# Bifurcation set of polynomial applications

A central problem in Mathematics is to describe the bifurcation locus of continuous (differentiable, analytic, polynomial) maps $F : M \to N$ between two topological (smooth, analytic, algebraic) manifolds. The bifurcation locus of $F$ is the smallest subset of $B(F) \subset N$ outside which $F$ is a locally trivial fibration. If $F$ is proper, then $F$ is a locally trivial fibration, hence $B(F)$ is empty, due to a well-known theorem due to Ehresmann. We are interested in non-proper maps $F$, especially polynomial maps $F : K^m \to K^n$, where $K$ is either $\mathbb{R}$ or $\mathbb{C}$, and $m > n$.

For polynomial maps $F : \mathbb{C}^2 \to \mathbb{C}$, M. Suzuki [Su] and, respectively, Hà H. V. and Lê D. T. [HL] have proven that $x \in \mathbb{C}$ is *not* a bifurcation point if and only if the Euler characteristic of the fibers $F^{-1}(t)$ is constant for $t$ varying in a neighborhood of $x$.
For real polynomial maps $F : \mathbb{R}^2 \to \mathbb{R}$, it was shown by M. Tibăr and A. Zaharia [TZ] that the constancy of the Euler characteristic does not suffice to detect the bifurcation points and that two new phenomena may appear: *vanishing at infinity* of connected components when $t$ approaches $x$, and the *splitting at infinity* of a connected component in several components.

Our project is based on the recent joint work done by the authors [JT1, JT2] and aims to find more general developments.

Namely, in a recent manuscript JT1 we have extended the above named results of Tibăr and Zaharia to the case of polynomial maps $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $n \geq 2$.

In the complex setting, Hà H.V. and Nguyen T.T. [HN] have shown an example of a polynomial map $F : \mathbb{C}^3 \to \mathbb{C}^2$ where the Euler characteristic of the fibers is constant, but the map $F$ is not a locally trivial fibration. Thus the Euler characteristic test is no more valid in dimensions higher that 2. In this new example the Betti numbers are not constant, which fact would give the hope that asking the Betti numbers constancy would force local triviality. Starting from a similar questions asked by Gurjar, we have produced an example of a polynomial map $F : \mathbb{C}^3 \to \mathbb{C}^2$ such that in a neighborhood of the origin $(0, 0) \in \mathbb{C}^2$ all points are regular values of $F$, the Betti numbers of $F^{-1}(t)$ are constant but still $F$ is not a locally trivial fibration [JT2].

Moreover, the main result in [JT2] gives a complete characterization of the bifurcation set $B(F)$ for polynomial maps $F : \mathbb{C}^{n+1} \to \mathbb{C}^n$, as follows:

**Theorem.** *Suppose that in a neighborhood of $x \in \mathbb{C}^n$ all points are regular values of $F$. Then $x$ is not a bifurcation point of $F$ if and only if the following two conditions are satisfied:*

*a) the Euler characteristic of $F^{-1}(t)$ is constant for $t$ varying in a neighborhood of $x$,*
*b) no connected component of $F^{-1}(t)$ vanishes at infinity as $t \to x$.*

In particular, if the fibers are connected and their Euler characteristic is constant then $F$ is a locally trivial fibration around each regular value. In fact, the above theorem was proved in a more general setting, namely for holomorphic submersions $F : M \to N$ between two complex manifolds such that $M$ is a Stein manifold and the Betti numbers of the fibers are finite. The main tools used in the proof were the results of Ilyashenko [Il] regarding uniformization of foliations by complex curves of Stein manifolds, the results of G. Meigniez from [Me] and various results regarding Runge domains in Stein manifolds.

In our project we carry out our investigations and study the bifurcation locus of polynomial maps $F : \mathbb{C}^m \to \mathbb{C}^n$ for general $m > n$. We propose to find sufficient conditions and, if possible, necessary and sufficient conditions for the local triviality of $F$ in terms of the topology of the fibers. At the same time we shall investigate whether it is possible to give an algebraic characterization for the non-vanishing condition that appears in the main results of [JT2] and [JT1].

# References


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