Two-level methods with optimal computing complexity for variational inequalities of the second kind

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Outline of the talk

• variational inequalities of the second kind

$$u \in K : \langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K$$

and the quasi-variational inequalities

$$u \in K : \langle F'(u), v - u\rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \text{ for any } v \in K$$

are equivalent with minimization problems

$$u \in K : F(u) + \varphi(u) \leq F(v) + \varphi(v), \text{ for any } v \in K$$

and

$$u \in K : F(u) + \varphi(u, u) \leq F(v) + \varphi(u, v), \text{ for any } v \in K.$$

respectively, where the functionals φ are not differentiable

• many mechanical problems are modeled with such inequalities: seepage flows by porous media, frictional contact problems, etc.

• L. B., R. Krause, *One- and two-level Schwarz methods for inequalities of the second kind and their application to frictional contact*, Numer. Math., 120, 4, 2012, pp. 573-599

we have introduced multiplicative two-level methods for such inequalities

• the methods introduced in this talk are an improvement of these methods

- by introduction of some level convex sets where we look for corrections
  - one avoid the use of the initial convex set (of the problem) for the finding of the corrections on the coarse discretization level (which introduces additional interpolations)
  - in this way, the iterations have an optimal computing complexity
- the convergence condition of the new algorithms for quasi-variational inequalities is similar with the existence and uniqueness condition of the solution of the inequality and does not depend anymore of the number of subdomains

• in this talk we introduce additive and multiplicative two-level methods for both variational inequalities of the second kind and quasi-variational inequalities

- introduce subspace correction algorithms corresponding to the methods in a general reflexive Banach space
- prove their global convergence by using some assumptions on
  - the construction of the level convex sets
  - the decomposition of the elements of the convex set in function of the domain decomposition
- estimate the error and explicitly write the convergence rate which essentially depends on a constant C_0 introduced by the stability condition of the domain decomposition in assumptions
- abstract algorithms become two-level Schwarz methods in the case of the finite element spaces, and we show that
  - assumptions introduced in the abstract framework hold for two-obstacle convex sets
  - explicitly write the constant C_0 depending on the mesh and domain decomposition parameters
  - we get that convergence rates of the two-level methods we have introduced depend very weakly on or, in certain cases, are totally independent of these parameters

General framework

for the subspace correction algorithms

- V - reflexive Banach space, $K \subset V$ - non empty closed convex subset
- closed subspaces of V:
  - V_0 - corresponding to the coarse discretization
  - $V_{11}, \ldots, V_{1m}$ - corresponding to the decomposition of the domain
- assumption on the choice of the convex sets where we look for the level corrections (the level convex sets depend on the current approximation in the algorithms)

Assumption (1)

We assume that for a given $w \in K$, we can recursively introduce the convex sets K_1 and K_0 as:
$0 \in K_1$, $K_1 \subset \{v_1 \in V : w + v_1 \in K\}$ *and, for a* $w_1 \in K_1$,
$0 \in K_0$, $K_0 \subset \{v_0 \in V_0 : w + w_1 + v_0 \in K\}$.

• assumptions on the decomposition of elements of the convex set K
- for the multiplicative algorithm

Assumption (2)

There exists a constant $C_0 > 0$ such that for any $u, w \in K$, any $w_{1i} \in V_{1i}$, $w_{11} + \ldots + w_{1i} \in K_1$, $i = 1, \ldots, m$, and any $w_0 \in K_0$, there exist $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$, which satisfy

$$u_{11} \in K_1 \text{ and } w_{11} + \ldots + w_{1i-1} + u_{1i} \in K_1,\ i = 2, \ldots, m,\ u_0 \in K_0$$
$$u - w = \sum_{i=1}^m u_{1i} + u_0 \text{ and } \sum_{i=1}^m ||u_{1i}|| + ||u_0|| \le C_0(||u - w|| + \sum_{i=1}^m ||w_{1i}|| + ||w_0||).$$

Convex sets K_1 and K_0 are constructed as in Assumption 1 using w and $w_1 = w_{11} + \ldots + w_{1m}$.

- for the additive algorithm

Assumption (3)

There exist a constant $C_0 > 0$ such that for any $u, w \in K$, there exist $u_{1i} \in V_{1i} \cap K_1$, $i = 1, \ldots, m$, and $u_0 \in K_0$, which satisfy

$$u - w = \sum_{i=1}^m u_{1i} + u_0 \text{ and } \sum_{i=1}^m ||u_{1i}|| + ||u_0|| \le C_0 ||u - w||.$$

Convex sets K_1 and K_0 are constructed as in Assumption 1 with the above w and $w_1 = 0$.

- conditions containing constant C_0 are named stability conditions of the decomposition.

• $F : V \to \mathbf{R}$ - Gâteaux differentiable functional:
- there exist $p, q > 1$, and for any $M > 0$ there exist α_M, $\beta_M > 0$ for which

$$\alpha_M ||v - u||^p \leq < F'(v) - F'(u), v - u >, \quad ||F'(v) - F'(u)||_{V'} \leq \beta_M ||v - u||^{q-1},$$

for any $u, v \in K$, $||u||, ||v|| \leq M$

$$\Downarrow$$

$1 < q \leq 2 \leq p$

F strictly convex functional

for any $u, v \in V$, $\|u\|, \|v\| \leq M$, we have

$$\alpha_M \|v - u\|^p \leq \langle F'(v) - F'(u), v - u \rangle \leq \beta_M \|v - u\|^q$$

and

$$\begin{aligned} \langle F'(u), v - u \rangle + \frac{\alpha_M}{p} \|v - u\|^p &\leq F(v) - F(u) \\ &\leq \langle F'(u), v - u \rangle + \frac{\beta_M}{q} ||v - u||^q \end{aligned}$$

Subspace correction algorithms

for variational inequalities of the second kind

- $\varphi : V \to \mathbf{R}$ convex lower semicontinuous functional
  - $F + \varphi$ coercive in the sense $F(v) + \varphi(v) \to \infty$, as $\|v\| \to \infty$, $v \in K$, if K is not bounded
  - technical assumption in the multiplicative case

$$\begin{aligned}
&\textstyle\sum_{i=1}^{m}[\varphi(w + \sum_{j=1}^{i-1} w_{1j} + u_{1i}) - \varphi(w + \sum_{j=1}^{i-1} w_{1j} + w_{1i})] \\
&+\varphi(w + w_1 + u_0) - \varphi(w + w_1 + w_0) \\
&\le \varphi(u) - \varphi(w + \textstyle\sum_{i=1}^{m} w_{1i} + w_0)
\end{aligned}$$

for $u, w \in K$, $u_{1i}, w_{1i} \in V_{1i}$ and $u_0, w_0 \in V_0$ as in Assumption 2, and $w_1 = \sum_{j=1}^{m} w_{1j}$.

  - technical assumption in the additive case,

$$\sum_{i=1}^{m} \varphi(w + u_{1i}) + \varphi(w + u_0) \le m\varphi(w) + \varphi(u)$$

for any $u, w \in K$, $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$ which satisfy Assumption 3

• variational inequality of the second kind

$$u \in K : \langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K \tag{1}$$

- problem has a unique solution

• to solve problem (1), we introduce two algorithms, one of multiplicative type and another one of the additive type

– iterations have an optimal computing complexity

• multiplicative algorithm for problem (1)

Algorithm (1)

We start the algorithm with an arbitrary $u^0 \in K$*. Assuming that after* $n \geq 0$ *iterations we have* $u^n \in K$*, we successively perform the following steps:*
- at the level 1, we construct the convex set K_1 *as in Assumption 1 with* $w = u^n$*. Then, we first write* $w_1^n = 0$*, and, for* $i = 1, \ldots, m$*, we successively calculate* $w_{1i}^{n+1} \in V_{1i}$*,* $w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1} \in K_1$*, the solution of the inequalities*

$$\begin{aligned}&\langle F'(u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1}\rangle \\ &+\varphi(u^n + w_1^{n+\frac{i-1}{m}} + v_{1i}) - \varphi(u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}) \geq 0,\end{aligned}$$

for any $v_{1i} \in V_{1i}$*,* $w_1^{n+\frac{i-1}{m}} + v_{1i} \in K_1$*, and write* $w_1^{n+\frac{i}{m}} = w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}$*,*
- at the level 0*, we construct the convex set* K_0 *as in Assumption 1 with* $w = u^n$ *and* $w_1 = w_1^{n+1}$*. Then, we calculate* $w_0^{n+1} \in K_0$*, the solution of the inequality*

$$\begin{aligned}&\langle F'(u^n + w_1^{n+1} + w_0^{n+1}), v_0 - w_0^{n+1}\rangle \\ &+\varphi(u^n + w_1^{n+1} + v_0) - \varphi(u^n + w_1^{n+1} + w_0^{n+1}) \geq 0,\end{aligned}$$

for any $v_0 \in K_0$*,*
- we write $u^{n+1} = u^n + w_1^{n+1} + w_0^{n+1}$*.*

• additive algorithm for problem (1)

Algorithm (2)

We start the algorithm with an $u^0 \in K$*. Assuming that after* $n \geq 0$ *iterations we have* $u^n \in K$*, we simultaneously perform, the following steps:*

- we construct the convex sets K_1 *and* K_0 *as in Assumption 1 with* $w = u^n$ *and* $w_1 = 0$*,*

- for $i = 1, \ldots, m$*, we simultaneously calculate:*

(a) $w_{1i}^{n+1} \in V_{1i} \cap K_1$*, the solutions of the inequalities*

$$\langle F'(u^n + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1} \rangle + \varphi(u^n + v_{1i}) - \varphi(u^n + w_{1i}^{n+1}) \geq 0,$$

for any $v_{1i} \in V_{1i} \cap K_1$*, write* $w_1^{n+1} = \sum_{i=1}^m w_{1i}^{n+1}$*,*

(b) $w_0^{n+1} \in K_0$*, the solution of the inequality*

$$\langle F'(u^n + w_0^{n+1}), v_0 - w_0^{n+1} \rangle + \varphi(u^n + v_0) - \varphi(u^n + w_0^{n+1}) \geq 0,$$

for any $v_0 \in K_0$*,*

Then, we write $u^{n+1} = u^n + \frac{r}{m+1}(w_1^{n+1} + w_0^{n+1})$*, with a fixed* $0 < r \leq 1$*.*

- by introduction of the level convex sets, the additional interpolations to check the coarse-grid constraints are avoided

Theorem (1)

Let V be a reflexive Banach space, $V_0, V_{11}, \cdots, V_{1m}$ some closed subspaces of V and K a non empty closed convex subset of V which satisfies the previous assumptions. Also, we assume that F is Gâteaux differentiable, φ is convex and lower semicontinuous and they have the above properties. Let $M = \sup\{||v|| : F(v) + \varphi(v) \leq F(u^0) + \varphi(u^0)\}$ where u^0 is the starting point in Algorithms 1 or 2. Then, the norms of the approximations of the solution u of problem (1) obtained from these algorithms are bounded by M and we have the following error estimations:
(i) if $p = q = 2$ we have

$$F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq (\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)],$$

$$\|u^n - u\|^2 \leq \tfrac{2}{\alpha_M} (\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)].$$

(ii) if $p > q$ we have

$$F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}},$$

$$\|u - u^n\|^p \leq \frac{p}{\alpha_M} \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}.$$

Constants $C_1 > 0$ and $C_2 > 0$ depend on the functional F, the solution u, the initial approximation u^0, m and the constant C_0.

Remark (1)

– *for Algorithm 1, constants C_1 and C_2 can be written as,*

$$C_1 = \beta_M(1+2C_0)(m+1)^{2-\frac{q}{p}}\left(\frac{p}{\alpha_M}\right)^{\frac{q}{p}}(F(u^0)-F(u)$$
$$+\varphi(u^0)-\varphi(u))^{\frac{p-q}{p(p-1)}} + \beta_M C_0(m+1)^{\frac{p-q+1}{p}}\frac{1}{\varepsilon^{\frac{1}{p-1}}}\left(\frac{p}{\alpha_M}\right)^{\frac{q-1}{p-1}}$$
$$C_2 = \frac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}+(q-1)C_1^{\frac{p-1}{q-1}}}$$

where $\varepsilon = \alpha_M/(p\beta_M C_0(m+1)^{\frac{p-q+1}{p}})$.
– *in the case of Algorithm 2, these constants can be written as,*

$$C_1 = \frac{m+1}{r}\left[1-\frac{r}{m+1}+(1+C_0)(m+1)\frac{\beta_M}{\frac{\alpha_M}{2}}+C_0^2(m+1)\left(\frac{\beta_M}{\frac{\alpha_M}{2}}\right)^2\right]$$
$$C_2 = \frac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}+(q-1)C_3^{\frac{p-1}{q-1}}}$$

where $C_3 = \frac{m+1-r}{r}[F(u^0)-F(u)+\varphi(u^0)-\varphi(u)]^{\frac{p-q}{p-1}} + \left(\frac{m+1}{r}\right)^{\frac{q}{p}}\frac{\beta_M(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p}}}\cdot$

$$(F(u^0)-F(u)+\varphi(u^0)-\varphi(u))^{\frac{p-q}{p(p-1)}} + \left(\frac{m+1}{r}\right)^{\frac{q-1}{p-1}}\frac{\beta_M^{\frac{p}{p-1}}C_0^{\frac{p}{p-1}}(m+1)^{q-1}}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p-1}}}$$

Subspace correction algorithms

for quasi-variational inequalities

- in this section we assume that $p = q = 2$
- $\varphi : V \times V \to \mathbf{R}$ functional (conditions imposed on the second argument of $\varphi(u,v)$ are similar with conditions of $\varphi(v)$ in the case of the variational inequalities of the second kind)
  - for any $u \in V$, $\varphi(u,\cdot) : V \to \mathbf{R}$ is convex and lower semicontinuous
  - if K is not bounded, $F + \varphi$ is coercive in the sense that $F(v) + \varphi(u,v) \to \infty$, as $\|v\| \to \infty,\ v \in K$, for any $u \in K$
  - for any $M > 0$ there exists $c_M > 0$ such that

$$|\varphi(v_1,w_2) + \varphi(v_2,w_1) - \varphi(v_1,w_1) - \varphi(v_2,w_2)| \le c_M \|v_1 - v_2\| \|w_1 - w_2\|$$

    for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $\|v_1\|,\ \|v_2\|,\ \|w_1\|\ \|w_2\| \le M$
  - technical assumption in the multiplicative case

$$\begin{aligned}&\textstyle\sum_{i=1}^m [\varphi(u, w + \sum_{j=1}^{i-1} w_{1j} + u_{1i}) - \varphi(u, w + \sum_{j=1}^{i-1} w_{1j} + w_{1i})]\\ &+\varphi(u, w + w_1 + u_0) - \varphi(u, w + w_1 + w_0) \le \varphi(u,v) - \varphi(u, w + \textstyle\sum_{i=1}^m w_{1i} + w_0)\end{aligned}$$

    for $u, w \in K$, $u_{1i}, w_{1i} \in V_{1i}$ and $u_0, w_0 \in V_0$ satisfying Assumption 2, and $w_1 = \sum_{j=1}^m w_{1j}$.
  - technical assumption in the additive case

$$\textstyle\sum_{i=1}^m \varphi(u, w + u_{1i}) + \varphi(u, w + u_0) \le m\varphi(u,w) + \varphi(u,u)$$

    for any $u, w \in K$, $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$ which satisfy Assumption 3

• quasi-variational inequality

$$u \in K : \langle F'(u), v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \text{ for any } v \in K. \quad (2)$$

- (existence and uniqueness condition of the solution) problem (2) has a unique solution if there exists a constant $\varkappa < 1$ such that

$$\frac{c_M}{\alpha_M} \leq \varkappa \text{ for any } M > 0. \quad (3)$$

• to solve problem (2), we introduce two algorithms, one of multiplicative type and another one of the additive type

- algorithms perform κ inner iterations at each iteration, where the first argument of φ is kept unchanged (having the value obtained at the previous iteration)
- convergence conditions are independent of the number of subspaces
- iterations have an optimal computing complexity

• multiplicative algorithm for problem (2)

Algorithm (3)

We start the algorithm with an arbitrary $u^0 \in K$*. Assuming that after* $n \geq 0$ *iterations, we have* $u^n \in K$*, we write* $\tilde{u}^n = u^n$ *and carry out the following two steps:*

1. We perform $\kappa \geq 1$ *multiplicative iterations, keeping the first argument of* φ *equal with* u^n*. We start with* $\tilde{u}^n$ *and having* $\tilde{u}^{n+k-1}$ *at iteration* $1 \leq k \leq \kappa$*, we successively calculate level corrections and compute* $\tilde{u}^{n+k}$*:*

– at the level 1 we construct the convex set K_1 *as in Assumption 1 with* $w = \tilde{u}^{n+k-1}$*. Then, we first write* $w_1^k = 0$*, and, for* $i = 1, \dots, m$*, we successively calculate* $w_{1i}^{k+1} \in V_{1i}$*,* $w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1} \in K_1$*, the solutions of the inequalities*

$$\langle F'(\tilde{u}^{n+k-1} + w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1}), v_{1i} - w_{1i}^{k+1}\rangle + \varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+\frac{i-1}{m}} + v_{1i}) - \varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1}) \geq 0,$$

for any $v_{1i} \in V_{1i}$*,* $w_1^{k+\frac{i-1}{m}} + v_{1i} \in K_1$*, and write* $w_1^{k+\frac{i}{m}} = w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1}$

– at the level 0*, we construct the convex set* K_0 *as in Assumption 1 with* $w = \tilde{u}^{n+k-1}$ *and* $w_1 = w_1^{k+1}$*. Then, we calculate* $w_0^{k+1} \in K_0$*, the solution of the inequality*

$$\langle F'(\tilde{u}^{n+k-1} + w_1^{k+1} + w_0^{k+1}), v_0 - w_0^{k+1}\rangle + \varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+1} + v_0) - \varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+1} + w_0^{k+1}) \geq 0,$$

for any $v_0 \in K_0$

– we write $\tilde{u}^{n+k} = \tilde{u}^{n+k-1} + w_1^{k+1} + w_0^{k+1}$*.*

2. We write $u^{n+1} = \tilde{u}^{n+\kappa}$

• additive algorithm for problem (2)

Algorithm (4)

We start the algorithm with an $u^0 \in K$. *Assuming that after* $n \geq 0$ *iterations, we have* $u^n \in K$, *we write* $\tilde{u}^n = u^n$ *and carry out the following two steps:*

1. We perform $\kappa \geq 1$ *additive iterations, keeping the first argument of* φ *equal with* u^n. *We start with* $\tilde{u}^n$ *and having* $\tilde{u}^{n+k-1}$ *at iteration* $1 \leq k \leq \kappa$, *we simultaneously calculate level corrections and compute* $\tilde{u}^{n+k}$*:*

− we construct the convex sets K_1 *and* K_0 *as in Assumption 1 with* $w = \tilde{u}^{n+k-1}$ *and* $w_1 = 0$,

− for $i = 1, \dots, m$,*we simultaneously calculate:*

(*a*) $w_{1i}^{k+1} \in V_{1i} \cap K_1$, *the solutions of the inequalities*

$$\begin{array}{l}\langle F'(\tilde{u}^{n+k-1} + w_{1i}^{k+1}), v_{1i} - w_{1i}^{k+1}\rangle \\ +\varphi(u^n, \tilde{u}^{n+k-1} + v_{1i}) - \varphi(u^n, \tilde{u}^{n+k-1} + w_{1i}^{k+1}) \geq 0,\end{array}$$

for any $v_{1i} \in V_{1i} \cap K_1$, *write* $w_1^{k+1} = \sum_{i=1}^m w_{1i}^{k+1}$, *and*

(*b*) $w_0^{k+1} K_0$, *the solution of the inequality*

$$\begin{array}{l}\langle F'(\tilde{u}^{n+k-1} + w_0^{k+1}), v_0 - w_0^{k+1}\rangle \\ +\varphi(u^n, \tilde{u}^{n+k-1} + v_0) - \varphi(u^n, \tilde{u}^{n+k-1} + w_0^{k+1}) \geq 0,\end{array}$$

for any $v_0 \in K_0$.

− we write $\tilde{u}^{n+k} = \tilde{u}^{n+k-1} + \frac{r}{m+1}(w_1^{k+1} + w_0^{k+1})$, *with a fixed* $0 < r \leq 1$.

2. We write $u^{n+1} = \tilde{u}^{n+\kappa}$

Theorem (2)

Let V be a reflexive Banach space, $V_0, V_{11}, \cdots, V_{1m}$ some closed subspaces of V, and K a non empty closed convex subset of V which satisfy the previous assumptions. Also, we assume that F is Gâteaux differentiable, φ is convex and lower semicontinuous in the second variable and they satisfy the conditions in this section. Let $M = \sup\{||v|| : F(v) + \varphi(u, v) \leq F(u^0) + \varphi(u, u^0)\}$ where u is the solution of problem (2) and u^0 is its initial approximation in Algorithms 3 or 4. On these conditions, if

$$\frac{c_M}{\alpha_M} < \frac{1}{2} \text{ for any } M > 0 \tag{4}$$

and κ satisfies $\left(\frac{C_1}{C_1+1}\right)^\kappa < \dfrac{1-2\frac{c_M}{\alpha_M}}{1+3\frac{c_M}{\alpha_M}+4\frac{c_M^2}{\alpha_M^2}+\frac{c_M^3}{\alpha_M^3}}$ then, the norms of the approximations of the solution u of problem (2) obtained from these algorithms are bounded by M, the two algorithms are convergent and we have the following error estimations:

$$F(u^{n+1}) + \varphi(u, u^{n+1}) - F(u) - \varphi(u, u) \leq [2\frac{c_M}{\alpha_M} + (\frac{C_1}{C_1+1})^\kappa(1 + 3\frac{c_M}{\alpha_M} + 4\frac{c_M^2}{\alpha_M^2} + \frac{c_M^3}{\alpha_M^3})]^n \cdot$$
$$[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)]$$

and

$$\|u^n - u\|^2 \leq \frac{2}{\alpha_M}[2\frac{c_M}{\alpha_M} + (\frac{C_1}{C_1+1})^\kappa(1 + 3\frac{c_M}{\alpha_M} + 4\frac{c_M^2}{\alpha_M^2} + \frac{c_M^3}{\alpha_M^3})]^n \cdot$$
$$[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)].$$

Constant $C_1 > 0$ depends on the functionals F and φ, the solution u, the initial approximation u^0, m and the constant C_0.

Remark (2)

− *constant C_1 can be written as*

$$C_1 = (m+1)[(1+2C_0)\frac{\beta_M}{\frac{\alpha_M}{2}} + C_0^2(\frac{\beta_M}{\frac{\alpha_M}{2}})^2]$$

for Algorithm 3 and as

$$C_1 = \frac{m+1}{r}[1 - \frac{r}{m+1} + (1+C_0)(m+1)\frac{\beta_M}{\frac{\alpha_M}{2}} + C_0^2(m+1)(\frac{\beta_M}{\frac{\alpha_M}{2}})^2]$$

for Algorithm 4

Remark (3)

Theorem 2 shows that:

1. *if the convergence condition (4) is satisfied and the number κ of the inner iterations is sufficiently large then Algorithms 3 and 4 are globally convergent and we have estimated the errors*
2. *convergence condition (4) is a little more restrictive than the existence and uniqueness condition of the solution (3) but they are of the same type*
3. *convergence condition (4) does not depend on the number m of subdomains*

Two-level methods

- two simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ of the domain $\Omega \subset \mathbf{R}^d$ of mesh sizes h and H
- both the families, of fine and coarse meshes, are regular and mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$
- domain Ω is decomposed as $\Omega = \bigcup_{i=1}^{m} \Omega_i$, the overlapping parameter will be denoted by δ
- $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$
- $\mathrm{diam}(\Omega_i) \leq CH$, constant C is a generic constant, independent of both meshes and of the overlapping parameter
- domain Ω may be different from $\Omega_0 = \cup_{\tau \in \mathcal{T}_H} \tau$, but we assume
  - if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$ then it also lies on $\partial\Omega$
  - $\mathrm{dist}(x, \Omega_0) \leq CH$ for any node x of $\mathcal{T}_h$

- piecewise linear finite element spaces
  - $V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\}$ corresponding to the domain Ω
  - $V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega \backslash \Omega_i\}$, for $i = 1, \ldots, m$, corresponding to the domain decomposition
  - $V_H^0 = \{v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0\}$ corresponding to the coarse decomposition, H-level, where the functions v are extended with zero in $\Omega \backslash \Omega_0$

- spaces V_h and V_h^i, $i = 1, \ldots, m$, and V_H^0 are considered as subspaces of $W^{1,s}$, for some fixed $1 < s < \infty$ (we denote by $\|\cdot\|_{0,s}$ the norm in L^s, and by $\|\cdot\|_{1,s}$ and $|\cdot|_{1,s}$ the norm and seminorm in $W^{1,s}$)

- problems (1) and (2) are considered in the space $V = V_h$ with the convex set of the form $K = \{v \in V_h : a \leq v \leq b\}$, where $a,\ b \in V_h$, $a \leq b$

- the two-level methods are obtained from the algorithms in the previous sections with $V_0 = V_H^0$, $V_{11} = V_h^1, \ldots, V_{1m} = V_h^m$

• technical conditions are satisfied by approximations of the functionals φ, obtained by numerical quadrature formulae in V_h, of the form

– for variational inequalities of the second kind

$$\varphi(v) = \sum_{k\in\mathcal{N}_h} s_k(h)\phi(v(x_k))$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is a continuous and convex function, $\mathcal{N}_h$ is the set of nodes of the mesh partition $\mathcal{T}_h$, and $s_k(h) \geq 0$, $k \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h

– for the quasi-variational inequalities

$$\varphi(u, v) = \sum_{k\in\mathcal{N}_h} s_k(h)\phi(u, v(x_k))$$

where $\phi : V_h \times \mathbf{R} \to \mathbf{R}$ is continuous, and, as above, $s_k(h) \geq 0$, $k \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. Also, we assume that $\phi(u, \cdot) : \mathbf{R} \to \mathbf{R}$ is convex for any $u \in V_h$

• level convex sets K_1 and K_0 can be constructed as in the following proposition

Proposition (1)

Assumption 1 holds for the convex sets K_1 and K_0 defined as

$$K_1 = [a_1, b_1],\ a_1 = a - w,\ b_1 = b - w,$$
$$K_0 = [a_0, b_0],\ a_0 = I_H(a_1 - w_1),\ b_0 = I_H(b_1 - w_1)$$

for any $w \in K$ and $w_1 \in K_1$, $I_H : V_h \to V_H^0$ being a nonlinear interpolation operator.

• the following proposition shows that the convergence rate of the algorithms depends very weakly (through constant C_0 in Assumptions 2 and 3) on the mesh and domain decomposition parameters and it is independent of them if H/δ and H/h are kept constant when $h \to 0$

Proposition (2)

Assumptions 2 and 3 for the convex sets K_1 and K_0 defined in Proposition 1 holds with the constant C_0 written as

$$C_0 = C(m+1)C_{d,s}(H,h)[1+(m-1)\tfrac{H}{\delta}]$$

where C is independent of the mesh and domain decomposition parameters, m is the number of subdomains and

$$C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or} \\ & 1 \le d < s \le \infty \\ (\ln \frac{H}{h} + 1)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ (\frac{H}{h})^{\frac{d-s}{s}} & \text{if } 1 \le s < d < \infty, \end{cases}$$

Remark (4)

The results have referred to problems in $W^{1,s}$ with Dirichlet boundary conditions. Similar results can be obtained for problems in $(W^{1,s})^d$ or problems with mixed boundary conditions.

Concluding remarks

- algorithms have been introduced as subspace correction methods in a reflexive Banach space for varaitional inequalities of the second kind and quasi-variational inequalities
- we have proved their global convergence and estimated the errors by making
  - an assumption on the construction of the level convex sets
  - assumptions on the decomposition of the elements of the convex set according with subspace decomposition
  - technical assumptions
- additive and multiplicative two-level Schwarz methods are obtained by using the finite element spaces
  - we have proved that assumptions made in the general framework as well as the technical assumptions hold for closed convex sets K of two-obstacle type
  - we explicitly write the dependence of the constant C_0 (introduced in the stability condition of the decomposition in assumptions) on the domain decomposition and mesh parameters
  - from Theorems 1 and 2, we can conclude
    - the two-level methods are globally convergent for variational inequalities of the second kind and quasi-variational inequalities
    - convergence rates essentially depend on the constant C_0
    - in view of the dependence of C_0 on the mesh and domain decomposition parameters, the convergence rate depends very weakly on the mesh and domain decomposition parameters, and it is even independent of them for some particular choices
    - the introduced methods have an optimal computing complexity per iteration