Une inéquation variationnelle implicite avec applications à des problèmes dynamiques de contact

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Introduction

- A. Signorini (1933, 1959), G. Fichera (1964, 1972) , G. Duvaut and J.L. Lions (1972) - static elastic problems with unilateral contact
- J.J. Telega (1991) - variational formulation of quasistatic elastic problems with unilateral contact and Coulomb friction
- L.E. Andersson (2000), M.C. and Rocca (2000, 2001) - mathematical analysis of quasistatic elastic problems with unilateral contact and local Coulomb friction
- J. Martins and J. T. Oden (1985, 1987) dynamic problems with normal compliance laws
- G. Lebeau, M. Schatzman (1984), J. U. Kim (1989) - a wave equation with unilateral boundary conditions.
- J. Muñoz-Rivera, R. Racke (1998) - dynamic frictionless problem in thermoelasticity with radial symmetry and unilateral contact conditions.
- J. Jarušek (1993) - dynamic unilateral contact problems with given friction, for viscoelastic bodies.

- K.L. Kuttler, M. Shillor (2001, 2004) - dynamic, bilateral or unilateral, contact with nonlocal friction.
- M. Sofonea and co-workers (1993,...) - contact problems with friction in viscoplasticity and viscoelasticity.
- P.D. Panagiotopoulos and co-workers (1983-1999), S. Migórski and co-workers (2005-...) - hemivariational inequalities and applications to contact problems.
- A. Petrov, M. Schatzman (2002, 2009) - dynamic viscoelastic problems with unilateral constraints.
- M.C., M. Raous, M. Schryve, (2006-2009) - dynamic problems in viscoelasticity coupling unilateral contact, friction and adhesion.
- P.J. Rabier and O.V. Savin (2000) - an intermediate pointwise contact condition in the static case.
- M.C. (2014, 2015) - dynamic contact with intermediate contact conditions between two viscoelastic bodies

Classical and Variational Formulations

We consider two viscoelastic bodies, characterized by a nonlinear constitutive law of Kelvin-Voigt type, which occupy the open, bounded and connected sets Ω^α of $\mathbb{R}^d$, $d = 2$ or 3, with Lipschitz boundaries $\Gamma^\alpha := \partial\Omega^\alpha$, $\alpha = 1, 2$.
Let Γ_U^α, Γ_F^α, $\Gamma_C^\alpha \subset \Gamma^\alpha$ be relatively open, mutually disjoint sets and assume $\text{meas}(\Gamma_U^\alpha) > 0$, $\alpha = 1, 2$.

Notations:
$\boldsymbol{u}^\alpha$, $u^\alpha = \left(u_1^\alpha, ..., u_d^\alpha\right) = \left(\bar{u}^\alpha, u_d^\alpha\right)$ - the displacement field,
$\boldsymbol{\varepsilon}^\alpha$, $\varepsilon^\alpha = \left(\varepsilon_{ij}\left(\boldsymbol{u}^\alpha\right)\right)$ - the infinitesimal strain tensor,
$\boldsymbol{\sigma}^\alpha$, $\sigma^\alpha = \left(\sigma_{ij}^\alpha\right)$ - the stress tensor in Ω^α, $\alpha = 1, 2$,
$\boldsymbol{f}_1 = (\boldsymbol{f}_1^1, \boldsymbol{f}_1^2)$ - the given body forces in $\Omega^1 \cup \Omega^2$,
$\boldsymbol{f}_2 = (\boldsymbol{f}_2^1, \boldsymbol{f}_2^2)$ - the tractions in $\Gamma_F^1 \cup \Gamma_F^2$,
$\boldsymbol{u}_0^\alpha$, $\boldsymbol{u}_1^\alpha$ - the initial displacements and velocities,
$\boldsymbol{U}^\alpha = \boldsymbol{0}$ - the prescribed displacement on Γ_U^α,
$\boldsymbol{\mathcal{A}}^\alpha$, $\boldsymbol{\mathcal{B}}^\alpha$ - the elasticity tensor and the viscosity tensor, respectively, corresponding to Ω^α, $\alpha = 1, 2$.

Assume that the solids are in contact, with a (sufficiently small) gap between the potential contact surfaces Γ_C^1 and Γ_C^2, described by an open, bounded subset Ξ of $\mathbb{R}^{d-1}$ such that Γ_C^1 and Γ_C^2 can be parametrized by two C^1 functions, $\varphi^1, \varphi^2 : \Xi \to \mathbb{R}$, satisfying

$$\varphi^1(\boldsymbol{\xi}) - \varphi^2(\boldsymbol{\xi}) \geq 0 \ \ \forall \boldsymbol{\xi} \in \Xi, \quad \Gamma_C^\alpha = \{\, (\boldsymbol{\xi}, \varphi^\alpha(\boldsymbol{\xi}))\,;\, \boldsymbol{\xi} \in \Xi\},\ \alpha = 1,\ 2.$$

We still denote the unit outward normal vector to Γ_C^α by $\boldsymbol{n}^\alpha : \Xi \to \mathbb{R}^d,\ \alpha = 1,\ 2$, the initial normalized gap by

$$g_0(\boldsymbol{\xi}) = \frac{\varphi^1(\boldsymbol{\xi}) - \varphi^2(\boldsymbol{\xi})}{\sqrt{1 + |\nabla\varphi^1(\boldsymbol{\xi})|^2}} \quad \forall \boldsymbol{\xi} \in \Xi,$$

and we use the following notations for the normal and tangential components of $\boldsymbol{v}^\alpha$, of the relative displacement and of $\boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha$:

$$\begin{aligned}
&\boldsymbol{v}^\alpha := \boldsymbol{v}^\alpha(\boldsymbol{\xi}, t) = \boldsymbol{v}(\boldsymbol{\xi}, \varphi^\alpha(\boldsymbol{\xi}), t),\ \ v_N^\alpha := v_N^\alpha(\boldsymbol{\xi}, t) = \boldsymbol{v}^\alpha(\boldsymbol{\xi}, \varphi^\alpha(\boldsymbol{\xi}), t) \cdot \boldsymbol{n}^\alpha(\boldsymbol{\xi}),\\
&v_N := v_N(\boldsymbol{\xi}, t) = v_N^1 + v_N^2,\ \ [v_N] := [v_N](\boldsymbol{\xi}, t) = v_N(\boldsymbol{\xi}, t) - g_0(\xi),\\
&\boldsymbol{v}_T^\alpha := \boldsymbol{v}_T^\alpha(\boldsymbol{\xi}, t) = \boldsymbol{v}^\alpha - v_N^\alpha \boldsymbol{n}^\alpha,\ \ \boldsymbol{v}_T := \boldsymbol{v}_T(\boldsymbol{\xi}, t) = \boldsymbol{v}_T^1 - \boldsymbol{v}_T^2,\\
&\sigma_N^\alpha := \sigma_N^\alpha(\boldsymbol{\xi}, t) = (\boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha) \cdot \boldsymbol{n}^\alpha,\quad \boldsymbol{\sigma}_T^\alpha := \boldsymbol{\sigma}_T^\alpha(\boldsymbol{\xi}, t) = \boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha - \sigma_N^\alpha \boldsymbol{n}^\alpha,
\end{aligned}$$

for all $\boldsymbol{\xi}$ in Ξ and for all $t \in [0, T]$.

The unilateral contact condition at time t can be written as $[u_N](\boldsymbol{\xi}, t) \leq 0 \quad \forall \boldsymbol{\xi} \in \Xi$.
The state variables are:
- the infinitesimal strain tensor $\boldsymbol{\varepsilon} := (\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2) = (\boldsymbol{\varepsilon}(\boldsymbol{u}^1), \boldsymbol{\varepsilon}(\boldsymbol{u}^2))$,
- the relative normal displacement $[u_N] = u_N^1 + u_N^2 - g_0$,
- the relative tangential displacement $\boldsymbol{u}_T = \boldsymbol{u}_T^1 - \boldsymbol{u}_T^2$.

Let $\mu = \mu(\xi, \dot{\boldsymbol{u}}_T)$ be the slip rate dependent coefficient of friction and assume that $\mu : \Xi \times \mathbb{R}^d \to \mathbb{R}_+$ is a bounded function such that for a.e. $\xi \in \Xi$ $\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant, denoted by C_μ, independent of ξ, and for every $v \in \mathbb{R}^d$ $\mu(\cdot, v)$ is measurable.

Consider also the following mappings:
- $\underline{\kappa}$, $\overline{\kappa} : \mathbb{R}^2 \to \mathbb{R}$ such that $\underline{\kappa}$ and $\overline{\kappa}$ are lower semicontinuous and upper semicontinuous, respectively, satisfying

$$\underline{\kappa}(s) \leq \overline{\kappa}(s) \text{ and } 0 \notin (\underline{\kappa}(s), \overline{\kappa}(s)) \ \forall s \in \mathbb{R}^2, \tag{1}$$

$$\exists\, r_0 \geq 0 \text{ such that } \max(|\underline{\kappa}(s)|, |\overline{\kappa}(s)|) \leq r_0 \ \forall s \in \mathbb{R}^2. \tag{2}$$

Classical formulation

Problem $\mathcal{P}_c$: Find $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0 = (\boldsymbol{u}_0^1, \boldsymbol{u}_0^2)$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 = (\boldsymbol{u}_1^1, \boldsymbol{u}_1^2)$ in $\Omega^1 \times \Omega^2$ and, for all $t \in (0, T)$,

$$\ddot{\boldsymbol{u}}^\alpha - \operatorname{div} \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{f}_1^\alpha \ \text{ in } \ \Omega^\alpha,$$

$$\boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{\mathcal{A}}^\alpha \varepsilon(\boldsymbol{u}^\alpha) + \boldsymbol{\mathcal{B}}^\alpha \varepsilon(\dot{\boldsymbol{u}}^\alpha) \ \text{ in } \ \Omega^\alpha,$$

$$\boldsymbol{u}^\alpha = \boldsymbol{0} \ \text{ on } \ \Gamma_U^\alpha, \quad \boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha = \boldsymbol{f}_2^\alpha \ \text{ on } \ \Gamma_F^\alpha, \ \alpha = 1, 2,$$

$$\boldsymbol{\sigma}^1 \boldsymbol{n}^1 + \boldsymbol{\sigma}^2 \boldsymbol{n}^2 = \boldsymbol{0} \ \text{ in } \ \Xi,$$

$$\underline{\kappa}([u_N], \dot{u}_N) \leq \sigma_N \leq \overline{\kappa}([u_N], \dot{u}_N) \ \text{ in } \ \Xi,$$

$$|\,\boldsymbol{\sigma}_T\,| \leq \mu(\dot{\boldsymbol{u}}_T)\,|\,\sigma_N\,| \ \text{ in } \ \Xi \ \text{ and}$$

$$\dot{\boldsymbol{u}}_T \neq \boldsymbol{0} \Rightarrow \boldsymbol{\sigma}_T = -\mu(\dot{\boldsymbol{u}}_T)|\sigma_N| \frac{\dot{\boldsymbol{u}}_T}{|\dot{\boldsymbol{u}}_T|}$$

where $\boldsymbol{\sigma}^\alpha = \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha)$, $\alpha = 1, 2$, $\sigma_N := \sigma_N^1$, $\boldsymbol{\sigma}_T := \boldsymbol{\sigma}_T^1$.

Example 1. (Adhesion and friction conditions)
Let $s_0 \geq 0$, $M \geq 0$ be two constants and $k_0 : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $k_0 \geq 0$ with $k_0(0) = 0$. For $s = (s_1, s_2) \in \mathbb{R}^2$, define

$$\underline{\kappa}(s) = \underline{\kappa}(s_1) = \begin{cases} 0 & \text{if} \quad s_1 \leq -s_0, \\ k_0(s_1) & \text{if} \quad -s_0 < s_1 < 0, \\ -M & \text{if} \quad s_1 \geq 0, \end{cases} \quad \overline{\kappa}(s) = \overline{\kappa}(s_1) = \begin{cases} 0 & \text{if} \quad s_1 < -s_0, \\ k_0(s) & \text{if} \quad -s_0 \leq s_1 \leq 0, \\ -M & \text{if} \quad s_1 > 0. \end{cases}$$

Example 2. (Friction condition)
In Example 1 we set $k_0 = s_0 = 0$ and define

$$\underline{\kappa}(s) = \underline{\kappa}_M(s_1) = \begin{cases} 0 & \text{if} \quad s_1 < 0, \\ -M & \text{if} \quad s_1 \geq 0, \end{cases} \quad \overline{\kappa}(s) = \overline{\kappa}_M(s_1) = \begin{cases} 0 & \text{if} \quad s_1 \leq 0, \\ -M & \text{if} \quad s_1 > 0. \end{cases}$$

The classical Signorini's conditions correspond, formally, to $M = +\infty$.
Example 3. (General normal compliance conditions)
Various normal compliance conditions, friction and adhesion laws can be obtained if $\underline{\kappa} = \overline{\kappa} = \kappa$, where $\kappa : \mathbb{R}^2 \to \mathbb{R}$ is some bounded Lipschitz continuous function with $\kappa(0,0) = 0$, so that σ_N is given by the relation $\sigma_N = \kappa([u_N], \dot{u}_N)$.

Variational formulation

We adopt the following notations:

$$\boldsymbol{H}^s(\Omega^\alpha) := H^s(\Omega^\alpha;\mathbb{R}^d),\ \alpha = 1,2,\ \boldsymbol{H}^s := \boldsymbol{H}^s(\Omega^1)\times \boldsymbol{H}^s(\Omega^2),$$

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle_{-s,s} = \langle \boldsymbol{v}^1,\boldsymbol{w}^1\rangle_{\boldsymbol{H}^{-s}(\Omega^1)\times \boldsymbol{H}^s(\Omega^1)} + \langle \boldsymbol{v}^2,\boldsymbol{w}^2\rangle_{\boldsymbol{H}^{-s}(\Omega^2)\times \boldsymbol{H}^s(\Omega^2)}$$

$$\forall\, \boldsymbol{v} = (\boldsymbol{v}^1,\boldsymbol{v}^2)\in \boldsymbol{H}^{-s},\ \forall\, \boldsymbol{w} = (\boldsymbol{w}^1,\boldsymbol{w}^2)\in \boldsymbol{H}^s,\ \ \forall\, s\in\mathbb{R}.$$

Define the Hilbert spaces $(\boldsymbol{H},\ |.|)$ with the associated inner product denoted by $(.\,,.)$, $(\boldsymbol{V},\ \|.\|)$ with the associated inner product (of $\boldsymbol{H}^1$) denoted by $\langle .\,,.\rangle$, and the closed convex cones $L^2_+(\Xi)$, $L^2_+(\Xi\times(0,T))$ as follows:

$$\boldsymbol{H} := \boldsymbol{H}^0 = L^2(\Omega^1;\mathbb{R}^d)\times L^2(\Omega^2;\mathbb{R}^d),\ \ \boldsymbol{V} := \boldsymbol{V}^1\times \boldsymbol{V}^2,\ \ \text{where}$$

$$\boldsymbol{V}^\alpha = \{\boldsymbol{v}^\alpha \in \boldsymbol{H}^1(\Omega^\alpha);\ \boldsymbol{v}^\alpha = \boldsymbol{0}\ \text{a.e. on } \Gamma_U^\alpha\},\ \alpha = 1,2,$$

$$L^2_+(\Xi) := \{\delta\in L^2(\Xi);\ \delta\geq 0\ \ \text{a.e. in}\ \ \Xi\},$$

$$L^2_+(\Xi\times(0,T)) := \{\eta\in L^2(0,T;L^2(\Xi));\ \eta\geq 0\ \ \text{a.e. in}\ \ \Xi\times(0,T)\}.$$

Let $\boldsymbol{A}, \boldsymbol{B} : \boldsymbol{H}^1 \to \boldsymbol{H}^1$ be two mappings defined by : $\forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1$

$\langle \boldsymbol{A}\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{A}^1 \boldsymbol{v}^1, \boldsymbol{w}^1 \rangle + \langle \boldsymbol{A}^2 \boldsymbol{v}^2, \boldsymbol{w}^2 \rangle$, $\langle \boldsymbol{B}\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{B}^1 \boldsymbol{v}^1, \boldsymbol{w}^1 \rangle + \langle \boldsymbol{B}^2 \boldsymbol{v}^2, \boldsymbol{w}^2 \rangle$, where

$\langle \boldsymbol{A}^\alpha \boldsymbol{v}^\alpha, \boldsymbol{w}^\alpha \rangle = \int_{\Omega^\alpha} \boldsymbol{\mathcal{A}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\, dx$, $\langle \boldsymbol{B}^\alpha \boldsymbol{v}^\alpha, \boldsymbol{w}^\alpha \rangle = \int_{\Omega^\alpha} \boldsymbol{\mathcal{B}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\, dx$.

Assume $\boldsymbol{f}_1^\alpha \in W^{1,\infty}(0,T; L^2(\Omega^\alpha; \mathbb{R}^d))$, $\boldsymbol{f}_2^\alpha \in W^{1,\infty}(0,T; L^2(\Gamma_F^\alpha; \mathbb{R}^d))$, $\alpha = 1, 2$,
$\boldsymbol{u}_0, \boldsymbol{u}_1 \in \boldsymbol{V}$, $g_0 \in L^2_+(\Xi)$, and define the following mappings:

$J : L^2(\Xi) \times (\boldsymbol{H}^1)^2 \to \mathbb{R}$, $J(\delta, \boldsymbol{v}, \boldsymbol{w}) = \int_\Xi \mu(\boldsymbol{v}_T)\, |\delta|\, |\boldsymbol{w}_T|\, d\xi \quad \forall\, \delta \in L^2(\Xi)$, $\forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1$,

$\boldsymbol{f} \in W^{1,\infty}(0,T; \boldsymbol{H}^1)$, $\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \sum_{\alpha=1,2} \int_{\Omega^\alpha} \boldsymbol{f}_1^\alpha \cdot \boldsymbol{v}^\alpha\, dx + \sum_{\alpha=1,2} \int_{\Gamma_F^\alpha} \boldsymbol{f}_2^\alpha \cdot \boldsymbol{v}^\alpha\, ds$

$\forall\, \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{H}^1$, $\forall\, t \in [0, T]$.

Assume the following compatibility conditions: $[u_{0N}] \leq 0$, $\overline{\kappa}([u_{0N}], u_{1N}) = 0$ a.e. in Ξ and $\exists\, \boldsymbol{p}_0 \in \boldsymbol{H}$ such that

$$(\boldsymbol{p}_0, \boldsymbol{w}) + \langle \boldsymbol{A}\boldsymbol{u}_0, \boldsymbol{w}\rangle + \langle \boldsymbol{B}\boldsymbol{u}_1, \boldsymbol{w}\rangle = \langle \boldsymbol{f}(0), \boldsymbol{w}\rangle \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}.$$

For every $\zeta = (\zeta_1, \zeta_2) \in L^2(0, T; (L^2(\Xi))^2)$ define the following sets:

$$\Lambda(\zeta) = \{\eta \in L^2(0, T; L^2(\Xi)); \underline{\kappa} \circ \zeta \leq \eta \leq \overline{\kappa} \circ \zeta \text{ a.e. in } \Xi \times (0, T)\},$$

$$\Lambda_+(\zeta) = \{\eta \in L^2_+(\Xi \times (0, T)); \underline{\kappa}_+ \circ \zeta \leq \eta \leq \overline{\kappa}_+ \circ \zeta \text{ a.e. in } \Xi \times (0, T)\},$$

$$\Lambda_-(\zeta) = \{\eta \in L^2_+(\Xi \times (0, T)); \overline{\kappa}_- \circ \zeta \leq \eta \leq \underline{\kappa}_- \circ \zeta \text{ a.e. in } \Xi \times (0, T)\},$$

where, for each $s \in \mathbb{R}$, $s_+ := \max(0, s)$ and $s_- := \max(0, -s)$ denote the positive and negative parts, respectively.

Problem $\mathcal{P}_{\boldsymbol{v}}^{1}$: Find $\boldsymbol{u} \in C^1([0,T];\boldsymbol{H}^{-\iota}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$, $\lambda \in \Lambda([u_N], \dot{u}_N)$ and

$$\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T)\rangle_{-\iota,\,\iota} - (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0) - \int_0^T (\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}})\,dt$$
$$+ \int_0^T \left\{ \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \langle \boldsymbol{B}\dot{\boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}\rangle - (\lambda, v_N - u_N)_{L^2(\Xi)} \right\} dt \qquad (3)$$
$$+ \int_0^T \{J(\lambda, \dot{\boldsymbol{u}}, \boldsymbol{v} + k\dot{\boldsymbol{u}} - \boldsymbol{u}) - J(\lambda, \dot{\boldsymbol{u}}, k\dot{\boldsymbol{u}})\}\,dt \geq \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle\,dt$$
$$\forall\, \boldsymbol{v} \in L^\infty(0,T;\boldsymbol{V}) \cap W^{1,2}(0,T;\boldsymbol{H}), \quad \text{where } 1 > \iota > \frac{1}{2},\ k > 0.$$

The Lagrange multiplier λ satisfies the relation $\lambda = \sigma_N$.
Let $\phi : (L^2_+(\Xi))^2 \times (\boldsymbol{V})^2 \to \mathbb{R}$ be defined by

$$\phi(\delta_1, \delta_2, \boldsymbol{v}, \boldsymbol{w}) = -(\delta_1 - \delta_2, w_N)_{L^2(\Xi)} + \int_\Xi \mu(\boldsymbol{v}_T)\,(\delta_1 + \delta_2)\,|\boldsymbol{w}_T|\,d\xi$$
$$\forall\,(\delta_1, \delta_2) \in (L^2_+(\Xi))^2,\ \forall\, \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2),\ \boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{V}.$$

Since $\eta \in \Lambda(\zeta)$ if and only if $(\eta_+, \eta_-) \in \Lambda_+(\zeta) \times \Lambda_-(\zeta)$, it follows that the variational problem P_v^1 is clearly equivalent with the following problem.
Problem $\mathcal{P}_v^2$: Find $\boldsymbol{u} \in C^1([0,T];\boldsymbol{H}^{-\iota}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$, $(\lambda_+, \lambda_-) \in \Lambda_+([u_N], \dot{u}_N) \times \Lambda_-([u_N], \dot{u}_N)$ and

$$\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T) \rangle_{-\iota,\iota} - (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0)$$
$$+ \int_0^T \{-(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}}) + \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u} \rangle + \langle \boldsymbol{B}\dot{\boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u} \rangle\}\, dt \tag{4}$$
$$+ \int_0^T \{\phi(\lambda_+, \lambda_-, \dot{\boldsymbol{u}}, \boldsymbol{v} + k\dot{\boldsymbol{u}} - \boldsymbol{u}) - \phi(\lambda_+, \lambda_-, \dot{\boldsymbol{u}}, k\dot{\boldsymbol{u}})\}\, dt \geq \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle\, dt$$
$$\forall\, \boldsymbol{v} \in L^\infty(0,T;\boldsymbol{V}) \cap W^{1,2}(0,T;\boldsymbol{H}).$$

General Existence Results

Let U_0, $(V_0, \|.\|, \langle .\,,.\rangle)$, $(U, \|.\|_U)$ and $(H_0, |.|, (.\,,.))$ be four Hilbert spaces such that U_0 is a closed linear subspace of V_0 dense in H_0, $V_0 \subset U \subseteq H_0$ with continuous embeddings and the embedding from V_0 into U is compact.
Let $B_r(\Xi)$, $B_r(\Xi_T)$ denote the closed balls with center 0 and radius r in $L^\infty(\Xi)$, $L^\infty(\Xi_T)$, respectively, where $\Xi_T := \Xi \times (0, T)$ and $r > 0$.
Let two functionals $F_{1,2} : V_0 \to \mathbb{R}$ differentiable on V_0 and assume that their derivatives $F'_{1,2} : V_0 \to V$, are strongly monotone and Lipschitz continuous, that is

$$\alpha_i\|u - v\|^2 \leq \langle F'_i(u) - F'_i(v), u - v\rangle, \ \|F'_i(u) - F'_i(v)\| \leq \beta_i\|u - v\|, \tag{5}$$

for all $u, v \in V_0$, $i = 1, 2$.
Let $I : (V_0)^2 \to (L^2(\Xi))^2$ and $\phi_0 : [0, T] \times (L^2_+(\Xi))^2 \times (V_0)^2 \to \mathbb{R}$ be two mappings satisfying the following conditions:

$$\begin{aligned} &\exists\, k_1 > 0 \text{ such that } \forall\, \bar{v}_1, \bar{v}_2 \in (V_0)^2, \\ &\|I(\bar{v}_1) - I(\bar{v}_2)\|_{(L^2(\Xi))^2} \leq k_1\|\bar{v}_1 - \bar{v}_2\|_{(U)^2}, \end{aligned} \tag{6}$$

$$\forall\, t \in [0, T],\ \forall\, \gamma_1,\, \gamma_2 \in L^2_+(\Xi),\ \forall\, v_1,\, v_2,\, v \in V_0,$$

$$\phi_0(t, \gamma_1, \gamma_2, v, v_1 + v_2) \leq \phi_0(t, \gamma_1, \gamma_2, v, v_1) + \phi_0(t, \gamma_1, \gamma_2, v, v_2), \tag{7}$$

$$\phi_0(t,\gamma_1,\gamma_2,v,\theta v_1) = \theta\,\phi_0(t,\gamma_1,\gamma_2,v,v_1), \quad \forall\,\theta \geq 0, \tag{8}$$

$$\forall\, v \in V_0, \quad \phi_0(0,0,0,0,v) = 0, \tag{9}$$

$$\phi_0(t,\gamma_1,\gamma_2,v,w) = 0, \quad \forall\, w \in U_0, \tag{10}$$

$$\begin{gathered}\forall\, r > 0,\ \exists\, k_2(r) > 0 \ \text{ such that } \ \forall\, t_1, t_2 \in [0,T], \\ \forall\,\gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2 \in L^2_+(\Xi) \cap B_r(\Xi),\ \forall\, v_1, v_2, w_1, w_2 \in V_0, \\ |\phi_0(t_1,\gamma_1,\gamma_2,v_1,w_1) - \phi_0(t_1,\gamma_1,\gamma_2,v_1,w_2) + \phi_0(t_2,\delta_1,\delta_2,v_2,w_2) - \phi_0(t_2,\delta_1,\delta_2,v_2,w_1)| \\ \leq k_2(r)(|t_1 - t_2| + \|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)} + \|v_1 - v_2\|_U)\|w_1 - w_2\|_U,\end{gathered} \tag{11}$$

$$\begin{gathered}\text{if } (\gamma_1^n,\gamma_2^n) \in (L^2_+(\Xi_T))^2 \ \text{ for all } n \in \mathbb{N} \\ \text{and } (\gamma_1^n,\gamma_2^n) \rightharpoonup (\gamma_1,\gamma_2) \text{ in } (L^2(0,T;L^2(\Xi)))^2, \ \text{then} \\ \int_0^T \phi_0(s,\gamma_1^n,\gamma_2^n,v,w)\,ds \to \int_0^T \phi_0(s,\gamma_1,\gamma_2,v,w)\,ds \quad \forall\, v,w \in L^2(0,T;V_0).\end{gathered} \tag{12}$$

Assume that $f_0 \in W^{1,\infty}(0,T;V_0)$, $u^0,\ u^1 \in V_0$ are given, and that the following compatibility conditions hold: $\overline{\kappa}(l(u^0,u^1)) = 0$ and $\exists\, p_0 \in H_0$ such that

$$(p_0, w) + \langle F_1'(u^0), w\rangle + \langle F_2'(u^1), w\rangle = \langle f_0(0), w\rangle \quad \forall\, w \in V_0. \tag{13}$$

Problem $\mathcal{Q}_1$: Find $u \in W_0,\ \lambda \in L^2(0,T;L^2(\Xi))$ such that $u(0)=u^0$, $\dot u(0) = u^1$, $(\lambda_+,\lambda_-) \in \Lambda_+(l(u,\dot u)) \times \Lambda_-(l(u,\dot u))$ and

$$\begin{aligned}
&\langle \dot u(T), v(T) - u(T)\rangle_{U'\times U} - (u_1, v(0)-u_0)\\
&+ \int_0^T \{-(\dot u, \dot v - \dot u) + \langle F_1'(u), v-u\rangle + \langle F_2'(\dot u), v-u\rangle\}\, dt \qquad (14)\\
&+ \int_0^T \{\phi_0(t,\lambda_+,\lambda_-,\dot u, v + k\dot u - u) - \phi_0(t,\lambda_+,\lambda_-,\dot u,k\dot u)\}\, dt \ge \int_0^T \langle f_0, v-u\rangle\, dt\\
&\forall\, v \in L^\infty(0,T;V_0) \cap W^{1,2}(0,T;H_0),
\end{aligned}$$

where $W_0 := C^1([0,T];U') \cap W^{1,2}(0,T;V_0)$.

We consider the following problem, which has the same solution u as the problem Q_1, and the solutions λ_1, λ_2 satisfy the relation $\lambda = \lambda_1 - \lambda_2$, where λ is a solution of Q_1 .
Problem $\mathcal{Q}_2$: Find $u \in W_0$, λ_1, $\lambda_2 \in L^2(0, T; L^2(\Xi))$ such that $u(0) = u^0$, $\dot{u}(0) = u^1$, $(\lambda_1, \lambda_2) \in \Lambda_+(I(u, \dot{u})) \times \Lambda_-(I(u, \dot{u}))$ and

$$\begin{aligned} &\langle \dot{u}(T), v(T) - u(T) \rangle_{U' \times U} - (u_1, v(0) - u_0) \\ &+ \int_0^T \{ -(\dot{u}, \dot{v} - \dot{u}) + \langle F_1'(u), v - u \rangle + \langle F_2'(\dot{u}), v - u \rangle \} \, dt \\ &+ \int_0^T \{ \phi_0(t, \lambda_1, \lambda_2, \dot{u}, v + k\dot{u} - u) - \phi_0(t, \lambda_1, \lambda_2, \dot{u}, k\dot{u}) \} \, dt \geq \int_0^T \langle f_0, v - u \rangle \, dt \\ &\forall\, v \in L^\infty(0, T; V_0) \cap W^{1,2}(0, T; H_0). \end{aligned} \tag{15}$$

Some auxiliary existence results

Lemma

Assume that (5), (7)-(9), (11), and (13) hold. If $r > 0$ *then, for each* $(\gamma_1, \gamma_2) \in (W^{1,\infty}(0,T;L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2 \cap (B_r(\Xi_T))^2$ *with* $\gamma_1(0) = \gamma_2(0) = 0$, *there exists a unique solution* $u = u_{(\gamma_1,\gamma_2)}$ *of the following evolution variational inequality: find* $u \in W^{2,2}(0,T;H_0) \cap W^{1,2}(0,T;V_0)$ *such that* $u(0) = u^0$, $\dot{u}(0) = u^1$, *and for almost all* $t \in (0,T)$

$$\begin{aligned}(\ddot{u}, v - \dot{u}) + \langle F_1'(u), v - \dot{u}\rangle + \langle F_2'(\dot{u}), v - \dot{u}\rangle \\ +\phi_0(t,\gamma_1,\gamma_2,\dot{u},v) - \phi_0(t,\gamma_1,\gamma_2,\dot{u},\dot{u}) \geq \langle f_0, v - \dot{u}\rangle \quad \forall\, v \in V_0.\end{aligned} \tag{16}$$

Lemma

Let $r > 0$, (γ_1, γ_2), $(\delta_1, \delta_2) \in (W^{1,\infty}(0, T; L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2 \cap (B_r(\Xi_T))^2$ such that $\gamma_1(0) = \gamma_2(0) = \delta_1(0) = \delta_2(0) = 0$ and let $u_{(\gamma_1,\gamma_2)}$, $u_{(\delta_1,\delta_2)}$ be the corresponding solutions of (16). Then there exists a constant $C_0 > 0$, independent of (γ_1, γ_2), (δ_1, δ_2), such that for all $t \in [0, T]$

$$\begin{aligned} & |\dot{u}_{(\gamma_1,\gamma_2)}(t) - \dot{u}_{(\delta_1,\delta_2)}(t)|^2 + \|u_{(\gamma_1,\gamma_2)}(t) - u_{(\delta_1,\delta_2)}(t)\|^2 \\ & \quad + \int_0^t \|\dot{u}_{(\gamma_1,\gamma_2)} - \dot{u}_{(\delta_1,\delta_2)}\|^2\, ds \\ & \leq C_0 \int_0^t \{\phi_0(s, \gamma_1, \gamma_2, \dot{u}_{(\gamma_1,\gamma_2)}, \dot{u}_{(\delta_1,\delta_2)}) - \phi_0(s, \gamma_1, \gamma_2, \dot{u}_{(\gamma_1,\gamma_2)}, \dot{u}_{(\gamma_1,\gamma_2)}) \\ & \quad + \phi_0(s, \delta_1, \delta_2, \dot{u}_{(\delta_1,\delta_2)}, \dot{u}_{(\gamma_1,\gamma_2)}) - \phi_0(s, \delta_1, \delta_2, \dot{u}_{(\delta_1,\delta_2)}, \dot{u}_{(\delta_1,\delta_2)})\}\, ds. \end{aligned} \tag{17}$$

A fixed point problem formulation

Theorem

For each $(\gamma_1, \gamma_2) \in (L^2_+(\Xi_T))^2 \cap (B_{r_0}(\Xi_T))^2$, *let* $(\gamma_1^n, \gamma_2^n)_n$ *be a sequence in* $(W^{1,\infty}(0,T;L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2 \cap (B_r(\Xi_T))^2$, *for some* $r > 0$, *such that* $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ *in* $(L^2(0,T;L^2(\Xi)))^2$, $\gamma_1^n(0) = \gamma_2^n(0) = 0$, *and let* $u_{(\gamma_1^n, \gamma_2^n)}$ *be the solution of (16) corresponding to* (γ_1^n, γ_2^n), *for every* $n \in \mathbb{N}$. *Then* $(u_{(\gamma_1^n, \gamma_2^n)})_n$ *is strongly convergent in* W_0, *its limit, denoted by* $u = u_{(\gamma_1, \gamma_2)}$, *is independent of the chosen sequence converging to* (γ_1, γ_2) *with the same properties as* $(\gamma_1^n, \gamma_2^n)_n$ *and is a solution of the following evolution variational inequality:* $u(0) = u^0$, $\dot{u}(0) = u^1$,

$$\begin{aligned}
&\langle \dot{u}(T), v(T) - u(T)\rangle_{U' \times U} - (u^1, v(0) - u^0) \\
&+ \int_0^T \{-(\dot{u}, \dot{v} - \dot{u}) + \langle F_1'(u), v - u\rangle + \langle F_2'(\dot{u}), v - u\rangle\}\, dt \\
&+ \int_0^T \{\phi_0(t, \gamma_1, \gamma_2, \dot{u}, v - u + k\dot{u}) - \phi_0(t, \gamma_1, \gamma_2, \dot{u}, k\dot{u})\}\, dt \geq \int_0^T \langle f_0, v - u\rangle\, dt \\
&\quad \forall\, v \in L^\infty(0,T;V_0) \cap W^{1,2}(0,T;H_0).
\end{aligned} \tag{18}$$

Let $\Phi : (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2 \to 2^{(L^2_+(\Xi_T)\cap B_{r_0}(\Xi_T))^2} \setminus \{\emptyset\}$ be the set-valued mapping defined by

$$\Phi(\gamma_1, \gamma_2) = \Lambda_+(I(u_{(\gamma_1,\gamma_2)}, \dot{u}_{(\gamma_1,\gamma_2)})) \times \Lambda_-(I(u_{(\gamma_1,\gamma_2)}, \dot{u}_{(\gamma_1,\gamma_2)})) \quad \text{for all } (\gamma_1, \gamma_2) \in (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2, \tag{19}$$

where $u_{(\gamma_1,\gamma_2)}$ is the solution of the variational inequality (18) which corresponds to (γ_1, γ_2) by the procedure described in the previous theorem.
It is easily seen that if (λ_1, λ_2) is a fixed point of Φ, i.e. $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$, then $(u_{(\lambda_1,\lambda_2)}, \lambda_1, \lambda_2)$ is a solution of the Problem Q_2.
We shall consider a new problem, which consists in finding a fixed point of the set-valued mapping Φ, called also multivalued function, which will provide a solution of Problem Q_1.

Definition

Let Y be a reflexive Banach space, D a weakly closed set in Y, and $F : D \to 2^Y \setminus \{\emptyset\}$ be a multivalued function. F is called sequentially weakly upper semicontinuous if $z_n \rightharpoonup z$, $y_n \in F(z_n)$ and $y_n \rightharpoonup y$ imply $y \in F(z)$.

Proposition

(Ky Fan) Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y, and $F : D \to 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $F(z)$ is convex for every $z \in D$. Then F has a fixed point.

Theorem

Assume that (1), (2), (5)-(12) and (13) hold. Then there exists $(\lambda_1, \lambda_2) \in (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2$ such that $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$. For each fixed point (λ_1, λ_2) of the set-valued mapping Φ, $(u_{(\lambda_1,\lambda_2)}, \lambda)$ with $\lambda = \lambda_1 - \lambda_2$ is a solution of the Problem Q_1.

Existence of a Solution to the Contact Problem

Theorem

Under the above assumptions there exists a solution of the Problem P_v^1.

Proof.

We prove that there exists at least a solution $(\boldsymbol{u}, \lambda_+, \lambda_-)$ of the Problem P_v^2 which will provide a solution $(\boldsymbol{u}, \lambda)$ of the Problem P_v^1 with $\lambda = \lambda_+ - \lambda_-$.
We apply the previous theorem to $U_0 = \boldsymbol{H}_0^1 = H_0^1(\Omega^1; \mathbb{R}^d) \times H_0^1(\Omega^2; \mathbb{R}^d)$, $V_0 = \boldsymbol{V}$, $U = \boldsymbol{H}^\iota$, $H_0 = \boldsymbol{H}$, $F_1' = \boldsymbol{A}$, $F_2' = \boldsymbol{B}$, $u^0 = \boldsymbol{u}_0$, $u^1 = \boldsymbol{u}_1$, $\phi_0 = \phi$, $f_0 = \boldsymbol{f}$ and to the mapping $I : \boldsymbol{V}^2 \to (L^2(\Xi))^2$ defined by $I(\boldsymbol{v}, \boldsymbol{w}) = ([v_N], w_N)\ \ \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$. □

Perspectives

- More complex contact interaction laws (in quasistatic or dynamic cases)
- Nonlinear quasistatic and dynamic unilateral contact problems with friction
- Hemivariational formulations
- Numerical analysis and solution methods (parareal methods...)