{\varepsilon}), l)) = \psi^{i-s}(e(u^{\varepsilon}), l) - \omega^{\varepsilon} l + I_{[l_0, +\infty[}(l)$$

$$\psi^{i-s}(e(u^{\varepsilon}), l) = \begin{cases} \frac{1}{2} \varepsilon b(l) e(u^{\varepsilon}) : e(u^{\varepsilon}) & \text{if } e^{s}(u^{\varepsilon}) \geq 0 \\ \frac{1}{2} (e^{s}(u^{\varepsilon}))^2 B^{\varepsilon}_{ijhk}(l) \delta_{ij} \delta_{hk} + \frac{1}{2} B^{\varepsilon}_{ijhk}(l) e^{d}(u^{\varepsilon}) : e^{d}(u^{\varepsilon}) & \text{if } e^{s}(u^{\varepsilon}) \leq 0 \end{cases}$$

$$\sigma^{\varepsilon}_{ij} = \begin{cases} \varepsilon b_{ijhk}(l) e_{hk}(u^{\varepsilon}) & \text{if } e^{s}(u^{\varepsilon}) \geq 0 \\ e^{s}(u^{\varepsilon}) B^{\varepsilon}_{ijhk}(l) \delta_{hk} + B^{\varepsilon}_{ijhk}(l) e^{d}_{hk}(u^{\varepsilon}) & \text{if } e^{s}(u^{\varepsilon}) \leq 0 \end{cases}$$

(bilinear in this case)

$$\phi(\dot{l}) = \frac{1}{2}\eta^{\varepsilon}\dot{l}^2 + I_{[0,+\infty[}(\dot{l})$$

$$\eta^{\varepsilon}\dot{l} = \begin{cases} \left(\omega^{\varepsilon} - \frac{1}{2}\varepsilon b_{,l}(l)e(u^{\varepsilon}) : e(u^{\varepsilon})\right)_+ & \text{if } e^s(u^{\varepsilon}) \geq 0 \\ \left(\omega^{\varepsilon} - \frac{1}{2}B^{\varepsilon}_{,l}(l)e(u^{\varepsilon}) : e(u^{\varepsilon})\right)_+ & \text{if } e^s(u^{\varepsilon}) \leq 0 \end{cases}$$

Context: Microcracked media with damage

$$\begin{cases} \mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon\, \mathbf{u}^1 + o(\varepsilon) \\ \sigma^\varepsilon = \sigma^0 + \varepsilon\, \sigma^1 + o(\varepsilon). \end{cases}$$

Matching asymptotic expansion

$$\begin{cases} \sigma^0_{ij,j} + f_i = 0 & \text{in } \Omega_\pm \\ \sigma^0_{ij} n_j = g_i & \text{on } S_g \\ u^0_i = 0 & \text{on } S_u \\ \sigma^0_{ij} = a^\pm_{ijhk} e_{hk}(u^0) & \text{in } \Omega_\pm \\ \sigma^0_{i2} = K^{22}_{ij}(l)\left[u^0_j\right]_+ + \tau^0 \delta_{i2} & \text{on } S \\ \left[u^0_2\right]\tau^0 = 0 & \text{on } S \\ \eta \dot{l} = \left(\omega - \frac{1}{2} K^{22}_{,l}(l)\left[u^0\right]_+ \cdot \left[u^0\right]_+\right)_+ & \text{on } S \end{cases}$$

$$K^\varepsilon = \begin{bmatrix} E_0 & \frac{\nu_0 L\varepsilon}{2l^2 C} & 0 \\ \frac{\nu_0 L\varepsilon}{2l^2 C} & \frac{L\varepsilon}{2l^2 C} & 0 \\ 0 & 0 & \frac{L\varepsilon}{l^2 C} \end{bmatrix}$$

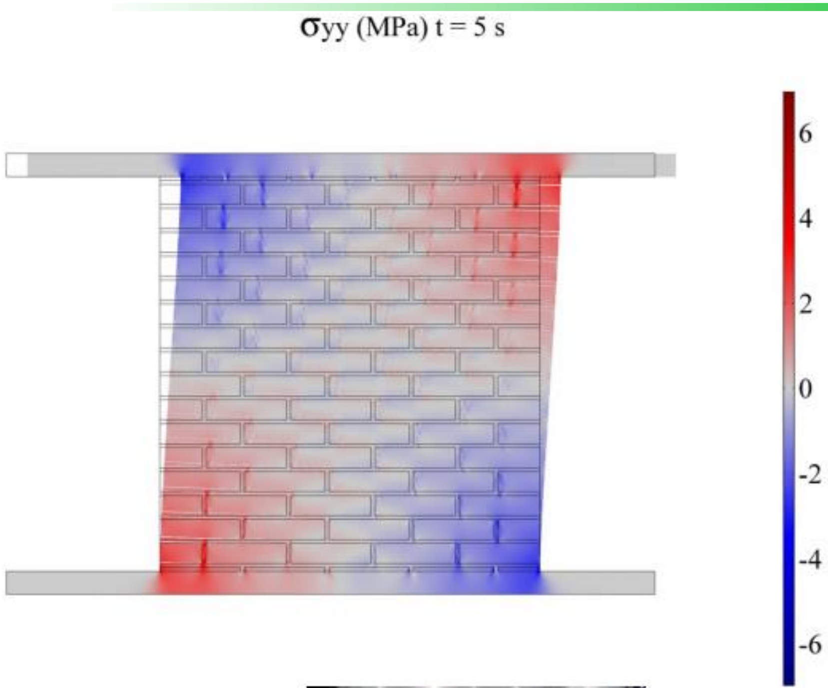
$$K^{22} = \begin{bmatrix} \frac{L}{2l^2 C} & 0 \\ 0 & \frac{L}{l^2 C} \end{bmatrix}$$

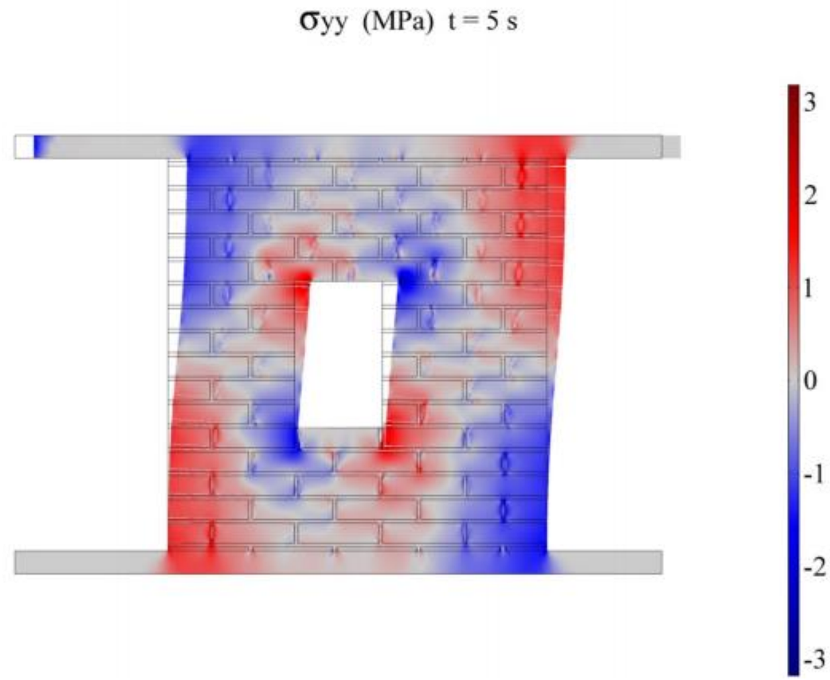
$$\eta \dot{l} = \left(\omega + \frac{L}{2l^3 C}\left[u^0_1\right]^2 + \frac{L}{l^3 C}\left[u^0_2\right]^2_+\right)_+$$

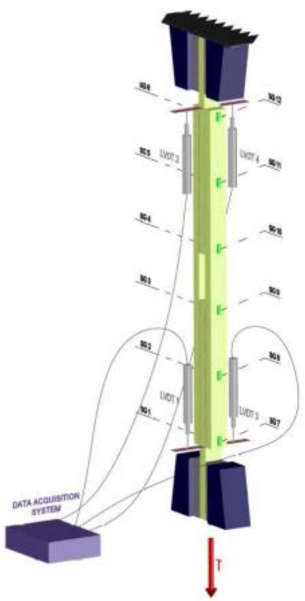
$\chi = l/l_0$

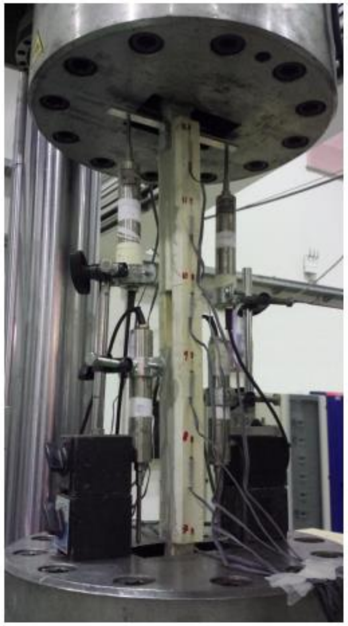
Damage variable

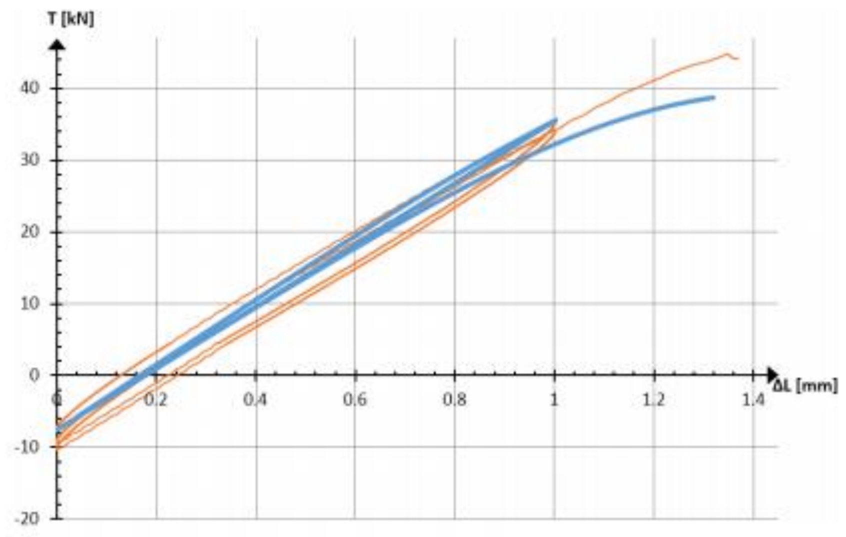
Numerical results



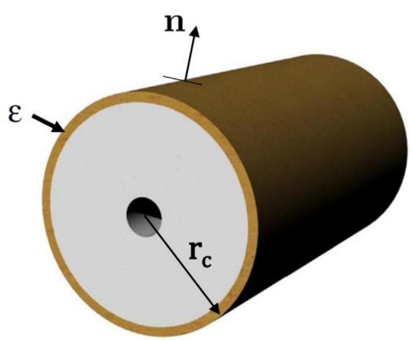






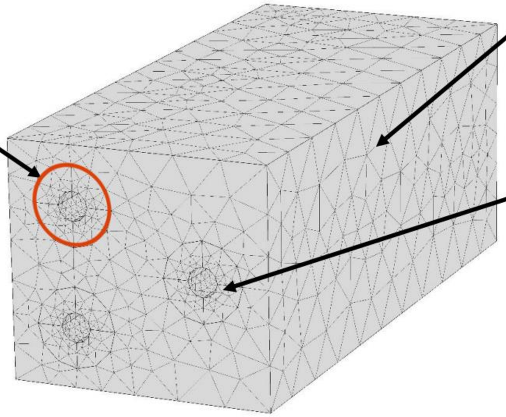


Numerical results



CEMENT LINE

$E^{cl} = 52$ MPa
$\nu^{cl} = 0.33$
$r_c = 0.068$ mm
$\varepsilon = 0.005$ mm

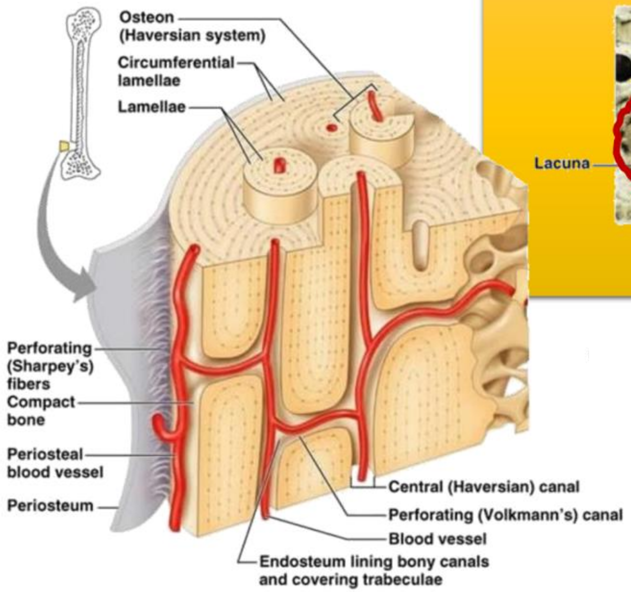


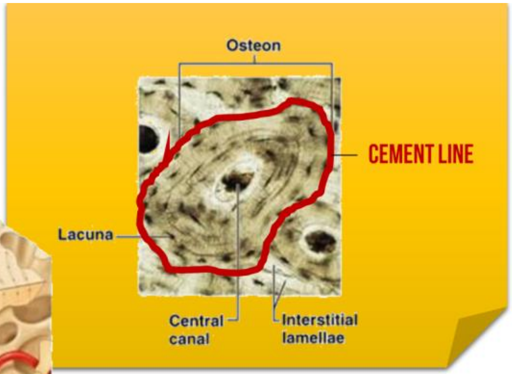
INTERSTITIAL BONE

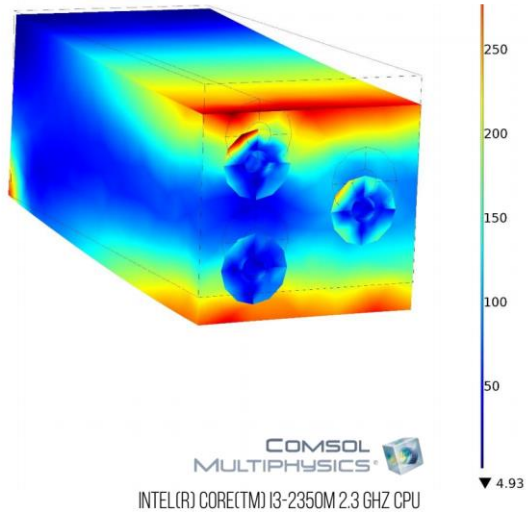
$E^{LB} = 8.85$ GPa
$\nu = 0.3$

OSTEONEAL BONE

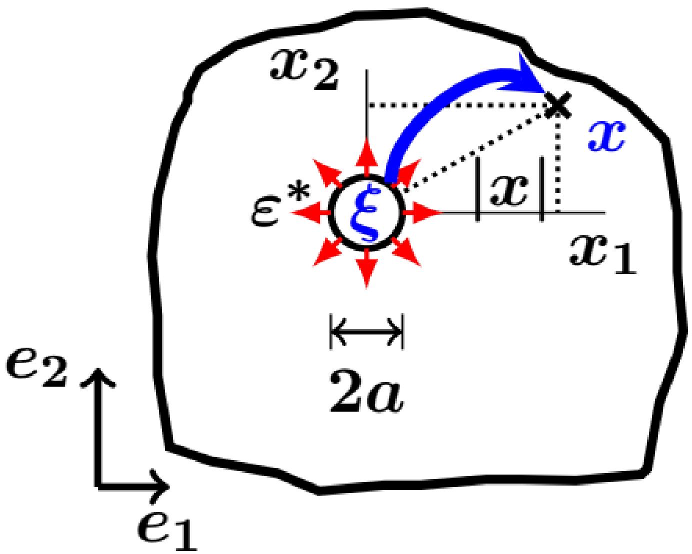
$E_l^o = 5$ GPa
$E_t^o = 150$ MPa
$G_l^o = 20$ GPa
$\nu = 0.3$







Non local aspects



Enrichment of the constitutive equations (softening behavior): taken into account at a distance interactions

Internal length

Gradient of the damage variable

Frémond, M., Nedjar, B. 1995
Mazars, J. 1984
Pijaudier-Cabot, G., Bazant, Z. 1987

Local damage

$$\eta\dot{\chi} = \left(\omega - \frac{1}{2}\varepsilon b_{,\chi}(\chi)e(u):e(u)\right)_+$$

$$\chi \in [0,1]$$

Equivalently

$$-(\dot{\chi} - \omega_s(\chi) - f) \in \partial I_{[0,1]}$$

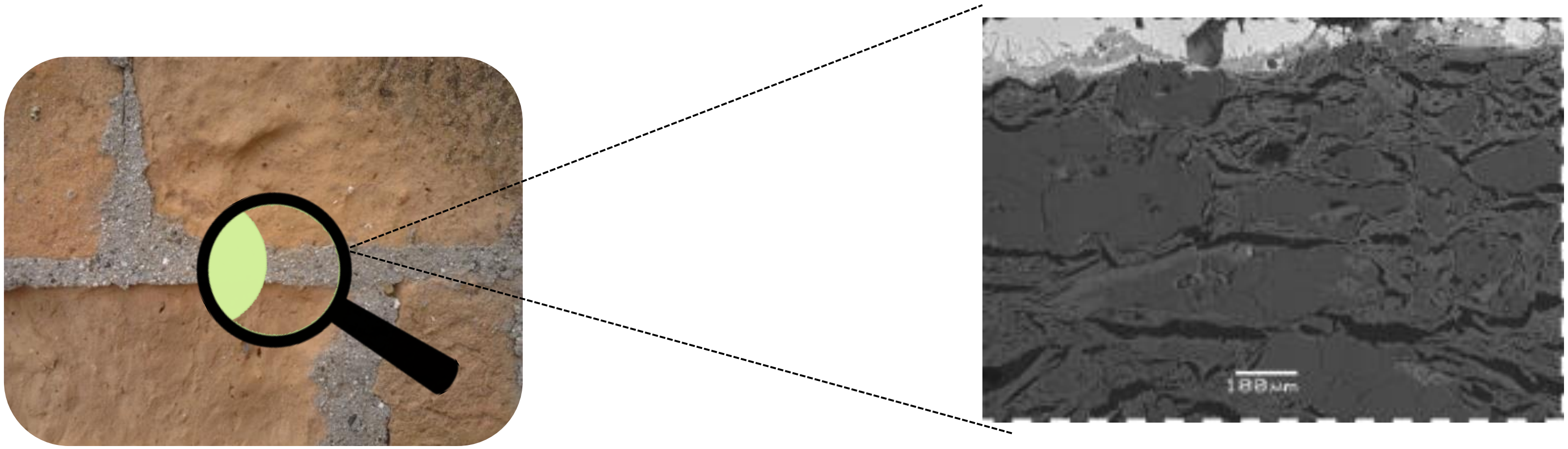
$$\partial I_{[0,1]}(x) = \begin{cases} \{0\} & if\ x \in]0,1[\\ \mathbb{R}^- & if\ x = 0, \\ \mathbb{R}^+ & if\ x = 1. \end{cases}$$

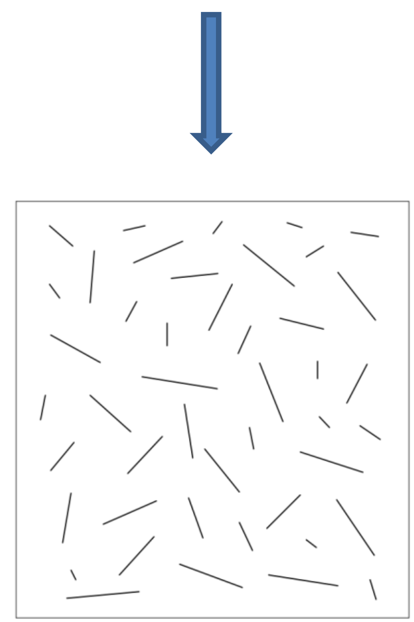
Non local damage version

$$-(\dot{\chi} - \omega_s(\chi) - f - \Delta\chi) \in \partial I_{[0,1]}$$

(Allen-Cahn type)

Random distribution of cracks





How to take into account this random distribution?

$$w_s(\chi) + f - \partial_t\left(\chi - \boxed{\int_0^t h(\chi)\,dW}\right) + \Delta\chi \;\in\; \partial I_{[0,1]}(\chi)$$

Brownian Motion

Mathematical study

$$\begin{cases} w_s(\chi) + f - \partial_t\Big(\chi - \int_0^t h(\chi)dW\Big) + \Delta\chi & \in & \partial I_{[0,1]}(\chi) & \text{in } \Omega \times D \times (0,T), \\ \chi(\omega, x, t=0) & = & \chi_0(x) & \omega \in \Omega, x \in D, \\ \nabla\chi.\mathbf{n} & = & 0 & \text{in } \Omega \times \partial D \times (0,T). \end{cases}$$

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $D \subset \mathbb{R}^d$, $d \geqslant 1$ and $T > 0$.
- χ the damage parameter, $0 \leqslant \chi \leqslant 1$.
- $w_s : \mathbb{R} \to [0, +\infty[$ a Lipschitz-continuous function with $w_s(0) = 0$.
- $f : \Omega \times D \times (0,T) \to \mathbb{R}$ a stochastic process in $L^2(\Omega \times (0,T) \times D)$.
- $h : \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous function with $h(0) = h(1) = 0$.
- $W = (W_t)_{0 \leqslant t \leqslant T}$ a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $\chi_0 : D \to \mathbb{R}$ the initial condition, $0 \leqslant \chi_0 \leqslant 1$ and $\chi_0 \in H^1(D)$.

Some elements of probability theory

- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a triplet composed by
  - A set Ω.
  - A sigma-field $\mathcal{F}$ on Ω.
  - A measure $\mathbb{P}$ on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$.
- A filtration associated with $\mathcal{F}$ is a family of sigma-fields $(\mathcal{F}_t)_{t \geqslant 0}$ satisfying
  - $\forall t \geqslant 0,\ \mathcal{F}_t \subset \mathcal{F}$.
  - $\forall s, t \geqslant 0,\ s \leqslant t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$.
- A random variable $X : \Omega \to \mathbb{R}$ is a $\mathbb{P}$-measurable application.
- The expectation of a random variable X is equal to $E[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega)$.
- A stochastic process $(X_t)_{0 \leqslant t \leqslant T}$ is a family of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

The brownian motion $W = (W_t)_{0\leqslant t\leqslant T}$

- $W = (W_t)_{0\leqslant t\leqslant T}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $W_0 = 0$ (standard brownian motion).
- $(\mathcal{F}_t)_{t\geqslant 0}$ the filtration associated with $\mathcal{F}$ and generated by the brownian motion W ($\mathcal{F}_t$ contains the "story" of W up to time t).
- $\forall t \in [0, T]$, $W_t : \Omega \to \mathbb{R}$ is a random variable $\mathcal{F}_t$-measurable.

$\forall s, t \in [0, T]$ with $t \geqslant s$

- $W_t - W_s \sim \mathcal{N}(0, t - s)$.
- $E\left[W_t - W_s\right] = 0$.
- $E\left[\left(W_t - W_s\right)^2\right] = t - s$.
- If X is a random variable $\mathcal{F}_s$-measurable then $E\left[(W_t - W_s)X\right] = 0$.

The Itô integral for a simple process

Set H a **Hilbert** space (for example $L^2(D)$ or $H^1(D)$).

Definition : simple process

$(\phi(t))_{0\leqslant t\leqslant T}$ is a simple process with values in H if there exist $0 = t_0 \leqslant \cdots \leqslant t_k \leqslant t_{k+1} = T$ and (k+1) random variables $\phi_0, \phi_1, \ldots, \phi_k : \Omega \to H$ such that

$$\begin{aligned} \phi(t) : \Omega &\to H \\ \omega &\mapsto \sum_{n=0}^{k} \phi_n 1_{[t_n, t_{n+1}[}(t). \end{aligned}$$

We denote by $S^2\big((0,T)\times\Omega; H\big)$ the set of simple processes with values in H.

The Itô integral of a simple process

$$\int_0^T \phi(s)dW(s) = \sum_{n=0}^{k} \int_{t_n}^{t_{n+1}} \phi(s)dW(s) = \sum_{n=0}^{k} \phi_n (W_{t_{n+1}} - W_{t_n}).$$

Properties of Itô integral

- **Zero average :** $E\Big[\int_0^T \phi(s)\,\mathrm{d}W(s)\Big] = 0.$
- **Itô isometry :**

$$E\Big[\|\int_0^T \phi(s)\,\mathrm{d}W(s)\|_H^2\Big] = E\Big[\int_0^T \|\phi(s)\|_H^2\,\mathrm{d}s\Big].$$

- **Linear continuity :** the application

$$\begin{aligned} S^2\big((0,T)\times\Omega;H\big) &\rightarrow C\big([0,T];L^2(\Omega;H)\big) \\ \phi &\mapsto \int_0^{\cdot} \phi(s)dW(s) \text{ is linear and continuous.} \end{aligned}$$

Extension of the Itô integral

To the predictable processes $X \in \mathcal{N}_W^2(0,T;H) \subset L^2\big((0,T)\times\Omega;H\big)$ using the density of $S^2\big((0,T)\times\Omega;H\big)$ in $L^2\big((0,T)\times\Omega;H\big)$ with the norm $E\big[\int_0^T \|.\|_X^2 ds\big]$.

Itô formula

Itô process

Every process with the form

$$X(t) = X(0) + \int_0^t A(s, X(s))ds + \int_0^t g(s, X(s))dW(s),$$

$\forall t \in [0, T]$ is called an Itô process.

Probabilistic writing : $dX(t) = A(t, X(t))dt + g(t, X(t))dW(t)$.

Derivation formula

$$\begin{array}{lrcl} \text{A smooth functional} & \psi & : & (0,T)\times H \to \mathbb{R} \\ & & & (t,v) \mapsto \psi(t,v) \\ \text{An Itô process} & \mathrm{d}X(t) & = & A(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W(t) \\ \textbf{Question :} & \mathrm{d}\psi(t,X) & = & ? \end{array}$$

Itô formula

$$\begin{aligned} \psi \quad &: \quad (0,T)\times H \to \mathbb{R} \\ &\quad\; (t,v) \mapsto \psi(t,v) \end{aligned}$$

$$\mathrm{d}X(t) = A(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W(t),$$

$$\mathrm{d}\psi(t,X) = \psi_t(t,X)dt + \psi_v(t,X)\,\mathrm{d}X(t)$$

Itô formula

$$\psi : (0,T)\times H \to \mathbb{R}$$
$$(t,v) \mapsto \psi(t,v)$$

$$\mathrm{d}X(t) = A(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W(t),$$

$$\begin{aligned}\mathrm{d}\psi(t,X) &= \psi_t(t,X)dt + \psi_v(t,X)\,\mathrm{d}X(t)\\ &= \psi_t(t,X)dt + \psi_v(t,X)\big(A(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W\big)\\ &+\frac{1}{2}\psi_{v,v}(t,X)g^2(t)dt.\end{aligned}$$

Some analysis automatisms that must be forgotten

- Itô integral is not an integral ! ! !
- Itô integral and absolute value :

$$\left|\int_0^T \varphi(s)dW(s)\right| \nleq \int_0^T |\varphi(s)|dW(s).$$

Some analysis automatisms that must be forgotten

- Itô integral is not an integral ! ! !
- Itô integral and absolute value :

$$\left|\int_0^T \varphi(s)dW(s)\right| \nleq \int_0^T |\varphi(s)|dW(s).$$

- Kind of Cauchy-Schwarz inequality ? NO !

$$\left|\int_0^T \varphi(s)\Psi(s)dW(s)\right| \nleq \left(\int_0^T \varphi^2(s)dW(s)\right)^{\frac{1}{2}} \left(\int_0^T \Psi^2(s)dW(s)\right)^{\frac{1}{2}}.$$

Some analysis automatisms that must be forgotten

- Itô integral is not an integral ! ! !
- Itô integral and absolute value :

$$\left|\int_0^T \varphi(s)dW(s)\right| \nleqslant \int_0^T |\varphi(s)|dW(s).$$

- Kind of Cauchy-Schwarz inequality ? NO !

$$\left|\int_0^T \varphi(s)\Psi(s)dW(s)\right| \nleqslant \left(\int_0^T \varphi^2(s)dW(s)\right)^{\frac{1}{2}} \left(\int_0^T \Psi^2(s)dW(s)\right)^{\frac{1}{2}}.$$

- Itô integral & order : $|\varphi(s)| \leqslant M \nRightarrow \int_0^T |\varphi(s)\Psi(s)|dW(s) \leqslant M \int_0^T |\Psi(s)|dW(s)$.

Some analysis automatisms that must be forgotten

- Itô integral is not an integral ! ! !
- Itô integral and absolute value :

$$\left|\int_0^T \varphi(s)dW(s)\right| \nleqslant \int_0^T |\varphi(s)|dW(s).$$

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$$\left|\int_0^T \varphi(s)\Psi(s)dW(s)\right| \nleqslant \left(\int_0^T \varphi^2(s)dW(s)\right)^{\frac{1}{2}} \left(\int_0^T \Psi^2(s)dW(s)\right)^{\frac{1}{2}}.$$

- Itô integral & order : $|\varphi(s)| \leqslant M \not\Rightarrow \int_0^T |\varphi(s)\Psi(s)|dW(s) \leqslant M \int_0^T |\Psi(s)|dW(s)$.
- If $V \hookrightarrow H$ compactly $\not\Rightarrow$ $L^2(\Omega, V) \hookrightarrow L^2(\Omega, H)$ compactly !

Tools of convex analysis : $\partial I_{[0,1]}(\chi)$?

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

Effective domain

The effective domain of a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is defined by :

$$\text{dom } f = \{x \in \mathbb{R} : f(x) < +\infty\}.$$

Example : The indicator function of $[0, 1]$

$$\begin{aligned} I_{[0,1]}(x) : \mathbb{R} &\to \overline{\mathbb{R}} \\ x &\mapsto \begin{cases} 0 & \text{if} \quad x \in [0,1], \\ +\infty & \text{else.} \end{cases} \end{aligned}$$

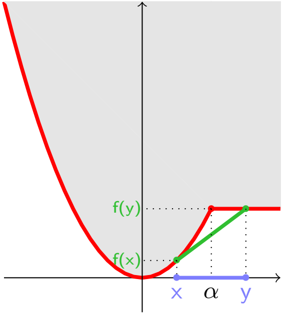
$$\text{Then, dom } I_{[0,1]} = [0, 1].$$

Convexity

A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is convex if its epigraph is convex in $\mathbb{R} \times \mathbb{R}$.
Equivalently if it satisfies

$$\forall x, y \in \text{dom } f, \forall t \in]0,1[, \; f\big((1-t)x + ty\big) \leqslant (1-t)f(x) + tf(y).$$

Examples :



The indicator function of $[0,1]$:

$$I_{[0,1]} : \mathbb{R} \to \overline{\mathbb{R}}$$
$$x \mapsto \begin{cases} 0 & \text{if} \quad x \in [0,1], \\ +\infty & \text{else.} \end{cases}$$

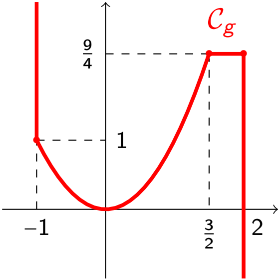
$$\text{dom } I_{[0,1]} = [0,1] \Rightarrow I_{[0,1]} \text{ convex.}$$

Multi-application or multivoque function

Consider two sets X and Y. A multi-application from X to Y is an application $g : X \to \mathcal{P}(Y)$. In other words,

$$\forall x \in X,\ g(x) = \varnothing \ \text{ or } \ g(x) = \{y\} \ \text{ or } \ g(x) = B,\ \text{with } y \in Y,\ B \subset Y.$$

Example :



$$g(x) = \begin{cases} [1, +\infty[& \text{if} \quad x = -1, \\ \{x^2\} & \text{if} \quad x \in \left[-1, \frac{3}{2}\right], \\ \left\{\frac{9}{4}\right\} & \text{if} \quad x \in \left[\frac{3}{2}, 2\right], \\ \left]-\infty, \frac{9}{4}\right] & \text{if} \quad x = 2, \\ \varnothing & \text{else.} \end{cases}$$

Subdifferentiability

Definition

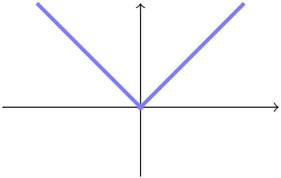
A convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that $\mathrm{dom}(f) \neq \emptyset$ is subdifferentiable at $x \in \mathrm{dom}(f)$ if there exists $x^* \in \mathbb{R}$ such that

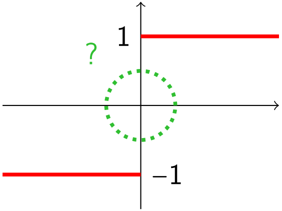
$$\forall y \in \mathbb{R},\ \ f(y) \geqslant f(x) + x^*(y - x).$$

Remarks

- The points x^* are called the sub-gradients of f at the point x.
- The set $\partial f(x) = \{x^* \in \mathbb{R} : x^* \text{ sub-gradient of } f \text{ at } x\}$ is called the sub-differential of f at x.
- If f is differentiable at x then $\partial f(x) = \{\nabla f(x)\}$.
- By convention, $\partial f(x) = \emptyset$ if $x \notin \mathrm{dom}\ f$.

Example 1 : $f : x \mapsto |x|$. Let us determine the subdifferential ∂f.





By definition, $x^* \in \partial f(0) \iff \forall y \in \mathbb{R},\ f(y) \geqslant f(0) + x^*(y-0)$

$\iff \forall y \in \mathbb{R},\ |y| \geqslant x^* y.$

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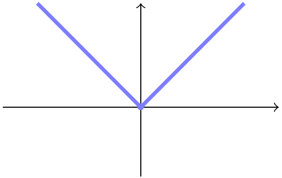
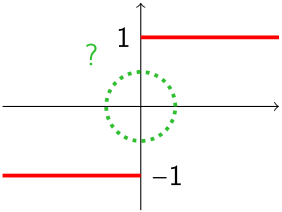
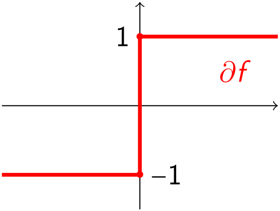




$$\text{By definition, } x^* \in \partial f(0) \iff \forall y \in \mathbb{R},\ f(y) \geqslant f(0) + x^*(y-0)$$
$$\iff \forall y \in \mathbb{R},\ |y| \geqslant x^* y.$$

- If $y \geqslant 0$: $y \geqslant x^* y \Leftrightarrow x^* \leqslant 1$.
- If $y \leqslant 0$: $-y \geqslant x^* y \Leftrightarrow x^* \geqslant -1$.

Thus, $\partial f(0) = [-1, 1]$.



Example 2 :

The indicator function :

$$I_{[0,1]} : \mathbb{R} \to \overline{\mathbb{R}}$$
$$x \mapsto \begin{cases} 0 & \text{if} \quad x \in [0,1], \\ +\infty & \text{else.} \end{cases}$$

By definition, $x^* \in \partial I_{[0,1]}(0) \quad \Leftrightarrow \quad \forall y \in \mathbb{R},\ I_{[0,1]}(y) \geqslant I_{[0,1]}(0) + x^*(y-0)$

$$\Leftrightarrow \quad \forall y \in \mathbb{R},\ I_{[0,1]}(y) \geqslant x^* y.$$

- If $y \in [0,1]$: $I_{[0,1]}(y) \geqslant x^* y \Leftrightarrow 0 \geqslant x^* y \Leftrightarrow x^* \leqslant 0$.
- If $y \notin [0,1]$: $I_{[0,1]}(y) = +\infty$ and so $I_{[0,1]}(y) \geqslant x^* y$.

Thus, $\partial I_{[0,1]}(0) = \mathbb{R}^-$.

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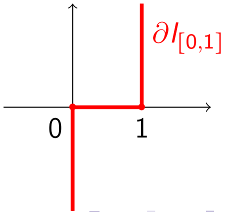
Thus, $\partial I_{[0,1]}(0) = \mathbb{R}^-$.

$$\partial I_{[0,1]} : [0,1] \to \mathcal{P}(\mathbb{R}), \quad x \mapsto \begin{cases} \{0\} & \text{if } x \in]0,1[, \\ \mathbb{R}^- & \text{if } x = 0, \\ \mathbb{R}^+ & \text{if } x = 1. \end{cases}$$



Our problem

Finding $0 \leqslant \chi \leqslant 1$ such that

$$w_s(\chi) + f - \partial_t\Big(\chi - \int_0^t h(\chi)dW\Big) + \Delta\chi \in \partial I_{[0,1]}(\chi)$$

$$\Leftrightarrow \begin{cases} w_s(\chi) + f - \partial_t\Big(\chi - \int_0^t h(\chi)dW\Big) + \Delta\chi \leqslant 0 & \text{if} \quad \chi = 0, \\ w_s(\chi) + f - \partial_t\Big(\chi - \int_0^t h(\chi)dW\Big) + \Delta\chi = 0 & \text{if} \quad \chi \in]0,1[, \\ w_s(\chi) + f - \partial_t\Big(\chi - \int_0^t h(\chi)dW\Big) + \Delta u \geqslant 0 & \text{if} \quad \chi = 1. \end{cases}$$

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Other writings

- PDE's researcher approach :

$$w_s(\chi) + f - \partial_t\Big(\chi - \int_0^t h(\chi)dW\Big) + \Delta\chi = \xi, \quad \text{with } \xi \in \partial I_{[0,1]}(\chi).$$

- Probability researcher approach :

$$d\chi = \big[w_s(\chi) + f + \Delta\chi - \xi\big]dt + h(\chi)dW, \quad \text{with } \xi \in \partial I_{[0,1]}(\chi).$$

Definition of a solution

Any pair $(\chi, \xi) \in \mathcal{N}_w^2(0, T, H^1(D)) \times \mathcal{N}_w^2(0, T, L^2(D))$ such that χ is in $L^\infty(0, T, L^2(\Omega, H^1(D))) \cap L^2(\Omega, C(0, T; L^2(D)))$ with $0 \leq \chi \leq 1$ is a solution to our stochastic problem if almost everywhere in $(0, T)$, $\mathbb{P}$-almost surely in Ω and for any v in $L^2(D)$

$$\int_D \partial_t \left(\chi - \int_0^{\cdot} h(\chi) dW(s) \right) v dx + \int_D \nabla \chi . \nabla v dx + \int_D \xi v dx = \int_D (w_s(\chi) + f) \, v dx,$$

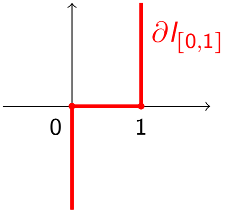
with $\chi(., 0) = \chi_0$ and $\xi \in \partial I_{[0,1]}(\chi)$.

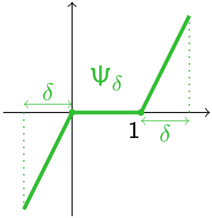
Sense of the initial condition

Since $\chi \in L^2(\Omega, C(0, T; L^2(D)))$, it satisfies the initial condition in the following sense

$$\mathbb{P}\text{-a.s in } \Omega, \ \chi(., 0) = \lim_{t \to 0} \chi(., t) \text{ in } L^2(D).$$

Yosida approximation





Intermediate problem

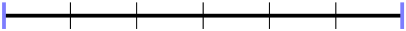
For a fixed $\delta > 0$, find χ_δ satisfying

$$\begin{cases} w_s(\chi_\delta) + f - \partial_t\left(\chi_\delta - \int_0^t h(\chi_\delta)\,dW\right) + \Delta\chi_\delta &= \Psi_\delta(\chi_\delta) & \text{in } \Omega \times D \times (0,T), \\ \chi_\delta(\omega, x, t = 0) &= \chi_0(x) & \omega \in \Omega, x \in D, \\ \nabla\chi_\delta.\mathbf{n} &= 0 & \text{in } \Omega \times \partial D \times (0,T), \end{cases}$$

in a « variational sense ».

Existence of χ_δ

Set $N \in \mathbb{N}^*$.



Discretization scheme : $\Delta t > 0$, $n \in \{0, \ldots, N\}$

- For a given $\chi_n \in L^2(\Omega, \mathcal{F}_{t_n}; H^1(D))$, find $\chi_{n+1} \in L^2(\Omega, \mathcal{F}_{t_{n+1}}; H^1(D))$:

$$\forall v \in H^1(D), \qquad \int_D \frac{\chi_{n+1} - \chi_n}{\Delta t} v dx + \int_D \big(\nabla \chi_{n+1} . \nabla v + \psi_\delta(\chi_{n+1}) v\big) dx$$

$$= \int_D (w_s(\chi_{n+1}) + f_n) v dx + \frac{W(t_{n+1}) - W(t_n)}{\Delta t} \int_D h(\chi_n) v dx.$$

Existence of χ_δ

Set $N \in \mathbb{N}^*$.

$t_0 = 0$ $\quad$ $t_n = n\Delta t$ $\quad$ $N\Delta t = T$

Discretization scheme : $\Delta t > 0$, $n \in \{0, \dots, N\}$

- For a given $\chi_n \in L^2(\Omega, \mathcal{F}_{t_n}; H^1(D))$, find $\chi_{n+1} \in L^2(\Omega, \mathcal{F}_{t_{n+1}}; H^1(D))$:

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$$= \int_D (w_s(\chi_{n+1}) + f_n) v dx + \frac{W(t_{n+1}) - W(t_n)}{\Delta t} \int_D h(\chi_n) v dx.$$

- **Linear problem :** for given S and χ_n, find χ_{n+1} such that $\forall v \in H^1(D)$,

$$\int_D \chi_{n+1} v dx + \Delta t \int_D \nabla \chi_{n+1}.\nabla v dx = \int_D \chi_n v dx - \int_D \psi_\delta(S) v dx$$

$$+\Delta t \int_D \big(w_s(S) + f_n\big) v dx + \big(W(t_{n+1}) - W(t_n)\big) \int_D h(\chi_n) v dx.$$

Existence of χ_δ : overview of the proof

1. Study of the operator $\mathbb{T}_n : S \mapsto \chi_{n+1}$: for a small Δ_t, $\mathbb{T}_n$ is a contraction.
2. Piecewise constant and affine approximations : $\forall t \in [0, T]$

$$\chi^{\Delta t}(t) = \sum_{k=0}^{N-1} \chi_{k+1} 1_{[t_k, t_{k+1})}(t), \ \tilde{\chi}^{\Delta t}(t) = \sum_{k=0}^{N-1} \Big[\frac{\chi_{k+1} - \chi_k}{\Delta t}(t - t_k) + \chi_k \Big] 1_{[t_k, t_{k+1})}(t)$$

satisfying

$$\begin{aligned} & \int_D \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) v dx + \int_D \nabla \chi^{\Delta t} . \nabla v dx + \int_D \psi_\delta(\chi^{\Delta t}) v dx \\ = \ & \int_D \big(w_s(\chi^{\Delta t}) + f_{\Delta t} \big) v dx, \forall v \in H^1(D). \end{aligned}$$

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$$\int_D \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) v dx + \int_D \nabla \chi^{\Delta t} . \nabla v dx + \int_D \psi_\delta(\chi^{\Delta t}) v dx$$
$$= \int_D \left(w_s(\chi^{\Delta t}) + f_{\Delta t} \right) v dx, \forall v \in H^1(D).$$

3. Estimates on the sequences $(\chi^{\Delta t})_{\Delta t}$ and $(\tilde{\chi}^{\Delta t})_{\Delta t}$ independent of Δt.
4. Extraction of subsequences weakly convergent and existence of weak limits $\chi_\delta, h_\delta, w_{s,\delta}$ and ζ_δ.

- **Convergence results when $\Delta_t \to 0$:**

$$\begin{aligned}
\chi^{\Delta t}, \tilde{\chi}^{\Delta t} &\overset{*}{\rightharpoonup} \chi_\delta \text{ in } L^\infty\left(0,T;L^2(\Omega,H^1(D))\right),\\
h(\chi^{\Delta t}) &\rightharpoonup h_\delta \text{ in } L^2\left(0,T;L^2(\Omega,H^1(D))\right),\\
\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t} &\rightharpoonup \chi_\delta - \int_0^\cdot h_\delta(s)dW(s) \text{ in } L^2\left(\Omega, H^1((0,T)\times D)\right),\\
\Psi_\delta(\chi^{\Delta t}) &\rightharpoonup \zeta_\delta \text{ in } L^2\left(0,T;L^2(\Omega,H^1(D))\right),\\
w_s(\chi^{\Delta t}) &\rightharpoonup w_{s,\delta} \text{ in } L^2(\Omega\times(0,T)\times D).
\end{aligned}$$

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h(\chi^{\Delta t}) &\rightharpoonup h_\delta \text{ in } L^2\left(0,T; L^2(\Omega, H^1(D))\right), \\
\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t} &\rightharpoonup \chi_\delta - \int_0^{\cdot} h_\delta(s)dW(s) \text{ in } L^2\left(\Omega, H^1((0,T)\times D)\right), \\
\Psi_\delta(\chi^{\Delta t}) &\rightharpoonup \zeta_\delta \text{ in } L^2\left(0,T; L^2(\Omega, H^1(D))\right), \\
w_s(\chi^{\Delta t}) &\rightharpoonup w_{s,\delta} \text{ in } L^2(\Omega\times(0,T)\times D).
\end{aligned}$$

- **Passage to the limit :** a.e in $(0,T)$, $\mathbb{P}$-a.s in Ω and $\forall v \in H^1(D)$

$$\int_D \partial_t\left(\chi_\delta - \int_0^{\cdot} h_\delta dW\right) v dx + \int_D \nabla\chi_\delta . \nabla v dx = \int_D \left(f + w_{s,\delta} - \zeta_\delta\right) v dx.$$

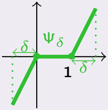
- **Delicate point** : Do we have $h_\delta = h(\chi_\delta)$, $w_{s,\delta} = w_s(\chi_\delta)$, $\zeta_\delta = \Psi_\delta(\chi_\delta)$?
YES ! How ? By showing that $\chi^{\Delta t} \to \chi_\delta$ in $L^2(\Omega\times(0,T)\times D)$.

Starting point : $\delta > 0$

A.e in $(0, T)$, $\mathbb{P}$-a.s in Ω and $\forall v \in H^1(D)$,

$$\int_D \partial_t \left(\chi_\delta - \int_0^\cdot h(\chi_\delta) dW\right) v dx + \int_D \nabla \chi_\delta . \nabla v dx + \int_D \Psi_\delta(\chi_\delta) v dx = \int_D (w_s(\chi_\delta) + f)\, v dx.$$

Estimates independent on $\delta > 0$



(χ_δ), $(\nabla \chi_\delta)$, $(\Delta \chi_\delta)$, $(\Psi_\delta(\chi_\delta))$, $\left(\partial_t(\chi_\delta - \int_0^\cdot h(\chi_\delta) dW)\right)$, $(w_s(\chi_\delta))$, $(-\frac{\chi_\delta^-}{\delta})$ and $(\frac{(\chi_\delta - 1)^+}{\delta})$ are bounded in $L^2(\Omega \times (0, T) \times D)$.

Starting point : $\delta > 0$

A.e in $(0, T)$, $\mathbb{P}$-a.s in Ω and $\forall v \in H^1(D)$,

$$\int_D \partial_t\left(\chi_\delta - \int_0^\cdot h(\chi_\delta)dW\right) v dx + \int_D \nabla\chi_\delta.\nabla v dx + \int_D \Psi_\delta(\chi_\delta) v dx = \int_D (w_s(\chi_\delta) + f)\, v dx.$$

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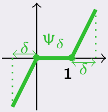
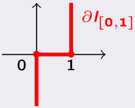


(χ_δ), $(\nabla\chi_\delta)$, $(\Delta\chi_\delta)$, $(\Psi_\delta(\chi_\delta))$, $\left(\partial_t(\chi_\delta - \int_0^\cdot h(\chi_\delta)dW)\right)$, $(w_s(\chi_\delta))$, $(-\frac{\chi_\delta^-}{\delta})$ and $(\frac{(\chi_\delta - 1)^+}{\delta})$ are bounded in $L^2(\Omega\times(0,T)\times D)$.

Convergence results when $\delta \to 0$



- There exist $\chi, \bar{h}, \xi$ and $\bar{w_s}$ such that $\forall v \in H^1(D)$:

$$\int_D \partial_t\left(\chi - \int_0^\cdot \bar{h}(s)dW(s)\right) v dx + \int_D \nabla\chi.\nabla v dx + \int_D \xi v dx = \int_D (\bar{w_s} + f)\, v dx.$$

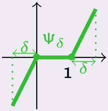
- $0 \leqslant \chi \leqslant 1$.
- **Remaining question** : do we have $\bar{h} = h(\chi)$, $\xi \in \partial I_{[0,1]}(\chi)$ and $\bar{w_s} = w_s(\chi)$?

Starting point : $\delta > 0$

A.e in $(0, T)$, $\mathbb{P}$-a.s in Ω and $\forall v \in H^1(D)$,

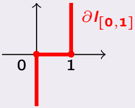
$$\int_D \partial_t \left(\chi_\delta - \int_0^{\cdot} h(\chi_\delta) dW \right) v dx + \int_D \nabla \chi_\delta . \nabla v dx + \int_D \Psi_\delta(\chi_\delta) v dx = \int_D (w_s(\chi_\delta) + f)\, v dx.$$

Estimates independent on $\delta > 0$



(χ_δ), $(\nabla \chi_\delta)$, $(\Delta \chi_\delta)$, $(\Psi_\delta(\chi_\delta))$, $\left(\partial_t(\chi_\delta - \int_0^{\cdot} h(\chi_\delta) dW)\right)$, $(w_s(\chi_\delta))$, $(-\frac{\chi_\delta^-}{\delta})$ and $(\frac{(\chi_\delta - 1)^+}{\delta})$ are bounded in $L^2(\Omega \times (0, T) \times D)$.

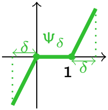
Convergence results when $\delta \to 0$



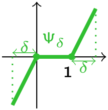
- There exist $\chi, \bar{h}, \xi$ and $\bar{w_s}$ such that $\forall v \in H^1(D)$:

$$\int_D \partial_t \left(\chi - \int_0^{\cdot} \bar{h}(s) dW(s) \right) v dx + \int_D \nabla \chi . \nabla v dx + \int_D \xi v dx = \int_D (\bar{w_s} + f)\, v dx.$$

- $0 \leqslant \chi \leqslant 1$.
- **Remaining question** : do we have $\bar{h} = h(\chi)$, $\xi \in \partial I_{[0,1]}(\chi)$ and $\bar{w_s} = w_s(\chi)$?



- $-\dfrac{\chi_\delta^-}{\delta} \rightharpoonup \xi_1$ in $L^2(\Omega \times (0,T) \times D)$, with $\xi_1 \leqslant 0$.
- $\dfrac{(\chi_\delta - 1)^+}{\delta} \rightharpoonup \xi_2$ in $L^2(\Omega \times (0,T) \times D)$, with $\xi_2 \geqslant 0$.
- $\Psi_\delta(\chi_\delta) = -\dfrac{\chi_\delta^-}{\delta} + \dfrac{(\chi_\delta - 1)^+}{\delta}$, thus $\xi = \xi_1 + \xi_2$.



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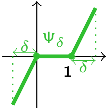
Using Itô formula, properties of superior and inferior limits one gets $\forall \alpha > 0$:

$$E\left[\int_Q e^{-\alpha s}\left\{\xi_2 - \xi\chi + \frac{1}{2}(h(\chi) - \bar{h})^2\right\}dxds\right] \leq 0.$$



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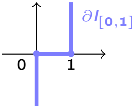
- Since $\xi_2 - \xi\chi = (1-\chi)\xi_2 - \xi_1\chi \geq 0$, thus $h(\chi) = \bar{h}$.
- Moreover, $h(\chi) = \bar{h}$ implies that $\xi_2 - \xi\chi = 0$, which gives
  - if $\chi = 0$ then $\xi_2 = 0$. So $\xi = \xi_1 \leqslant 0$ and thus $\xi \in \mathbb{R}^-$.
  - if $\chi = 1$ then $\xi_1 = 0$. So $\xi = \xi_2 \geqslant 0$ and thus $\xi \in \mathbb{R}^+$.
  - if $0 < \chi < 1$, then $\xi_1 = \xi_2 = 0$ and $\xi = 0$.



  Finally $\xi \in \partial I_{[0,1]}(\chi)$.

- One shows that $\chi_\delta \to \chi$ in $L^2(\Omega \times (0,T) \times D)$ thus $\bar{w}_s = w_s(\chi)$.

Uniqueness of the solution

(χ, ξ), $(\hat{\chi}, \hat{\xi})$ two pairs of solutions with $\chi(0,.) = \hat{\chi}(0,.) = \chi_0$

$$\begin{aligned} d\chi &= \left[w_s(\chi) + f + \Delta\chi - \xi\right]dt + h(\chi)dW, \\ d\hat{\chi} &= \left[w_s(\hat{\chi}) + f + \Delta\hat{\chi} - \hat{\xi}\right]dt + h(\hat{\chi})dW, \\ \text{thus } d(\chi - \hat{\chi}) &= \left[w_s(\chi) - w_s(\hat{\chi}) + \Delta(\chi - \hat{\chi}) - (\xi - \hat{\xi})\right]dt + \left(h(\chi) - h(\hat{\chi})\right) dW. \end{aligned}$$

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Itô Formula to the process $\chi - \hat{\chi}$ and the function $F(s, v) = e^{-\alpha s}\|v\|^2$ with $\alpha > 0$

$$\frac{1}{2}e^{-\alpha T}\|(\chi - \hat{\chi})(T)\|^2 - \frac{1}{2}\|(\chi - \hat{\chi})(0)\|^2 + \int_0^T \int_D (\xi - \hat{\xi})(\chi - \hat{\chi})dxds$$

$$+\frac{\alpha}{2}\int_0^T \int_D e^{-\alpha s}|\chi - \hat{\chi}|^2 dxds + \int_0^T \int_D e^{-\alpha s}|\nabla(\chi - \hat{\chi})|^2 dxds$$

$$= \int_0^T \int_D e^{-\alpha s}\left(w_s(\chi) - w_s(\hat{\chi})\right)(\chi - \hat{\chi})dxds + \frac{1}{2}\int_0^T \int_D e^{-\alpha s}\left(h(\chi) - h(\hat{\chi})\right)^2 dxds$$

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Itô Formula to the process $\chi - \hat{\chi}$ and the function $F(s,v) = e^{-\alpha s}||v||^2$ with $\alpha > 0$

$$\frac{1}{2}e^{-\alpha T}||(\chi-\hat{\chi})(T)||^2 - \frac{1}{2}||\underbrace{(\chi-\hat{\chi})(0)}_{=0}||^2 + \int_0^T\int_D \underbrace{(\xi-\hat{\xi})(\chi-\hat{\chi})}_{\geqslant 0}\,dxds$$

$$+\frac{\alpha}{2}\int_0^T\int_D e^{-\alpha s}|\chi-\hat{\chi}|^2dxds + \int_0^T\int_D e^{-\alpha s}|\nabla(\chi-\hat{\chi})|^2dxds$$

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$$+ \int_0^T\int_D e^{-\alpha s}(\chi-\hat{\chi})\Big(h(\chi)-h(\hat{\chi})\Big)dxdW.$$

Uniqueness of the solution

By taking the expectation

$$\frac{1}{2}e^{-\alpha T}E\left[\|(\chi-\hat{\chi})(T)\|^2\right]+\frac{\alpha}{2}E\left[\int_0^T\int_D e^{-\alpha s}|\chi-\hat{\chi}|^2dxds\right]+E\left[\int_0^T\int_D e^{-\alpha s}|\nabla(\chi-\hat{\chi})|^2dxds\right]$$

$$\leqslant E\left[\int_0^T\int_D e^{-\alpha s}\left(w_s(\chi)-w_s(\hat{\chi})\right)(\chi-\hat{\chi})dxds\right]+\underbrace{E\left[\int_0^T\int_D e^{-\alpha s}(\chi-\hat{\chi})\big(h(\chi)-h(\hat{\chi})\big)dxdW\right]}_{=0}$$

$$+\frac{1}{2}E\left[\int_0^T\int_D e^{-\alpha s}\left(h(\chi)-h(\hat{\chi})\right)^2dxds\right].$$

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$$+\frac{1}{2}E\left[\int_0^T\int_D e^{-\alpha s}\left(h(\chi)-h(\hat{\chi})\right)^2dxds\right].$$

Then

$$\frac{1}{2}e^{-\alpha T}E\left[\|(\chi-\hat{\chi})(T)\|^2\right]+\frac{\alpha}{2}E\left[\int_0^T\int_D e^{-\alpha s}|\chi-\hat{\chi}|^2dxds\right]+E\left[\int_0^T\int_D e^{-\alpha s}|\nabla(\chi-\hat{\chi})|^2dxds\right]$$

$$\leqslant C_{w_s}E\left[\int_0^T\int_D e^{-\alpha s}|\chi-\hat{\chi}|^2dxds\right]+\frac{C_h^2}{2}E\left[\int_0^T\int_D e^{-\alpha s}|\chi-\hat{\chi}|^2dxds\right].$$

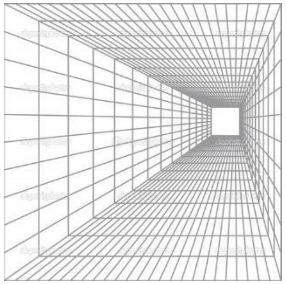
Uniqueness of the solution

$$\frac{1}{2}e^{-\alpha T}E\left[\|(\chi-\hat{\chi})(T)\|^2\right]+E\left[\int_0^T\int_D e^{-\alpha s}|\nabla(\chi-\hat{\chi})|^2dxds\right]$$
$$\leqslant \left(C_{w_s}+\frac{C_h^2}{2}-\frac{\alpha}{2}\right)E\left[\int_0^T\int_D e^{-\alpha s}|\chi-\hat{\chi}|^2dxds\right].$$

- By choosing $\alpha>0$ such that $\left(C_{w_s}+\frac{C_h^2}{2}-\frac{\alpha}{2}\right)\leqslant 0$ one gets $\chi=\hat{\chi}$.
- Going back to the equation, one also concludes that $\xi=\hat{\xi}$.

Conclusions and perspectives

- Deterministic: asymptotic expansions, models of damaged imperfect interfaces + numerical implementation
- Stochastic: mathematical results on damage model



Stochastics

- « Couplings » damage model/ equilibrium equations
- Asymptotic expansions
- Other constitutive equations
- Numerics