PROJET DE RECHERCHES: *ALMOST FIBRATIONS*

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Recent publications related to the present project:

(1) D.Andrica, D.Mangra and C.Pintea, *The circular Morse-Smale characteristic of closed surfaces*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 57(105) (2014), no. 3, 235-242.
(2) Z.Balogh, C.Pintea and H.Rohner, *Size of tangencies to non-involutive distributions*, Indiana Univ. Math. J. 60 (2011), no. 6, 2061-2092.
(3) FPZ L. Funar, C. Pintea, and P. Zhang, *Examples of smooth maps with finitely many critical points in dimensions (4, 3), (8, 5) and (16, 9)*, Proc. Amer. Math. Soc. 138 (2010), 355–365.
(4) L. Funar, *Global classification of isolated singularities in dimensions* $(4, 3)$ *and* $(8, 5)$, Ann. Scuola Norm. Sup. Pisa CI. Sci. (5), Vol. X (2011), 819–861.
(5) FP L.Funar and C.Pintea, *Manifolds which admit maps with finitely many critical points into spheres of small dimensions*, 21p., Michigan Math. J., to appear, axiv:1611.04344.

Context. The problem of constructing a resolution of singularities of projective varieties is one of the most fundamental obstructions to our understanding of their analytic, arithmetic and geometric properties. A class of n-dimensional projective varieties is *dominant* if for every projective n-dimensional variety X, there exists Y from that class and a surjective morphism $Y \to X$. In this terminology, a result of de Jong says that smooth projective varieties form a dominant class over any field and in any dimension. A conjecture due to Bogomolov and Husemöller (see [3]) claims that the class of n-dimensional smooth projective varieties X, endowed with a tower of smooth fibrations $X \to X_1 \to \cdots \to X_n$, with $\dim X_i = n - i$, is dominant, for any n.

We consider here a topological version of this problem, in which we consider smooth maps $M \to N$ between compact manifolds with only isolated singularities, called almost fibrations. Our aim is to understand whether there exist genuine almost fibrations and whether towers of almost fibrations could be dominant. This is related to problems on real singularities and generalized open book decompositions.

State of the art. We set $\varphi(M, N)$ for the minimum number of critical points of a smooth map $M \to N$ between compact manifolds, which extends the F-category defined and studied by Takens in [13]. Following the work of Farber (see [4, 5]) we have:

$$\varphi(M, S^1) = \begin{cases} \varphi(M, \mathbb{R}), & \text{if } H^1(M, \mathbb{Z}) = 0; \\ 0, & \text{if } M \text{ fibers over } S^1; \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

More precisely for any non-zero class ξ in $H^1(M, \mathbb{Z})$ there exists a function $f : M \to S^1$ in the homotopy type prescribed by ξ with at most one critical point. This was extended in [5] to closed 1-forms in a prescribed non-zero class in $H^1(M, \mathbb{R})$ having at most one zero. The question on whether there is a closed non-singular 1-form (i.e. a fibration over S^1 for integral classes) was answered by Thurston in dimension 3 (see [14]) and Latour for $\dim(M) \geq 6$ (see [11]). Notice that $\varphi(M, \mathbb{R}) \leq \dim M + 1$ (see [13]).

In [1] the authors found that $\varphi(M^m, N^n) \in \{0, 1, \infty\}$, when $0 \leq m - n \leq 2$, except for the exceptional pairs of dimensions $(m, n) \in \{(2, 2), (4, 3), (4, 2)\}$. Further, if $m - n = 3$ and there exists a smooth function $M^m \to N^n$ with finitely many critical points, all of them cone-like, then $\varphi(M^m, N^n) \in \{0, 1\}$ except for the exceptional pairs of dimensions $(m,n) \in \{(5,2),(6,3),(8,5)\}$. On the other hand in [7] the authors provided many nontrivial examples and showed that $\varphi(M^m, S^n)$ can take arbitrarily large even values for $m = 2n-2$, $n \in \{3,5,9\}$; these examples were classified in [6] for $n \in \{3,5\}$.

In [8] the authors show that there are examples of manifolds M^m with *nontrivial* (i.e. finite nonzero) $\varphi(M^m, S^{[\frac{m}{2}]-k})$, for $m \geq 6$, $m \geq 2k \geq 0$ where, when present, the superscripts denote the dimensions of the corresponding manifolds and to describe how to construct all of them for $(m,n) = (6,3)$. They were buliding on [6], were one outlined a method for constructing manifolds with finite $\varphi(M^6, S^3)$ using generalized *Hopf links*, which was further detailed in [2]. In some sense these provide other high dimensional analogs of Lefschetz fibrations. The main technical ingredient is the explicit description of a family of fibered links in spheres, starting from Haefliger's classification of links in stable range.

Questions. These new open book decompositions might be useful for the study of (generalized) Pfaffian equations (see [9]), in a similar way to Giroux' description of contact structures in dimension 3. Our project aims at a better understanding of these almost fibrations. For instance we don't know whether critical points can merge together, while there are lower bounds for $\varphi(M^{2n-2}, S^n)$ (see [7]). No nontrivial examples are known for $m < 2n-2$, though as polynomials maps with isolated singularities do exist ([12]) for $m - n \geq 4$. We plan to construct such examples in any co-dimension larger than 4.

A particularly exciting case is the dimension 4, where we wish to analyse the corresponding problems in the case of holomorphic/Lagrangian fibrations at least in the case of elliptic fibrations. Using some deformations of the torus almost fibrations of the sphere S^4 over S^2 we plan to prove that $\varphi(S^4, S^2) = 1$. Further, we wish to show that a smooth closed 4-manifold M has finite $\varphi(M^4, S^2)$ if and only it admits an *achiral* Lefschetz fibration over the 2-sphere. Moreover, if $\pi_1(M^4) = 0$ this could only happen if M^4 has a handlebody decompoition without index one handles (namely it is geometrically simply connected). There are examples of smooth 4-manifolds which are not achiral Lefschetz fibrations, although none of them is simply connected.

A related question is to determine the possible topology of fibers of almost fibrations. More specifically, we wish to state necessary and sufficient conditions for a given embedded submanifold $F^{m-n} \subset M^m$ to be (contained in) the fiber of a smooth map $M^m \to N^n$ with small critical sets (e.g. finite, or empty etc). This and related problems have been extensively studied in foliation theory and the theory of integrable submanifolds (see [10]).

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